Questions related to SECTION 8.1

1. (a) Find the next two terms of the sequence.
   (b) Find a recurrence relation that generates the relation.
   (c) Find an explicit formula for the general nth term of the sequence.

   i. $\left\{1, \frac{1}{2},, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, ...\right\}$

   (a) $\frac{1}{32}$ and $\frac{1}{64}$

   (b) $a_1 = 1; \qquad a_{n+1} = \frac{a_n}{2}$

   (c) $a_n = \frac{1}{2^{n-1}}$

   ii. $\left\{1, -2, 3, -4, 5, ...\right\}$

   (a) -6 and 7

   (b) $a_1 = 1; \qquad a_{n+1} = (-1)^n(|a_n| + 1)$

   (c) $a_n = (-1)^{n+1}n$

2. Write the terms a_1, a_2, a_3 and a_4 of the following sequences. If the sequence appears to converge, make a conjecture about its limit. If the sequence diverges, explain why.

   (a) $a_n = \dfrac{(-1)^n}{n}; \quad n = 1, 2, 3, ...$

   $$a_1 = -1 \quad , \quad a_2 = \frac{1}{2} \quad , \quad a_3 = -\frac{1}{3} \quad and \quad a_4 = \frac{1}{4}$$

   This sequence converges to 0 since each term is smaller in absolute value than the preceding term and they get arbitrarily close to 0.

   (b) $a_n = 1 - 10^{-n}; \quad n = 1, 2, 3, ...$

   $$a_1 = 0.9 \quad , \quad a_2 = 0.99 \quad , \quad a_3 = 0.999 \quad and \quad a_4 = 0.9999$$

   This sequence converges to 1.

(c) $a_{n+1} = \dfrac{a_n^2}{10}; \quad a_0 = 1$

Rewrite the recurrence as $a_{n+1} = 10^{-1}a_n^2$. Then we have

$$a_0 = 1 \quad , \quad a_1 = 10^{-1} = \frac{1}{10} \quad , \quad a_2 = 10^{-1}(10^{-1})^2 = 10^{-3} = \frac{1}{1000} \quad ,$$
$$a_3 = 10^{-1}(10^{-3})^2 = 10^{-7} = \frac{1}{10000000},$$
$$a_4 = 10^{-1}(10^{-7})^2 = 10^{-15} = \frac{1}{1000000000000000}$$

This sequence converges to 0.

(d) $a_{n+1} = 0.5a_n(1 - a_n); \quad a_0 = 0.8$

$$a_0 = 0.8 \quad , \quad a_1 = 0.5 \cdot 0.8(1 - 0.8) = 0.5 \cdot 0.8 \cdot 0.2 = 0.08 \quad ,$$
$$a_2 = 0.5 \cdot 0.08(1 - 0.08) = 0.5 \cdot 0.08 \cdot 0.92 = 0.0368 \quad ,$$
$$a_3 = 0.5 \cdot 0.0368(1 - 0.0368) = 0.01772288$$
$$a_4 = 0.5 \cdot 0.01772288(1 - 0.01772288) \approx 0.0087$$

3. Consider the following recurrence relations.

   (a) Find the terms a_0, a_1, a_2 and a_3 of the sequence.

   (b) If possible, find an explicit formula for the n^{th} term of the sequence.

   i. $a_{n+1} = a_n + 2; \quad a_0 = 3$

   ii. $a_{n+1} = 2a_n + 1; \quad a_0 = 0$

   i.

$$a_0 = 3 \quad , \quad a_1 = 5 \quad , \quad a_2 = 7 \quad and \quad a_3 = 9$$
$$a_n = 2n + 3$$

   ii.

$$a_0 = 0 \quad , \quad a_1 = 1 \quad , \quad a_2 = 3 \quad , \quad a_3 = 7 \quad and \quad a_4 = 15$$
$$a_n = 2^n - 1$$

Questions related to SECTION 8.2

1. Find the limit of the following sequences or determine that the limit does not exist.

   (a) $\left\{\dfrac{3n^3-1}{2n^3+1}\right\}$

   Dividing the numerator and the denominator by n^3 we get :

$$\lim_{n\to\infty}\frac{3-n^{-3}}{2+n^{-3}}=\frac{3}{2}$$

   (b) $\left\{\left(1+\dfrac{2}{n}\right)^n\right\}$

   Find the limit of the logarithm of the expression, which is $n\ln\left(1+\dfrac{2}{n}\right)$, using L'Hopital's rule.

$$\lim_{n\to\infty} n\ln\left(1+\frac{2}{n}\right)=\lim_{n\to\infty}\frac{\ln\left(1+\frac{2}{n}\right)}{\frac{1}{n}}=\lim_{n\to\infty}\frac{\frac{1}{1+\frac{2}{n}}\left(-\frac{2}{n^2}\right)}{-\frac{1}{n^2}}=\lim_{n\to\infty}\frac{2}{1-\frac{2}{n}}=2$$

   Thus the limit of the original expression is e^2.

   (c) $\left\{\sqrt{\left(1+\dfrac{1}{2n}\right)^n}\right\}$

   Take the logarithm of the expression and use L'Hopital's rule.

$$\lim_{n\to\infty}\frac{n}{2}\ln\left(1+\frac{1}{2n}\right)=\lim_{n\to\infty}\frac{\ln\left(1+\frac{1}{2n}\right)}{\frac{2}{n}}=\lim_{n\to\infty}\frac{\frac{1}{1+\frac{1}{2n}}\left(-\frac{1}{2n^2}\right)}{-\frac{2}{n^2}}=\frac{1}{4}$$

   Thus the limit of the original expression is $e^{\frac{1}{4}}$.

   (d) $\left\{\dfrac{\ln\left(\frac{1}{n}\right)}{n}\right\}$

   Since $\ln\left(\dfrac{1}{n}\right)=-\ln n$, we get

$$\lim_{n\to\infty}\frac{\ln\left(\frac{1}{n}\right)}{n}=\lim_{n\to\infty}\frac{-\ln n}{n}$$

   Then by applying L'Hopital's rule we get :

$$\lim_{n\to\infty}\frac{\ln\left(\frac{1}{n}\right)}{n}=\lim_{n\to\infty}\frac{-\ln n}{n}=\lim_{n\to\infty}\frac{-\frac{1}{n}}{1}=-\lim_{n\to\infty}\frac{1}{n}=0$$

(e) $\left\{ \left(\frac{1}{n}\right)^{1/n} \right\}$

Find the limit of the logarithm of the expression, which is $\frac{1}{n}\ln\left(\frac{1}{n}\right)$, using L'Hopital's rule.

$$\lim_{n\to\infty} \frac{1}{n}\ln\left(\frac{1}{n}\right) = \lim_{n\to\infty} \frac{-\ln\, n}{n} = \lim_{n\to\infty} \frac{-\frac{1}{n}}{1} = \lim_{n\to\infty} \frac{-1}{n} = 0$$

Thus the limit of the original expression is e^0.

(f) $\left\{ \left(1 - \frac{4}{n}\right)^{n} \right\}$

Find the limit of the logarithm of the expression, which is $n\ln\left(1-\frac{4}{n}\right)$, using L'Hopital's rule.

$$\lim_{n\to\infty} n\ln\left(1-\frac{4}{n}\right) = \lim_{n\to\infty} \frac{\ln\left(1-\frac{4}{n}\right)}{\frac{1}{n}} = \lim_{n\to\infty} \frac{\frac{1}{1-\frac{4}{n}}\left(\frac{4}{n^2}\right)}{-\frac{1}{n^2}} = \lim_{n\to\infty} \frac{-4}{1-\frac{4}{n}} = -4$$

Thus the limit of the original expression is e^{-4}.

(g) $a_n = e^{-n}\cos\, n$

The sequence is

$$a_n = e^{-n}\cos\, n = \frac{\cos\, n}{e^n}$$

The numerator of the sequence is bounded by 1 and the denominator increases without any bound, so:

$$\lim_{n\to\infty} e^{-n}\cos\, n = \lim_{n\to\infty} \frac{\cos\, n}{e^n} = 0$$

(h) $a_n = \frac{\ln\, n}{n^{1.1}}$

Using L'Hopital's rule, we have

$$\lim_{n\to\infty} \frac{\ln\, n}{n^{1.1}} = \lim_{n\to\infty} \frac{\frac{1}{n}}{(1.1)n^{0.1}} = \lim_{n\to\infty} \frac{1}{(1.1)n^{1.1}} = 0$$

Questions related to SECTION 8.3

1. Evaluate the following geometric sums.

   (a) $\sum_{k=0}^{20} \left(\frac{2}{5}\right)^{2k}$

   We have $a_1 = 1$, $r = \frac{4}{25}$ and $n = 21$

   $$S = \frac{a_1(1-r^n)}{1-r} = 1 \cdot \frac{1-\left(\frac{4}{25}\right)^{21}}{1-\frac{4}{25}} = \frac{25^{21}-4^{21}}{25^{21}-4\cdot 25^{20}} \approx 1.1905$$

   (b) $\sum_{k=4}^{12} 2^k$

   We have $a_1 = 16$, $r = 2$ and $n = 9$

   $$S = \frac{a_1(1-r^n)}{1-r} = 16 \cdot \frac{1-2^9}{1-2} = 511 \cdot 16 = 8176$$

   (c) $\sum_{k=0}^{9} \left(-\frac{3}{4}\right)^{k}$

   We have $a_1 = 1$, $r = -\frac{3}{4}$ and $n = 10$

   $$S = \frac{a_1(1-r^n)}{1-r} = 1 \cdot \frac{1-\left(-\frac{3}{4}\right)^{10}}{1+\frac{3}{4}} = \frac{4^{10}-3^{10}}{4^{10}+3\cdot 4^9} = \frac{141361}{262144} \approx 0.5392$$

   (d) $\sum_{k=0}^{20} (-1)^k$ We have $a_1 = 1$, $r = -1$ and $n = 21$

   $$S = \frac{a_1(1-r^n)}{1-r} = 1 \cdot \frac{1-(-1)^{21}}{1+1} = 1$$

2. For the following telescoping series, find a formula for the n^{th} term of the sequence of the partial sums $\{S_n\}$. Then evaluate $\lim_{n\to\infty} S_n$ to obtain the value of the series or state that the series diverges.

   (a) $\sum_{k=1}^{\infty} \left(\frac{1}{k+2} - \frac{1}{k+3}\right)$

   When we write the terms of the sum we get:

   $$\sum_{k=1}^{\infty} \left(\frac{1}{k+2} - \frac{1}{k+3}\right) = \left(\frac{1}{3}-\frac{1}{4}\right) + \left(\frac{1}{4}-\frac{1}{5}\right) + \left(\frac{1}{5}-\frac{1}{6}\right) + \left(\frac{1}{6}-\frac{1}{7}\right) + \ldots$$

It is clear that second term of each summand cancels with the first term of the succeeding summand,so

$$S_n = \frac{1}{3} - \frac{1}{n+3} = \frac{n}{3n+9} \quad \Rightarrow \quad \lim_{n\to\infty} \frac{n}{3n+9} = \frac{1}{3}$$

