Questions related to SECTION 8.1

1. (a) Find the next two terms of the sequence.
(b) Find a recurrence relation that generates the relation.
(c) Find an explicit formula for the general nth term of the sequence.

i. \( \left\{ \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots \right\} \)
   
   (a) \( \frac{1}{32} \) and \( \frac{1}{64} \)
   
   (b) \( a_1 = 1; \quad a_{n+1} = \frac{a_n}{2} \)
   
   (c) \( a_n = \frac{1}{2^{n-1}} \)

ii. \( \{1, -2, 3, -4, 5, \ldots\} \)
   
   (a) \( -6 \) and \( 7 \)
   
   (b) \( a_1 = 1; \quad a_{n+1} = (-1)^n(|a_n| + 1) \)
   
   (c) \( a_n = (-1)^{n+1}n \)

2. Write the terms \( a_1, a_2, a_3 \) and \( a_4 \) of the following sequences. If the sequence appears to converge, make a conjecture about its limit. If the sequence diverges, explain why.

(a) \( a_n = \frac{(-1)^n}{n}; \quad n = 1, 2, 3, \ldots \)

\[
  a_1 = -1, \quad a_2 = \frac{1}{2}, \quad a_3 = -\frac{1}{3} \quad \text{and} \quad a_4 = \frac{1}{4}
\]

This sequence converges to 0 since each term is smaller in absolute value than the preceding term and they get arbitrarily close to 0.

(b) \( a_n = 1 - 10^{-n}; \quad n = 1, 2, 3, \ldots \)

\[
  a_1 = 0.9, \quad a_2 = 0.99, \quad a_3 = 0.999 \quad \text{and} \quad a_4 = 0.9999
\]

This sequence converges to 1.
(c) \( a_{n+1} = \frac{a_n^2}{10} \), \( a_0 = 1 \)

Rewrite the recurrence as \( a_{n+1} = 10^{-1}a_n^2 \). Then we have

\[
\begin{align*}
a_0 &= 1, \\
a_1 &= 10^{-1} = \frac{1}{10}, \\
a_2 &= 10^{-1}(10^{-1})^2 = 10^{-3} = \frac{1}{1000}, \\
a_3 &= 10^{-1}(10^{-3})^2 = 10^{-7} = \frac{1}{10000000}, \\
a_4 &= 10^{-1}(10^{-7})^2 = 10^{-15} = \frac{1}{1000000000000000}
\end{align*}
\]

This sequence converges to 0.

(d) \( a_{n+1} = 0.5a_n(1 - a_n) \); \( a_0 = 0.8 \)

\[
\begin{align*}
a_0 &= 0.8, \\
a_1 &= 0.5 \cdot 0.8(1 - 0.8) = 0.5 \cdot 0.8 \cdot 0.2 = 0.08, \\
a_2 &= 0.5 \cdot 0.08(1 - 0.08) = 0.5 \cdot 0.08 \cdot 0.92 = 0.0368, \\
a_3 &= 0.5 \cdot 0.0368(1 - 0.0368) = 0.01772288, \\
a_4 &= 0.5 \cdot 0.01772288(1 - 0.01772288) \approx 0.0087
\end{align*}
\]

3. Consider the following recurrence relations.

(a) Find the terms \( a_0, a_1, a_2 \) and \( a_3 \) of the sequence.

(b) If possible, find an explicit formula for the \( n^{th} \) term of the sequence.

i. \( a_{n+1} = a_n + 2; \) \( a_0 = 3 \)

ii. \( a_{n+1} = 2a_n + 1; \) \( a_0 = 0 \)

i.

\[
\begin{align*}
a_0 &= 3, \\
a_1 &= 5, \\
a_2 &= 7 \\
a_3 &= 9
\end{align*}
\]

\( a_n = 2n + 3 \)

ii.

\[
\begin{align*}
a_0 &= 0, \\
a_1 &= 1, \\
a_2 &= 3, \\
a_3 &= 7 \\
a_4 &= 15
\end{align*}
\]

\( a_n = 2^n - 1 \)
Questions related to SECTION 8.2

1. Find the limit of the following sequences or determine that the limit does not exist.

   (a) \( \left\{ \frac{3n^3 - 1}{2n^3 + 1} \right\} \)

   Dividing the numerator and the denominator by \( n^3 \) we get:

   \[
   \lim_{n \to \infty} \frac{3 - n^{-3}}{2 + n^{-3}} = \frac{3}{2}
   \]

   (b) \( \left\{ \left( 1 + \frac{2}{n} \right)^n \right\} \)

   Find the limit of the logarithm of the expression, which is \( n \ln \left( 1 + \frac{2}{n} \right) \), using L'Hopital's rule.

   \[
   \lim_{n \to \infty} n \ln \left( 1 + \frac{2}{n} \right) = \lim_{n \to \infty} \frac{\ln \left( 1 + \frac{2}{n} \right)}{\frac{1}{n}} = \lim_{n \to \infty} \frac{\frac{1}{1 + \frac{2}{n}} \left( -\frac{2}{n^2} \right)}{-\frac{1}{n^2}} = \lim_{n \to \infty} \frac{2}{1 - \frac{2}{n}} = 2
   \]

   Thus the limit of the original expression is \( e^2 \).

   (c) \( \left\{ \sqrt{\left( 1 + \frac{1}{2n} \right)^n} \right\} \)

   Take the logarithm of the expression and use L'Hopital's rule.

   \[
   \lim_{n \to \infty} \frac{n}{2} \ln \left( 1 + \frac{1}{2n} \right) = \lim_{n \to \infty} \frac{\ln \left( 1 + \frac{1}{2n} \right)}{\frac{2}{n}} = \lim_{n \to \infty} \frac{\frac{1}{1 + \frac{1}{2n}} \left( -\frac{1}{2n^2} \right)}{-\frac{2}{n^2}} = \frac{1}{4}
   \]

   Thus the limit of the original expression is \( e^{\frac{1}{4}} \).

   (d) \( \left\{ \frac{\ln \left( \frac{1}{n} \right)}{n} \right\} \)

   Since \( \ln \left( \frac{1}{n} \right) = -\ln n \), we get

   \[
   \lim_{n \to \infty} \frac{\ln \left( \frac{1}{n} \right)}{n} = \lim_{n \to \infty} \frac{-\ln n}{n}
   \]

   Then by applying L'Hopital's rule we get:

   \[
   \lim_{n \to \infty} \frac{\ln \left( \frac{1}{n} \right)}{n} = \lim_{n \to \infty} \frac{-\ln n}{n} = \lim_{n \to \infty} \frac{-\frac{1}{n}}{1} = -\lim_{n \to \infty} \frac{1}{n} = 0
   \]
(e) \{ \left( \frac{1}{n} \right)^{1/n} \}

Find the limit of the logarithm of the expression, which is \( \frac{1}{n} \ln \left( \frac{1}{n} \right) \), using L’Hopital’s rule.

\[
\lim_{n \to \infty} \frac{1}{n} \ln \left( \frac{1}{n} \right) = \lim_{n \to \infty} \frac{-\ln n}{n} = \lim_{n \to \infty} \frac{-1}{n} = 0
\]

Thus the limit of the original expression is \( e^0 \).

(f) \{ \left( 1 - \frac{4}{n} \right)^n \}

Find the limit of the logarithm of the expression, which is \( n \ln \left( 1 - \frac{4}{n} \right) \), using L’Hopital’s rule.

\[
\lim_{n \to \infty} n \ln \left( 1 - \frac{4}{n} \right) = \lim_{n \to \infty} \frac{\ln \left( 1 - \frac{4}{n} \right)}{1/n} = \lim_{n \to \infty} \frac{1}{1-\frac{4}{n^n}} \left( \frac{4}{n^2} \right) = \lim_{n \to \infty} \frac{-4}{1 - \frac{4}{n^n}} = -4
\]

Thus the limit of the original expression is \( e^{-4} \).

(g) \( a_n = e^{-n} \cos n \)

The sequence is

\[
a_n = e^{-n} \cos n = \frac{\cos n}{e^n}
\]

The numerator of the sequence is bounded by 1 and the denominator increases without any bound, so:

\[
\lim_{n \to \infty} e^{-n} \cos n = \lim_{n \to \infty} \frac{\cos n}{e^n} = 0
\]

(h) \( a_n = \frac{\ln n}{n^{.1}} \)

Using L’Hopital’s rule, we have

\[
\lim_{n \to \infty} \frac{\ln n}{n^{.1}} = \lim_{n \to \infty} \frac{\frac{1}{n}}{(1.1)n^{0.1}} = \lim_{n \to \infty} \frac{1}{(1.1)n^{1.1}} = 0
\]
Questions related to SECTION 8.3

1. Evaluate the following geometric sums.

(a) $\sum_{k=0}^{20} \left( \frac{2}{5} \right)^{2k}$

We have $a_1 = 1$, $r = \frac{4}{25}$ and $n = 21$

$$S = \frac{a_1 (1 - r^n)}{1 - r} = \frac{1 - \left( \frac{4}{25} \right)^{21}}{1 - \frac{4}{25}} = \frac{25^{21} - 4^{21}}{25^{21} - 4^{21}} \approx 1.1905$$

(b) $\sum_{k=4}^{12} 2^k$

We have $a_1 = 16$, $r = 2$ and $n = 9$

$$S = \frac{a_1 (1 - r^n)}{1 - r} = \frac{16 - 2^9}{1 - 2} = 511 \cdot 16 = 8176$$

(c) $\sum_{k=0}^{9} \left( -\frac{3}{4} \right)^k$

We have $a_1 = 1$, $r = -\frac{3}{4}$ and $n = 10$

$$S = \frac{a_1 (1 - r^n)}{1 - r} = \frac{1 - \left( -\frac{3}{4} \right)^{10}}{1 + \frac{3}{4}} = \frac{4^{10} - 3^{10}}{4^{10} + 3^4} = \frac{141361}{262144} \approx 0.5392$$

(d) $\sum_{k=0}^{20} (-1)^k$ We have $a_1 = 1$, $r = -1$ and $n = 21$

$$S = \frac{a_1 (1 - r^n)}{1 - r} = \frac{1 - (-1)^{21}}{1 + 1} = 1$$

2. For the following telescoping series, find a formula for the $n^{th}$ term of the sequence of the partial sums $\{S_n\}$. Then evaluate $\lim_{n \to \infty} S_n$ to obtain the value of the series or state that the series diverges.

(a) $\sum_{k=1}^{\infty} \left( \frac{1}{k+2} - \frac{1}{k+3} \right)$

When we write the terms of the sum we get:

$$\sum_{k=1}^{\infty} \left( \frac{1}{k+2} - \frac{1}{k+3} \right) = \left( \frac{1}{3} - \frac{1}{4} \right) + \left( \frac{1}{4} - \frac{1}{5} \right) + \left( \frac{1}{5} - \frac{1}{6} \right) + \left( \frac{1}{6} - \frac{1}{7} \right) + \ldots$$
It is clear that second term of each summand cancels with the first term of the succeeding summand, so

$$S_n = \frac{1}{3} - \frac{1}{n+3} = \frac{n}{3n + 9} \Rightarrow \lim_{n \to \infty} \frac{n}{3n + 9} = \frac{1}{3}$$