

Questions related to SECTION 4.1

For the function

$$f(x) = \frac{x}{(x^2 + 1)^2} \quad \text{on } [-2, 2]$$

- Find the critical points of f on the given interval.
- Determine the absolute extreme values of f on the given interval.
- Use a graphing utility to confirm your conclusions.

$$f'(x) = \frac{1 \cdot (x^2 + 1)^2 - x \cdot 2(x^2 + 1)2x}{(x^2 + 1)^4} = \frac{1 - 3x^2}{(x^2 + 1)^3}$$

at critical points $f'(x) = 0$ i.e.

$$\frac{1 - 3x^2}{(x^2 + 1)^3} = 0 \quad \Rightarrow \quad 1 - 3x^2 = 0$$

Thus we get $x = \pm\sqrt{\frac{1}{3}}$ as the critical points.

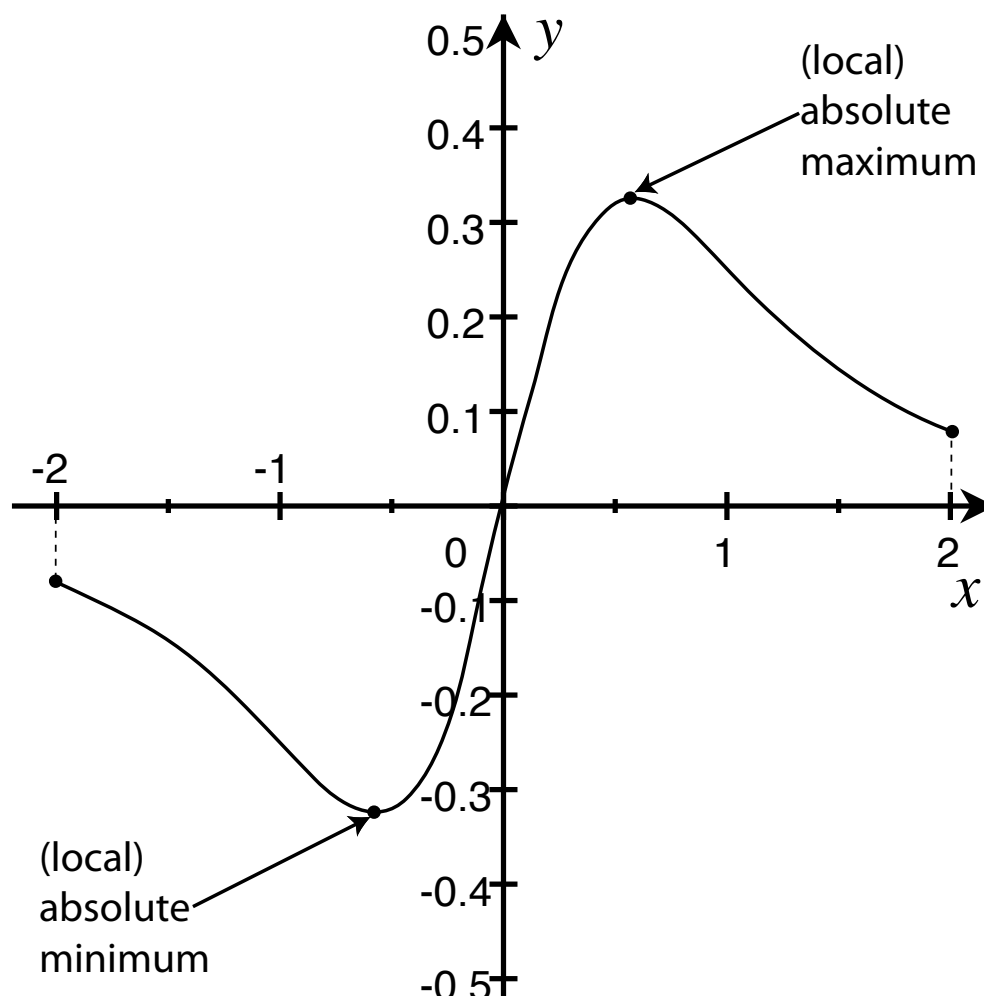
$$\begin{aligned} f(-2) &= -0.08 & f(2) &= 0.08 \\ f\left(\sqrt{\frac{1}{3}}\right) &= \frac{3\sqrt{3}}{16} \approx 0.325 & f\left(-\sqrt{\frac{1}{3}}\right) &= \frac{-3\sqrt{3}}{16} \approx -0.325 \end{aligned}$$

Thus the absolute maximum value of f on the given interval is $\frac{3\sqrt{3}}{16}$ and the absolute minimum value of f on the given interval is $\frac{-3\sqrt{3}}{16}$

In order to draw the graph of f we can use the first derivative Test.

x	$x < -\sqrt{\frac{1}{3}}$	$-\sqrt{\frac{1}{3}}$	$-\sqrt{\frac{1}{3}} < x < \sqrt{\frac{1}{3}}$	$\sqrt{\frac{1}{3}}$	$x > \sqrt{\frac{1}{3}}$
$f'(x)$	< 0	0	> 0	0	< 0
$f(x)$	\searrow	-0.325	\nearrow	0.325	\searrow

Then by using those values we can draw the graph of f as follows:



Questions related to SECTION 4.2

1. Find the intervals on which f is increasing and decreasing. Superimpose the graphs of f and f' to verify your work.

$$f(x) = x^4 - 4x^3 + 4x^2$$

$$f'(x) = 4x^3 - 12x^2 + 8x = 4x(x^2 - 3x + 2) = 4x(x - 2)(x - 1)$$

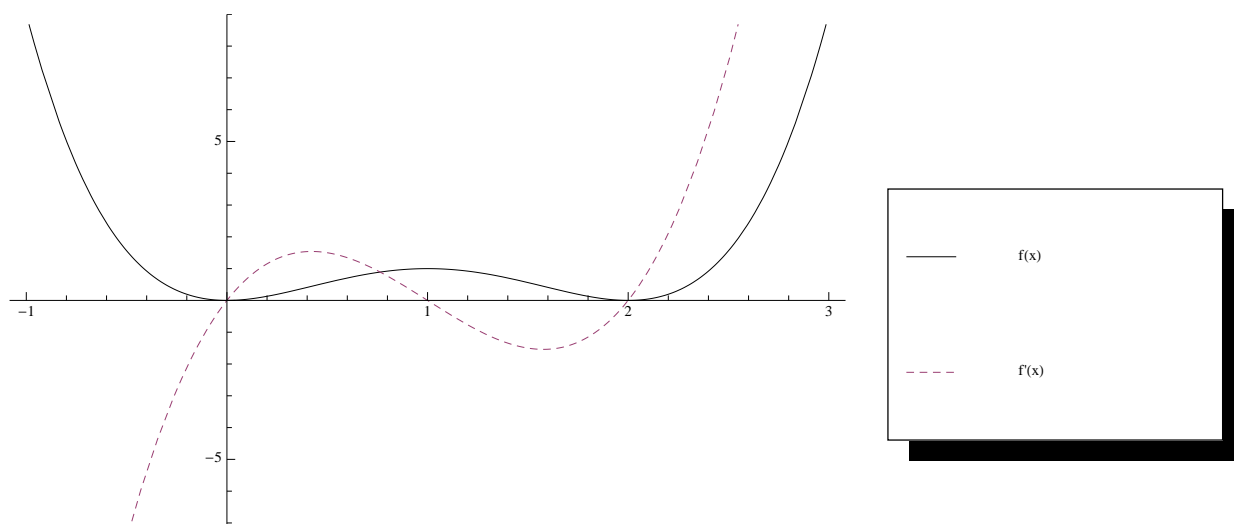
Thus $f'(x)$ is zero at $x = 0, 1, 2$.

On $(-\infty, 0)$ $f' < 0$ so f is decreasing on $(-\infty, 0)$

On $(0, 1)$ $f' > 0$ so f is increasing on $(0, 1)$

On $(1, 2)$ $f' < 0$ so f is decreasing on $(1, 2)$

On $(2, \infty)$ $f' > 0$ so f is increasing on $(2, \infty)$



2. (a) Locate the critical points of the given function.
- (b) Use the first derivative test to locate the local maximum and minimum values.
- (c) Identify the absolute minimum and maximum values of the function on the given interval.

$$f(x) = x^{\frac{2}{3}}(x - 4) \quad x \in [-5, 5]$$

$$f'(x) = \frac{2}{3}x^{-\frac{1}{3}}(x - 4) + x^{\frac{2}{3}} = \frac{2(x - 4)}{3x^{\frac{1}{3}}} + x^{\frac{2}{3}} = \frac{2x - 8 + 3x^{\frac{2}{3}}x^{\frac{1}{3}}}{3x^{\frac{1}{3}}} = \frac{5x - 8}{3x^{\frac{1}{3}}}$$

$f'(x)$ is not defined at $x = 0$ and $f'(x) = 0$ at $x = \frac{8}{5}$. Because $x = 0$ and $x = \frac{8}{5}$ are in the domain of the function, these points are the critical points of f .

x	$x < 0$	0	$0 < x < \frac{8}{5}$	$\frac{8}{5}$	$x > \frac{8}{5}$
$f'(x)$	> 0	N.D	< 0	0	> 0
$f(x)$	\nearrow	0	\searrow	-3.28	\nearrow

Local maximum at $x = 0$ and $f(0) = 0$

Local minimum at $x = \frac{8}{5}$ and $f\left(\frac{8}{5}\right) \approx -3.28$

$$f(-5) \approx -26.32 \quad f(0) = 0 \quad f\left(\frac{8}{5}\right) \approx -3.28 \quad f(5) \approx 2.92$$

Thus we have absolute maximum value of f on $[-5, 5]$ as 2.92 and absolute minimum value of f as -26.32

3. Locate the critical points of the following functions. Then use the second derivative test to determine whether they correspond to local minima or local maxima or whether the test is inconclusive.

$$p(x) = \frac{x - 4}{x^2 + 20}$$

$$p'(x) = \frac{x^2 + 20 - 2x(x - 4)}{(x^2 + 20)^2} = \frac{-x^2 + 8x + 20}{(x^2 + 20)^2} = -\frac{(x + 2)(x - 10)}{(x^2 + 20)^2}$$

Critical Points:

$$-p'(x) = 0$$

$$-\frac{(x + 2)(x - 10)}{(x^2 + 20)^2} = 0 \quad \Rightarrow \quad x = -2 \quad \text{and} \quad x = 10$$

- $p'(x)$ not defined : $p'(x)$ is defined for all real x .

Thus the critical points are

$$\left(-2, -\frac{1}{4}\right) \quad \text{and} \quad \left(10, \frac{1}{20}\right)$$

Then in order to use the second derivative test let's find $p''(x)$

$$\begin{aligned} p''(x) &= \frac{(-2x + 8)(x^2 + 20)^2 - 2(x^2 + 20)2x(-x^2 + 8x + 20)}{(x^2 + 20)^4} = \\ &= \frac{(-2x + 8)(x^2 + 20) - 4x(-x^2 + 8x + 20)}{(x^2 + 20)^3} = \frac{2(x^3 - 12x^2 - 60x + 80)}{(x^2 + 20)^3} \end{aligned}$$

Second Derivative Test:

Note that if $p''(x) > 0$ then x is a local minimum and if $p''(x) < 0$ then x is a local maximum.

$$\begin{aligned} p''(-2) > 0 &\quad \Rightarrow \quad x = -2 \quad \text{is a local minimum} \\ p''(10) < 0 &\quad \Rightarrow \quad x = 10 \quad \text{is a local maximum} \end{aligned}$$

Questions related to SECTION 4.3

1. Sketch the graph of $f(x) = x^4 - 6x^2$

(a) Domain: $(-\infty, \infty)$

(b) x - and y - intercepts:

$$x\text{-intercept: } y = 0 \Leftrightarrow x^4 - 6x^2 = 0 \Leftrightarrow x = 0, \quad x = -\sqrt{6}, \quad x = \sqrt{6}$$

So the x -intercepts are $(0, 0)$, $(-\sqrt{6}, 0)$, $(\sqrt{6}, 0)$

$$y\text{-intercept: } x = 0 \Leftrightarrow y = 0$$

So the y -intercept is $(0, 0)$.

(c) Critical Points and Local Extreme Values:

i. Find $f'(x)$

$$f'(x) = 4x^3 - 12x = 4x(x^2 - 3)$$

ii. Find all critical points:

$$f'(x) = 0 : f'(x) = 4x(x^2 - 3) = 0 \Leftrightarrow x = 0, \quad x = -\sqrt{3}, \quad x = \sqrt{3}$$

$f'(x)$ is not defined: $f'(x)$ is defined for all real x .

iii. Increasing and Decreasing Intervals:

x	$x <$	$-\sqrt{3}$	$< x <$	0	$< x <$	$\sqrt{3}$	$< x$
$f'(x)$	$-$	0	$+$	0	$-$	0	$+$
$f(x)$	\searrow	loc. min	\nearrow	loc. max	\searrow	loc. min	\nearrow

iv. Evaluate $f(x)$ at each critical point:

local minimums at $(-\sqrt{3}, -9)$, $(\sqrt{3}, -9)$ and local maximum at $(0, 0)$

(d) Determine Concavity and Points of Inflection:

i. Find $f''(x)$

$$f''(x) = 12x^2 - 12 = 12(x^2 - 1) = 12(x - 1)(x + 1)$$

ii. Find all potential points of inflection:

$$f''(x) = 0 : f''(x) = 12(x - 1)(x + 1) = 0 \Leftrightarrow x = 1, \quad x = -1$$

$f''(x)$ is not defined: $f''(x)$ is defined for all real x .

iii. Intervals of Concavity:

x	$x <$	-1	$< x <$	1	$< x$
$f''(x)$	$+$	0	$-$	0	$-$
$f(x)$	\cup	infl. point	\cap	infl. point	\cup

iv. The points of inflection are : $(-1, -5)$ and $(1, -5)$

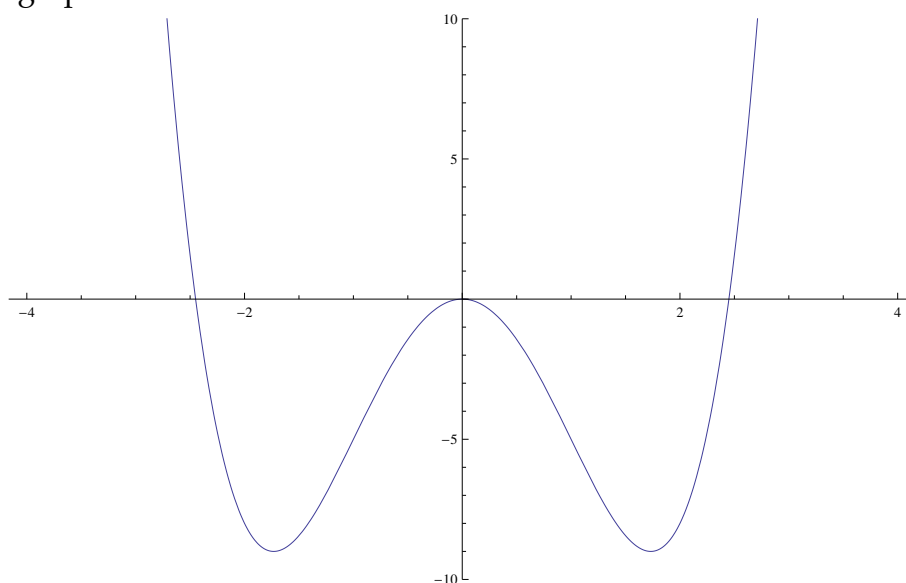
(e) Asymptotes:

Horizontal Asymptote: None

Vertical Asymptote: None

Oblique Asymptote: None

(f) Sketch the graph:



(g) The range of $f(x)$ is $[-9, \infty)$.

2. Sketch the graph of $y = \frac{(x+1)^2}{1+x^2}$

(a) Domain: $x \in \mathbb{R}$

(b) x - and y - intercepts:

$$x\text{-intercept: } y = 0 \Leftrightarrow \frac{(x+1)^2}{1+x^2} = 0 \Leftrightarrow (x+1)^2 = 0 \Leftrightarrow x = -1$$

So the x -intercept is $(-1, 0)$.

$$y\text{-intercept: } x = 0 \Leftrightarrow y = \frac{(0+1)^2}{1+0^2} \Leftrightarrow y = 1$$

So the y -intercept is $(0, 1)$.

(c) Critical Points and Local Extreme Values:

i. Find $f'(x)$

$$f'(x) = \frac{(1+x^2) \cdot 2(x+1) - (x+1)^2 \cdot 2x}{(1+x^2)^2} = \frac{2(1-x^2)}{(1+x^2)^2}$$

ii. Find all critical points:

$$f'(x) = 0 : f'(x) = \frac{2(1-x^2)}{(1+x^2)^2} = 0 \Leftrightarrow x = -1 \text{ and } x = 1$$

$f'(x)$ is not defined: $f'(x)$ is defined for all real x .

iii. Increasing and Decreasing Intervals:

x	$x < -1$	-1	$-1 < x < 1$	1	$x > 1$
$f'(x)$	$-$	0	$+$	0	$-$
$f(x)$	\searrow	loc. min.	\nearrow	loc. max.	\searrow

iv. Evaluate $f(x)$ at each critical point:

local minimum at $(-1, 0)$ and local maximum at $(1, 2)$

(d) Determine Concavity and Points of Inflection:

i. Find $f''(x)$

$$f''(x) = \frac{(1+x^2)^2 \cdot 2(-2x) - 2(1-x^2)[2(1+x^2) \cdot 2x]}{(1+x^2)^4} = \frac{4x(x^2-3)}{(1+x^2)^3}$$

ii. Find all potential points of inflection:

$$f''(x) = 0 : f''(x) = \frac{4x(x^2 - 3)}{(1 + x^2)^3} = 0 \Leftrightarrow x = 0, x = \pm\sqrt{3}$$

$f''(x)$ is not defined: $f''(x)$ is defined for all real x .

iii. Intervals of Concavity:

x	$x <$	$-\sqrt{3}$	$< x <$	0	$< x <$	$\sqrt{3}$	$< x$
$f''(x)$	$-$	0	$+$	0	$-$	0	$+$
$f(x)$	\cap	infl. point	\cup	infl. point	\cap	infl. point	\cup

iv. The points of inflection are : $(-\sqrt{3}, 0.1340)$, $(0, 1)$ and $(\sqrt{3}, 1.8660)$

(e) Asymptotes:

Horizontal Asymptote:

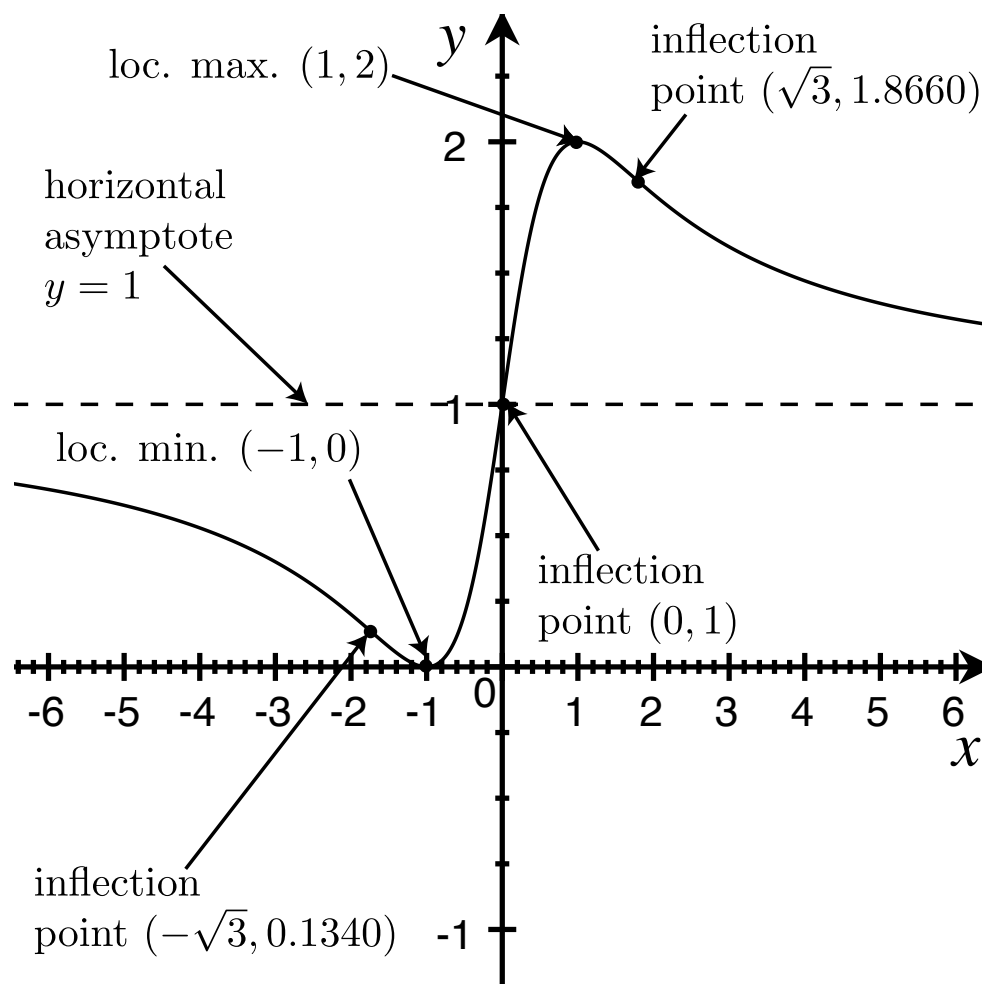
$$\lim_{x \rightarrow \pm\infty} \frac{(x+1)^2}{1+x^2} = \lim_{x \rightarrow \pm\infty} \frac{x^2 + 2x + 1}{1+x^2} = \lim_{x \rightarrow \pm\infty} \frac{x^2 \left(1 + \frac{2}{x} + \frac{1}{x^2}\right)}{x^2 \left(1 + \frac{1}{x^2}\right)} = 1$$

So we have a horizontal asymptote at $y = 1$.

Vertical Asymptote: None

Oblique Asymptote: None

(f) Sketch the graph:



(g) The range of $f(x)$ is $[0, 2]$.