CHAPTER 3

EXACT ONE-DIMENSIONAL SOLUTIONS

3.1 Introduction

- Exact solutions for simple cases are presented.
- Objective is to:
  - Understand the physical significance of each term in the equations of continuity, Navier-Stokes, and energy.
  - Identify the conditions under which certain terms can be neglected.
  - General procedure: first determine the flow field and then the temperature field.

3.2 Simplification of the Governing Equations

- Simplifying Assumptions
  (1) Laminar flow
  (2) Constant properties
  (3) Parallel streamlines (fully developed flow):
    \[ v = 0 \] (3.1)

Continuity (two-dimensional, constant density):

\[ \frac{\partial u}{\partial x} = 0 \] (3.2)

It follows that

\[ \frac{\partial^2 u}{\partial x^2} = 0 \] (3.3)

- (3.1)-(3.3) result in significant simplifications.

(4) Negligible axial variation of temperature. For axial flow, this condition leads to

\[ \frac{\partial T}{\partial x} = 0 \] (3.4)

- (3.4) is exact for certain channel flows and a reasonable approximation for others. The following are conditions that may lead to the validity of (3.4):
  (i) Parallel streamlines.
  (ii) Far away from the entrance region of a channel (infinitely long channels).
(iii) Uniform surface conditions.

- If (3.4) is valid everywhere:
  \[
  \frac{\partial^2 T}{\partial x^2} = 0 \tag{3.5}
  \]

- Rotating flows, Fig. 3.2: The streamlines are concentric circles
  \[
  \frac{\partial T}{\partial \theta} = 0 \tag{3.6}
  \]

and
  \[
  \frac{\partial^2 T}{\partial \theta^2} = 0 \tag{3.7}
  \]

3.3 Exact Solutions

- Applications of simplifications of Section 3.2.

3.3.1 Couette Flow

- This is shear driven flow
- Fluid is set in motion by moving channel surface
- Streamlines are parallel
- No axial variation
- Many terms in the Navier-Stokes equations and energy equation drop out

Example 3.1: Couette Flow with Dissipation

- Upper plate moves with velocity \( U_o \)
- Moving plate at temperature \( T_o \)
- Taking into consideration dissipation
- Determine temperature distribution and the rate of heat transfer at the moving plate
- Assume laminar flow

Solution

- Review all assumptions
- Analysis
  - Starting with the energy equation
\[
\rho c_p \left( \frac{\partial T}{\partial t} + u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} + w \frac{\partial T}{\partial z} \right) = k \left( \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} \right) + \mu \Phi 
\]  
(2.19b)

**Dissipation:**

\[
\Phi = 2 \left[ \left( \frac{\partial u}{\partial x} \right)^2 + \left( \frac{\partial v}{\partial y} \right)^2 + \left( \frac{\partial w}{\partial z} \right)^2 \right] + \left[ \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right)^2 + \left( \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \right)^2 + \left( \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} \right)^2 \right] - 2 \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right)^2 
\]  
(2.17)

- **Simplification of the above equations:**

1. **Incompressible fluid:**
   \[
   \frac{\partial \rho}{\partial t} = \frac{\partial \rho}{\partial x} = \frac{\partial \rho}{\partial y} = \frac{\partial \rho}{\partial z} = 0 
   \]  
   (a)

2. **Infinite plates:**
   \[
   \frac{\partial \rho}{\partial x} = \frac{\partial \rho}{\partial z} = 0 
   \]  
   (b)

3. **Continuity and no-slip condition:**
   \[
   v = 0 
   \]  
   (f)

- **Navier-Stokes equation (2.10x):**
   \[
   \rho \left( \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} \right) = \rho g_x - \frac{\partial p}{\partial x} + \mu \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right) 
   \]  
   (2.10x)

Introduce the above simplifications into (2.10x):

\[
\frac{d^2 u}{dy^2} = 0 
\]  
(j)

Solution to (j) is

\[
u = C_1 y + C_2 
\]  
(k)

Use boundary conditions on \( u \) to determine the two constants. Solution becomes

\[
\frac{u}{U_o} = \frac{y}{H} 
\]  
(3.8)

**Dissipation function simplifies to:**

\[
\Phi = \left( \frac{\partial u}{\partial y} \right)^2 
\]  
(n)
Energy equation (2.19b) simplifies to

\[ k \frac{d^2T}{dy^2} + \mu \frac{U_o^2}{H^2} = 0 \]  

(p)

Solution is

\[ \frac{T - T_o}{\frac{\mu U_o^2}{k}} = \frac{1}{2} \left( 1 - \frac{y^2}{H^2} \right) \]  

(3.9)

Heat flux at the moving plate:

\[ q''(H) = -k \frac{dT(H)}{dy} \]

Use (3.9)

\[ q''(H) = \frac{\mu U_o^2}{H} \]  

(3.10)

Example 3.2: Flow in a Tube at Uniform Surface Temperature

- Study this example with attention to physical conditions and how they lead to simplifications of the governing equations.
- Follow the procedure of Example 3.1 above

3.3.3 Rotating Flow

- Note that all angular variation of velocity, pressure and temperature vanish in Concentric rotating flows.

Example 3.3: Lubrication Oil Temperature in Rotating Shaft

- Study this example with attention to physical conditions and how they lead to simplifications of the governing equations.
- Follow the procedure of Example 3.1