CHAPTER 6

HEAT TRANSFER IN CHANNEL FLOW

6.1 Introduction

(1) Laminar vs. turbulent flow

- Flow through tubes, transition Reynolds number Re_{D_t} is

$$Re_{D_t} = \frac{\bar{u}D}{\nu} \approx 2300 \tag{6.1}$$

(2) Entrance vs. fully developed region

- Classification based on velocity and temperature profiles:

(i) Entrance region
(ii) Fully developed region

(3) Surface boundary conditions

- Two common boundary conditions::

(i) Uniform surface temperature
(ii) Uniform surface heat flux

(4) Objective.

- Objective depends on surface thermal boundary condition:

(i) Uniform surface temperature. Determine axial variation of:

(1) Mean fluid temperature
(2) Heat transfer coefficient
(3) Surface heat flux

(ii) Uniform surface flux. Determine axial variation of:

(1) Mean fluid temperature
(2) Heat transfer coefficient
(3) Surface temperature

6.2 Hydrodynamic and Thermal Regions: General Features

- Fluid enters with uniform velocity and temperature.

(1) Entrance region. Extends from the inlet to the section where the boundary layer thickness reaches the center of channel.

(2) Fully developed region. This zone follows the entrance region.

6.2.1 Flow Field

(1) Entrance Region (Developing Flow, $0 \le x \le L_h$).

- Name: *hydrodynamic entrance region.*
- Length: L_h (*hydrodynamic entrance length*).
- Streamlines are not parallel.
- Core velocity u_c increases with distance
- Pressure decreases with distance ($dp/dx < 0$).
- $\delta < D/2$.

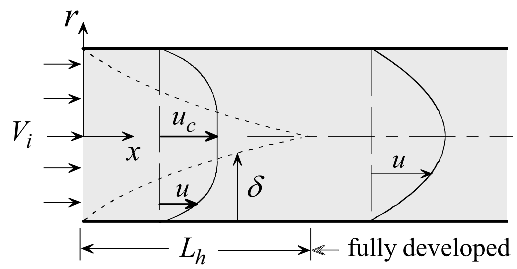


Fig. 6.1

(2) Fully Developed Flow Region. $x \ge L_h$

- Streamlines are parallel ($v_r = 0$).
- $\partial u / \partial x = 0$ for two-dimensional incompressible fluid.

6.2.2 Temperature Field

(1) Entrance Region (Developing Temperature, $0 \le x \le L_t$)

- Name: *Thermal entrance region*.
- Length: L_t (*thermal entrance length*).
- Core temperature T_c is uniform, $T_c = T_i$.
- $\delta_t < D/2$

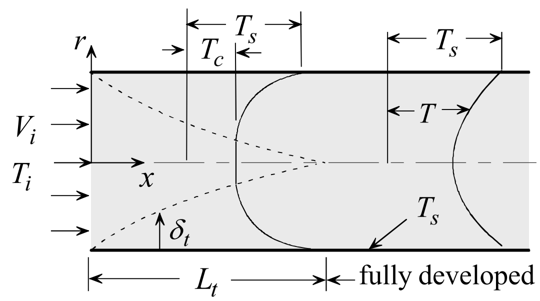


Fig. 6.2

(2) Fully Developed Temperature Region. $x \ge L_t$

- Temperature varies radially and axially, $\partial T / \partial x \ne 0$.

6.3 Hydrodynamic and Thermal Entrance Lengths

6.3.1 Scale Analysis

(1) Hydrodynamic Entrance Length L_h.

- Starting with external flow result (4.16)

$$\frac{\delta}{x} \sim \frac{1}{\sqrt{Re_x}} \tag{4.16}$$

- Applying (4.16) to a tube at $x = L_h$:

$$\delta \sim D \text{ and } Re_{L_h} = Re_D \frac{L_h}{D} \tag{b}$$

- Substituting (b) into (4.16) and rearranging

$$\left(\frac{L_h / D}{Re_D}\right)^{1/2} \sim 1 \qquad (6.2)$$

(2) Thermal Entrance Length L_t.

- Starting with external flow result (4.24)

$$\delta_t \sim L Re_L^{-1/2} Pr^{-1/2} \qquad (4.24)$$

- Applying (4.24) at $x = L_t$:

$$\delta_t \sim D \text{ and } Re_{L_t} = Re_D \frac{L_t}{D} \qquad (b)$$

Substituting (b) into (4.24) and rearranging

$$\left(\frac{L_t / D}{Re_D Pr}\right)^{1/2} \sim 1 \qquad (6.3)$$

- (6.2) and (6.3) give

$$\frac{L_t}{L_h} \sim Pr \qquad (6.4)$$

6.3.2 Analytic and Numerical Solutions: Laminar Flow

(1) Hydrodynamic Entrance Length L_h.

- Results for L_h:

$$\frac{L_h}{D_e} = C_h Re_{D_e} \qquad (6.5)$$

- Table 6.1 gives C_h
- Compare with scaling:

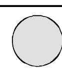
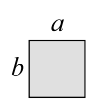
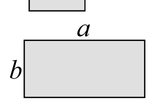
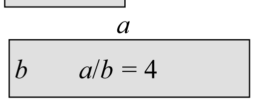
$$\left(\frac{L_h / D}{Re_D}\right)^{1/2} \sim 1 \qquad (6.2)$$

Rewrite (6.5)

$$\left(\frac{L_h / D_e}{Re_{D_e}}\right)^{1/2} = (C_h)^{1/2} \qquad (a)$$

Example: Rectangular channel, aspect ratio 2, Table 6.1 gives $C_h = 0.085$. Substituting this value into (a), gives

Table 6.1
Entrance length coefficients C_h and C_t [1]

geometry	C_h	C_t	
		uniform surface flux	uniform surface temperature
	0.056	0.043	0.033
 $a/b = 1$	0.09	0.066	0.041
 $a/b = 2$	0.085	0.057	0.049
 $a/b = 4$	0.075	0.042	0.054
	0.011	0.012	0.008

$$\left(\frac{L_h / D_e}{Re_{D_e}}\right)^{1/2} = (0.085)^{1/2} = 0.29 \qquad \text{(b)}$$

Scaling replaces 0.29 by unity.

(2) Thermal Entrance Length L_t .

- L_t depends on surface boundary conditions: Two cases: (i) Uniform surface temperature. (ii) Uniform surface flux.
- Solution

$$\frac{L_t}{D_e} = C_t Pr Re_D \qquad (6.6)$$

- Table 6.1 gives C_t for both cases.
- Compare with scaling. Rewrite (6.6)

$$\left(\frac{L_t / D_e}{Pr Re_D}\right)^{1/2} = (C_t)^{1/2} \qquad \text{(c)}$$

Scaling gives

$$\left(\frac{L_t / D}{Re_D Pr}\right)^{1/2} \sim 1 \qquad (6.3)$$

Example: Rectangular channel, aspect ratio 2, Table 6.1 gives $C_t = 0.049$. Substituting this value into (c), gives

$$\left(\frac{L_t / D_e}{Pr Re_{D_e}}\right)^{1/2} = (0.049)^{1/2} = 0.22 \qquad \text{(d)}$$

Scaling replaces 0.22 be unity.

- Turbulent flow: $L = L_h = L_t$

$$\frac{L}{D} \approx 10 \qquad (6.7)$$

6.4 Channels with Uniform Surface Heat Flux q''_s

- Inlet mean temperature: $T_{mi} = T_m(0)$.
- Determine:
  (1) Total heat transfer rate q_s.
  (2) Mean temperature variation $T_m(x)$.
  (3) Surface temperature variation $T_s(x)$.
- Total heat transfer rate

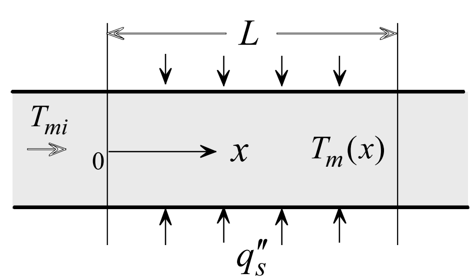


Fig. 6.3

$$q_s = q''_s A_s = q''_s P x \tag{6.8}$$

A_s = surface area

P = perimeter

- Mean temperature $T_m(x)$. Conservation of energy between inlet and section x:

$$q_s = q''_s P x = m c_p [T_m(x) - T_{mi}]$$

or

$$T_m(x) = T_{mi} + \frac{q''_s P}{m c_p} x \tag{6.9}$$

- Surface temperature $T_s(x)$. Newton's law of cooling gives

$$q''_s = h(x)[T_s(x) - T_m(x)]$$

or

$$T_s(x) = T_m(x) + \frac{q''_s}{h(x)}$$

Using (6.9)

$$T_s(x) = T_{mi} + q''_s \left[\frac{Px}{m c_p} + \frac{1}{h(x)} \right] \tag{6.10}$$

NOTE:

- Determining $T_s(x)$ requires knowing $h(x)$.
- To determine $h(x)$: Must know if:
  - Flow is Laminar or turbulent.
  - Entrance or fully developed region

6.5 Channels with Uniform Surface Temperature

- Inlet mean temperature: $T_{mi} = T_m(0)$.
- Determine:

(1) Mean temperature variation $T_m(x)$.

(2) Total heat transfer rate q_s between $x = 0$ and location x.

(3) Surface heat flux variation $q''_s(x)$.

- Mean temperature variation $T_m(x)$.

Conservation of energy to element

$$dq_s = m\, c_p\, dT_m \tag{a}$$

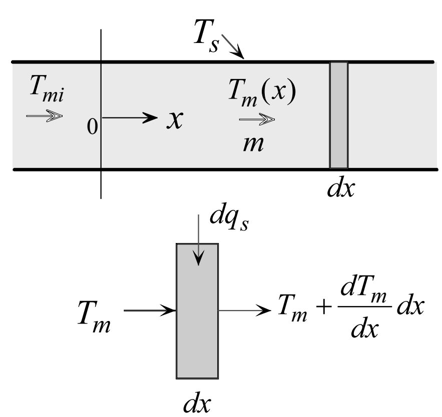


Fig. 6.4

Newton's law:

$$dq_s = h(x)[T_s - T_m(x)]Pdx \tag{b}$$

Combine (a) and (b)

$$\frac{dT_m}{T_s - T_m(x)} = \frac{P}{m\,c_p}h(x)dx \tag{c}$$

Integrating (c)

$$\ln\left[\frac{T_m(x) - T_s}{T_{mi} - T_s}\right] = -\frac{P}{m\ c_p}\int_0^x h(x)dx \tag{6.11}$$

Definite $\bar{h}$

$$\bar{h} = \frac{1}{x}\int_0^x h(x)dx \tag{6.12}$$

(6.12) into (6.11), solve $T_m(x)$

$$T_m(x) = T_s + (T_{mi} - T_s)\exp[-\frac{P\bar{h}}{m\,c_p}x] \tag{6.13}$$

NOTE:

- Determining $T_m(x)$ requires knowing $h(x)$.
- To determine $h(x)$: Must know if:
  - Flow is laminar or turbulent.
  - Entrance or fully developed region
- Heat transfer rate. Conservation of energy:

$$q_s = m\,c_p[T_m(x) - T_{mi}] \tag{6.14}$$

- Surface heat flux.: Newton's law:

$$q_s''(x) = h(x)[T_s - T_m(x)] \tag{6.15}$$

6.6 Determination of Heat Transfer Coefficient $h(x)$ and Nusselt Number Nu_D

6.6.1 Scale Analysis

- Estimate $h(x)$ and Nu_D.
- Tube: radius r_o, surface temperature T_s, mean temperature T_m.
- Fourier's law and Newton's law:

$$h = \frac{-k\dfrac{\partial T(r_o, x)}{\partial r}}{T_m - T_s} \tag{6.16}$$

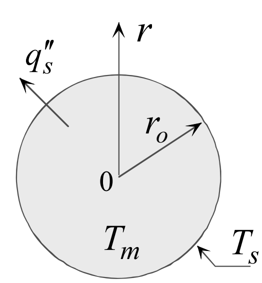


Fig. 6.5

Scaling (6.16)

$$h \sim \frac{k\dfrac{(T_m - T_s)}{\delta_t}}{T_m - T_s}$$

or

$$h \sim \frac{k}{\delta_t} \tag{6.17}$$

The Nusselt number

$$Nu_D = \frac{hD}{k}$$

Use (6.17)

$$Nu_D \sim \frac{D}{\delta_t} \tag{6.18}$$

- Fully developed region: $\delta_t(x) \sim D$, equation (6.18) gives

$$Nu_D \sim 1 \quad \text{(fully developed)} \tag{6.19}$$

- Entrance region: Need to scale δ_t ($\delta_t(x) < D$).
  - For external flow

$$\delta_t \sim x\,Pr^{-1/2} Re_x^{-1/2} \tag{4.24}$$

  - (4.24) into (6.18)

$$Nu_D \sim \frac{D}{x} Pr^{1/2} Re_x^{1/2} \tag{c}$$

  - Expressing Re_x in terms of Re_D

$$Re_x = \frac{\bar{u}x}{\nu} = \frac{\bar{u}D}{\nu}\frac{x}{D} = Re_D \frac{x}{D} \tag{d}$$

Substitute (d) into (c)

$$Nu_D \sim \left(\frac{D}{x}\right)^{1/2} Pr^{1/2} Re_D^{1/2} \tag{6.20a}$$

Rewrite

$$\frac{Nu_D}{\left(\dfrac{PrRe_D}{x/D}\right)^{1/2}} \sim 1 \tag{6.20b}$$

- Scaling estimates (6.19) and (6.20) will be compared with exact solutions.

6.6.2 Basic Considerations for the Analytical Determination of Heat Flux, Heat Transfer Coefficient and Nusselt Number

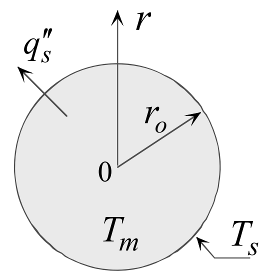


Fig. 6.5

- Need to determine velocity and temperature distribution.
- Assume: fully developed velocity
- Neglect axial conduction
- Section outline:
  - Definitions
  - Governing equations for determining:

    (i) Surface heat flux
    (ii) Heat transfer coefficient
    (iii) Nusselt number

(1) Fourier's law and Newton's law.

- Surface heat flux. Fourier's law gives surface heat flux q''_s

$$q''_s = -k\frac{\partial T(x, r_o)}{\partial r} \tag{a}$$

Define dimensionless variables

$$\theta = \frac{T - T_s}{T_i - T_s}, \quad \xi = \frac{x/D}{Re_D Pr}, \quad R = \frac{r}{r_o}, \quad v_x^* = v\frac{v_x}{\bar{u}}, \quad v_r^* = \frac{v_r}{\bar{u}}, \quad Re_D = \frac{\bar{u}D}{\nu} \tag{6.21}$$

Substitute into (a)

$$q''_s(\xi) = \frac{k}{r_o}(T_s - T_i)\frac{\partial\theta(\xi,1)}{\partial R} \tag{6.22}$$

- Heat transfer coefficient. Define h

$$h(\xi) = \frac{q''_s}{T_m - T_s} \tag{6.23}$$

Combine (6.22) and (6.23)

$$h(\xi) = \frac{k(T_s - T_i)}{r_o(T_m - T_s)}\frac{\partial\theta(\xi,1)}{\partial R} = -\frac{k}{r_o}\frac{1}{\theta_m(\xi)}\frac{\partial\theta(\xi,1)}{\partial R} \tag{6.24}$$

where θ_m is defined as

$$\theta_m \equiv \frac{T_m - T_s}{T_i - T_s} \tag{6.25}$$

- Nusselt number. Define:

$$Nu(\xi) = \frac{h(\xi)D}{k} = \frac{h(\xi)2r_o}{k} \tag{6.26}$$

(6.24) into (6.26)

$$Nu(\xi) = \frac{-2}{\theta_m(\xi)}\frac{\partial\theta(\xi,1)}{\partial R} \tag{6.27}$$

(2) The Energy Equation. Review assumptions on energy equation (2.24).

$$\rho c_p \left(v_r \frac{\partial T}{\partial r} + v_z \frac{\partial T}{\partial z} \right) = k \left[\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial T}{\partial r} \right) + \frac{\partial^2 T}{\partial z^2} \right] \tag{2.24}$$

Replace z by x, use dimensionless variables:

$$v_x^* \frac{\partial \theta}{\partial \xi} + 2Re_D Pr\, v_r^* \frac{\partial \theta}{\partial R} = \frac{4}{R} \frac{\partial}{\partial R} \left(R \frac{\partial \theta}{\partial R} \right) + \frac{1}{(Re_D Pr)^2} \frac{\partial^2 \theta}{\partial \xi^2} \tag{6.28}$$

where

$$Pe = Re_D Pr\,, \quad \text{Peclet number} \tag{6.29}$$

- Neglect conduction for

$$Pe = PrRe_D \geq 100 \tag{6.30}$$

Thus, under such conditions, (6.28) becomes

$$v_x^* \frac{\partial \theta}{\partial \xi} + 2Re_D Pr\, v_r^* \frac{\partial \theta}{\partial R} = \frac{4}{R} \frac{\partial}{\partial R} \left(R \frac{\partial \theta}{\partial R} \right) \tag{6.31}$$

(3) Mean (Bulk) Temperature T_m. Define:

$$m c_p T_m = \int_0^{r_o} \rho c_p v_x T \, 2\pi r dr \tag{a}$$

Mass flow rate m is given by

$$m = \int_0^{r_o} \rho v_x 2\pi r dr \tag{b}$$

(b) into (a), assume constant properties

$$T_m = \frac{\int_0^{r_o} v_x T r dr}{\int_0^{r_o} v_x r dr} \tag{6.32a}$$

Dimensionless form:

$$\theta_m = \frac{T_m - T_s}{T_i - T_s} = \frac{\int_0^1 v_x^* \theta R dR}{\int_0^1 v_x^* R dR} \tag{6.32b}$$

6.7 Heat Transfer Coefficient in the Fully Developed Temperature Region

6.7.1 Definition of Fully Developed Temperature Profile

- Far away from the entrance ($x/d > 0.05Re_D Pr$), temperature profile becomes *fully developed.*

- To define fully developed temperature, introduce the dimensionless temperature ϕ

$$\phi = \frac{T_s(x) - T(r,x)}{T_s(x) - T_m(x)} \tag{6.33}$$

- Fully developed temperature is defined as a profile in which ϕ is independent of x:

$$\phi = \phi(r) \tag{6.34}$$

(6.34) gives

$$\frac{\partial \phi}{\partial x} = 0 \tag{6.35}$$

(6.33) and (6.35) give

$$\frac{\partial \phi}{\partial x} = \frac{\partial}{\partial x}\left[\frac{T_s(x) - T(r,x)}{T_s(x) - T_m(x)}\right] = 0 \tag{6.36a}$$

Expand and use the definition of ϕ in (6.33)

$$\frac{dT_s}{dx} - \frac{\partial T}{\partial x} - \phi(r)\left[\frac{dT_s}{dx} - \frac{dT_m}{dx}\right] = 0 \tag{6.36b}$$

6.7.2 Heat Transfer Coefficient and Nusselt Number

- Examine h and Nu in the fully developed region.
- Fourier's and Newton's law:

$$h = \frac{-k\dfrac{\partial T(r_o,x)}{\partial r}}{T_m - T_s} \tag{6.16}$$

Use (6.33) to eliminate $\partial T(r_o,x)/\partial r$. (6.16) gives

$$h = -k\frac{d\phi(r_o)}{dr} = \text{constant} \tag{6.37}$$

IMPORTANT CONCLUSOIN:

THE HEAT TRANSFER COEFFICIENT IN THE FULLY DEVELOPED REGION IS CONSTANT INDEPENDET OF LOCATION.

- Nusselt number

$$Nu_D = \frac{hD}{k} = -D\frac{d\phi(r_o)}{dr} \tag{6.38}$$

  - Scaling estimate based on limiting case of entrance region:

$$Nu_D \sim 1 \quad \text{(fully developed)} \tag{6.19}$$

  - Scale estimate based on fully developed region:

Scale $\partial T(r_o,x)/\partial r$ as

$$\frac{\partial T(r_o,x)}{\partial r} \sim \frac{T_s - T_m}{D}$$

Substitute into (6.16)

$$h \sim \frac{k}{D} \tag{6.39}$$

Substitute (6.39) into (6.38)

$$Nu_D \sim 1 \quad \text{(fully developed)} \tag{6.40}$$

6.7.3 Fully Developed Region for Tubes at Uniform Surface flux

- Determine:

(i) Surface temperature $T_s(x)$.
(ii) Heat transfer coefficient.

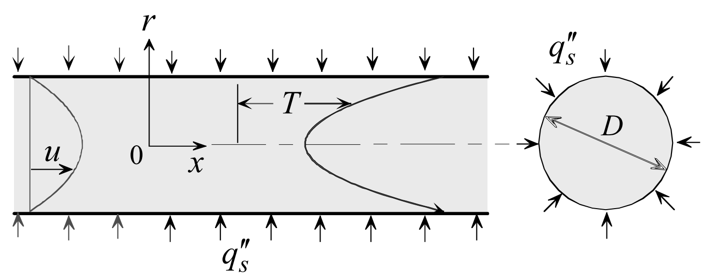


Fig. 6.6

- Newton's law

$$q_s'' = h[T_s(x) - T_m(x)] \tag{a}$$

Since q_s'' and h are constant it follows that

$$T_s(x) - T_m(x) = \text{constant} \tag{b}$$

Differentiate

$$\frac{dT_s}{dx} = \frac{dT_m}{dx}\,. \tag{c}$$

(c) into (6.36b)

$$\frac{\partial T}{\partial x} = \frac{dT_s}{dx} \tag{d}$$

(c) and (d)

$$\frac{\partial T}{\partial x} = \frac{dT_s}{dx} = \frac{dT_m}{dx} \quad \text{(for constant } q_s''\text{)} \tag{6.41}$$

- Unknowns: $T(r,x)$, $T_m(x)$ and $T_s(x)$
- Conservation of energy:

$$q_s'' P dx + mc_p T_m = mc_p\left[T_m + \frac{dT_m}{dx}dx\right]$$

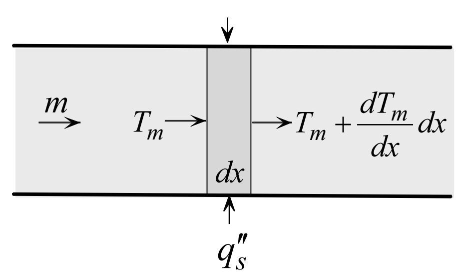


Fig. 6.7

or

$$\frac{dT_m}{dx} = \frac{q_s'' P}{mc_p} = \text{constant} \tag{6.42}$$

Substitute (6.42) into (6.41)

$$\frac{\partial T}{\partial x} = \frac{dT_s}{dx} = \frac{dT_m}{dx} = \frac{q_s'' P}{mc_p} = \text{constant} \tag{6.43}$$

Integrate(6.43)

$$T_m(x) = \frac{q_s'' P}{mc_p} x + C_1 \tag{e}$$

Use inlet condition

$$T_m(0) = T_{mi} \tag{f}$$

Solution (e) becomes

$$T_m(x) = T_{mi} + \frac{q_s'' P}{mc_p} x \tag{6.44}$$

- Need to determine $T(r,x)$ and $T_s(x)$. This requires solving the differential form of the energy equation.
- Set $v_r = 0$ in energy equation (2.24)

$$\rho c_p v_x \frac{\partial T}{\partial x} = \frac{k}{r} \frac{\partial}{\partial r}\left(r \frac{\partial T}{\partial r} \right) \tag{6.45}$$

Fully developed flow axial velocity

$$v_x = 2\bar{u}\left[1 - \frac{r^2}{r_o^2} \right] \tag{6.46}$$

(6.43) and (6.46) into (6.45)

$$\rho c_p 2\bar{u}\left[1 - \frac{r^2}{r_o^2} \right] \frac{q_s'' P}{\dot{m} c_p} = \frac{k}{r} \frac{\partial}{\partial r}\left(r \frac{\partial T}{\partial r} \right) \tag{g}$$

However, $m = \pi r_o^2 \rho \bar{u}$ and $P = 2\pi r_o$, equation (g) becomes

$$\frac{4q_s''}{r_o}\left[1 - \frac{r^2}{r_o^2} \right] = \frac{k}{r} \frac{\partial}{\partial r}\left(r \frac{\partial T}{\partial r} \right) \tag{6.47}$$

Boundary conditions are:

$$\frac{\partial T(0,x)}{\partial r} = 0 \tag{6.48a}$$

$$k \frac{\partial T(r_o,x)}{\partial r} = q_s'' \tag{6.48b}$$

Integrate (6.47)

$$\frac{4}{r_o} q_s'' \left[\frac{r^2}{2} - \frac{r^4}{4r_o^2} \right] = kr \frac{\partial T}{\partial r} + f(x) \tag{h}$$

Boundary condition (6.48a) gives $f(x) = 0$. Equation (h) becomes

$$\frac{\partial T}{\partial r} = \frac{4q_s''}{kr_o} \left[\frac{r}{2} - \frac{r^3}{4r_o^2} \right]$$

Integrate again

$$T(r,x) = \frac{4q_s''}{kr_o} \left[\frac{r^2}{4} - \frac{r^4}{16r_o^2} \right] + g(x) \tag{6.49}$$

The integration "constant" is $g(x)$. Use $T_m(x)$ to determine $g(x)$. Substitute (6.46) and (6.49) into (6.32a)

$$T_m(x) = \frac{7}{24} \frac{r_o q_s''}{k} + g(x) \tag{6.50}$$

Equate (6.44) and (6.50) gives $g(x)$

$$g(x) = T_{mi} - \frac{7}{24} \frac{r_o q_s''}{k} + \frac{P q_s''}{m c_p} x \tag{6.51}$$

(6.51) into (6.49)

$$T(r,x) = T_{mi} + \frac{4q_s''}{kr_o} \left[\frac{r^2}{4} - \frac{r^4}{16r_o^2} \right] - \frac{7}{24} \frac{r_o q_s''}{k} + \frac{P q_s''}{m c_p} x \tag{6.52}$$

Set $r = r_o$ in (6.52) to obtain $T_s(x)$

$$T_s(x) = T_{mi} + \frac{11}{24} \frac{r_o q_s''}{k} + \frac{P q_s''}{m c_p} x \tag{6.53}$$

(6.44), (6.52) and (6.53) into (6.33) gives $\phi(r)$

$$\phi(r) = 1 - \frac{24}{11} \frac{1}{r_o^2} \left[r^2 - \frac{r^4}{4r_o^2} \right] + \frac{24}{11} \frac{P q_s''}{m c_p} x + \frac{7}{11} x \tag{6.54}$$

Differentiate (6.54) and substitute into (6.38) gives

$$Nu_D = \frac{48}{11} = 4.364 \tag{6.55}$$

NOTE:

- (6.55) applies to laminar fully developed velocity and temperature in tubes with uniform surface heat flux.
- The Nusselt number is independent of Reynolds and Prandtl numbers.
- Scaling gives Nusselt as

$$Nu_D \sim 1 \tag{6.40}$$

This compares favorable with (6.55).

6.7.4 Fully Developed Region for Tubes at Uniform Surface Temperature

- Determine: Nusselt number
- Solve the energy equation for the fully developed region
- Neglect axial conduction and dissipation.
- Energy equation: set $v_r = 0$ in (2.24)

$$\rho c_p v_x \frac{\partial T}{\partial x} = \frac{k}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) \tag{6.45}$$

- Boundary conditions

$$\frac{\partial T(0,x)}{\partial r} = 0 \tag{6.56a}$$

$$T(r_o, x) = T_s \tag{6.56b}$$

- Axial velocity for fully developed flow is

$$v_x = 2\bar{u}\left[1 - \frac{r^2}{r_o^2}\right] \tag{6.46}$$

- Use (6.36a) to Eliminate $\partial T / \partial x$ in (6.45)

$$\frac{\partial \phi}{\partial x} = \frac{\partial}{\partial x}\left[\frac{T_s(x) - T(r,x)}{T_s(x) - T_m(x)}\right] = 0 \tag{6.36a}$$

For uniform $T_s(x) = T_s$, above gives

$$\frac{\partial T}{\partial x} = \frac{T_s - T(r,x)}{T_s - T_m(x)}\frac{dT_m}{dx} \tag{6.57}$$

(6.46) and (6.57) into (6.45)

$$2\rho c_p \bar{u}\left[1 - \frac{r^2}{r_o^2}\right]\frac{T_s - T(r,x)}{T_s - T_m(x)}\frac{dT_m}{dx} = \frac{k}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) \tag{6.58}$$

Solution: (6.58) was solved using an infinite power series. Solution gives the Nusselt number as

$$Nu_D = 3.657 \tag{6.59}$$

6.7.5 Nusselt Number for Laminar Fully Developed Velocity and Temperature in Channels of Various Cross-Sections

- Table 6.2 lists Nusselt numbers for channels of various cross-sections.
- Two cases: (1) uniform surface heat flux and (2) uniform surface temperature.
- Nusselt number of Non-circular channels is based on the equivalent diameter.
- Scaling estimate:

$$Nu_D \sim 1 \quad \text{(fully developed)} \tag{6.40}$$

- Table 6.2: Nusselt number ranges from 2.46 to 8.235.

6.8 Thermal Entrance Region: Laminar Flow through Tubes

6.8.1 Uniform Surface Temperature: Graetz Solution

- Laminar flow.
- Fully developed inlet velocity.
- Neglect axial conduction ($Pe > 100$).
- Uniform surface temperature T_s.

Fully developed flow:

$$v_r = 0 \tag{3.1}$$

Axial velocity

$$v_z = \frac{1}{4\mu}\frac{dp}{dz}(r^2 - r_o^2) \tag{3.12}$$

(3.12) expressed in dimensionless form

$$v_x^* = \frac{v_x}{\bar{u}} = 2(1 - R^2) \tag{6.61}$$

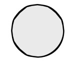
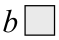
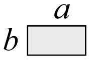
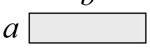
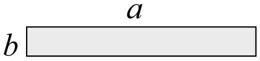
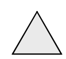
(3.1) and (6.61) into energy equation (6.31)

$$\frac{1}{2}\left(1 - R^2\right)\frac{\partial\theta}{\partial\xi} = \frac{1}{R}\frac{\partial}{\partial R}\left(R\frac{\partial\theta}{\partial R}\right) \tag{6.62}$$

- Boundary conditions

Table 6.2

Nusselt number for laminar fully developed conditions in channels of various cross-sections [3]

Channel geometry	$\frac{a}{b}$	**Nusselt number Nu_D** Uniform surface flux	Uniform surface temperature
		4.364	3.657
b  a	1	3.608	2.976
b  a	2	4.123	3.391
a  b	4	5.331	4.439
b  a	8	6.49	5.597
	∞	8.235	7.541
		3.102	2.46

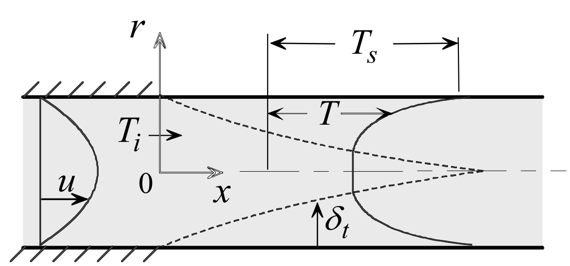


Fig. 6.8

$$\frac{\partial\theta(\xi,0)}{\partial R} = 0 \tag{6.63a}$$

$$\theta(\xi,1) = 0 \tag{6.63b}$$

$$\theta(0,R) = 1 \tag{6.63c}$$

- Analytic and numerical solutions to this problem have been obtained.
- Review analytic solution leading to:

(i) Mean temperature, $\theta_m(\xi)$

$$\theta_m(\xi) = 8\sum_{n=0}^{\infty}\frac{G_n}{\lambda_n^2}\exp(-2\lambda_n^2\xi) \tag{6.66}$$

(ii) Local Nusselt number, $Nu(\xi)$

$$Nu(\xi) = \frac{\sum_{n=0}^{\infty} G_n \exp(-2\lambda_n^2\xi)}{2\sum_{n=0}^{\infty}\frac{G_n}{\lambda_n^2}\exp(-2\lambda_n^2\xi)} \tag{66.7}$$

(iii) Average Nusselt number, $\overline{Nu}(\xi)$

$$\overline{Nu}(\xi) = \frac{\overline{h}(\xi)D}{k} \tag{f}$$

RESULTS

- Table 6.3 lists values of λ_n and G_n for $0 \le n \le 10$. Table 6.4 gives $Nu(\xi)$ and $\overline{Nu}(\xi)$ at selected values of the axial distance ξ.
- Fig. 6.9 gives the variation of $Nu(\xi)$ and $\overline{Nu}(\xi)$ along a tube.

Table 6.3
Uniform surface temperature [4]

n	λ_n	G_n
0	2.70436	0.74877
1	6.67903	0.54383
2	10.67338	0.46286
3	14.67108	0.41542
4	18.66987	0.38292
5	22.66914	0.35869
6	26.66866	0.33962
7	30.66832	0.32406
8	34.66807	0.31101
9	38.66788	0.29984
10	42.66773	0.29012

Table 6.4
Local and average Nusselt number for tube at uniform surface temperature [5]

$\xi = \frac{x/D}{Re_D Pr}$	$Nu(\xi)$	$\overline{Nu}(\xi)$
0	∞	∞
0.0005	12.8	19.29
0.002	8.03	12.09
0.005	6.00	8.92
0.02	4.17	5.81
0.04	3.77	4.86
0.05	3.71	4.64
0.1	3.66	4.15
∞	3.66	3.66

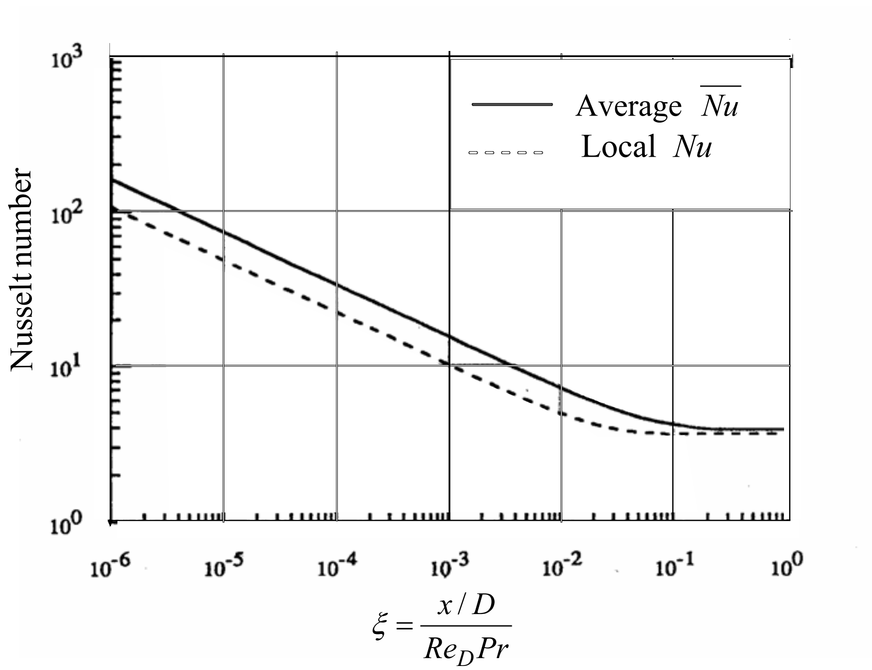


Fig. 6.9 Local and average Nusselt number for tube at uniform surface temepratu re [4]

NOTE:

(1) The average Nusselt number is greater than the local Nusselt number.

(2) Asymptotic value of Nusselt number of 3.657 is reached at $\xi \approx 0.05$. Thus

$$Nu(\infty) = 3.657 \tag{6.69}$$

(3) Evaluate fluid properties at the mean temperatures $\overline{T}_m$, defined as

$$\overline{T}_m = \frac{T_{mi} + T_{mo}}{2} \tag{6.70}$$

6.8.2 Uniform Surface Heat Flux

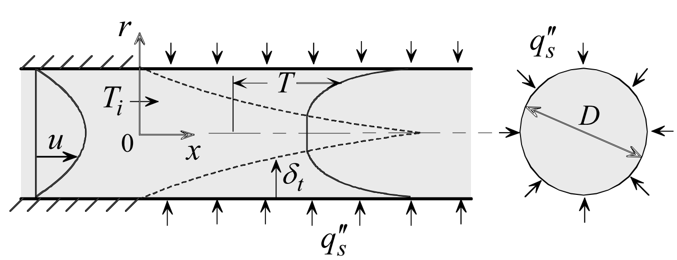


Fig. 6.10

- Repeat Graetz entrance problem replacing the uniform surface temperature with uniform heat flux.
- Inlet velocity is fully developed.
- Energy equation is

$$\frac{1}{2}\left(1 - R^2\right)\frac{\partial \theta}{\partial \xi} = \frac{1}{R}\frac{\partial}{\partial R}\left(R\frac{\partial \theta}{\partial R}\right) \tag{6.62}$$

Boundary conditions

$$\frac{\partial\theta(\xi,0)}{\partial R}) = 0 \qquad (6.71a)$$

$$\frac{\partial\theta(\xi,1)}{\partial R} = \frac{q_s'' r_o}{k(T_i - T_s)} \qquad (6.71b)$$

$$\theta(0,R) = 1 \qquad (6.71c)$$

- Solution.
- Local Nusselt number:

$$Nu(\xi) = \frac{hx}{k} = \left[\frac{11}{48} + \frac{1}{2}\sum_{n=1}^{\infty} A_n \exp(-2\beta_n^2 \xi)\right]^{-1} \qquad (6.72)$$

- The average Nusselt number is given by

$$\overline{Nu}(\xi) = \frac{hx}{k} = \left[\frac{11}{48} + \frac{1}{2}\sum_{n=1}^{\infty} A_n \frac{1-\exp(-2\beta_n^2 \xi)}{2\beta_n^2 \xi}\right]^{-1} \qquad (6.73)$$

- The eigenvalues β_n^2 and the constant A_n are listed in Table 6.5
- Limiting case: $\xi = \infty$ (fully developed)

$$Nu(\infty) = \frac{48}{11} = 4.364 \qquad (6.74)$$

Table 6.5
Uniform surface flux [4]

n	β_n^2	A_n
1	25.6796	0.198722
2	83.8618	0.069257
3	174.1667	0.036521
4	296.5363	0.023014
5	450.9472	0.016030
6	637.3874	0.011906
7	855.8495	0.009249
8	1106.3290	0.007427
9	1388.8226	0.006117
10	1703.3279	0.005141

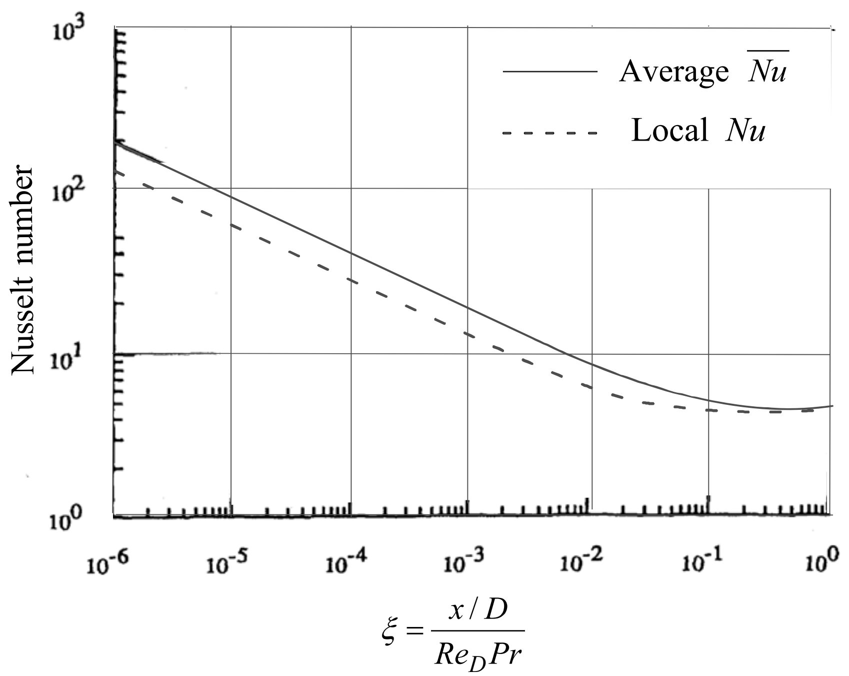


Fig. 6.11 Local and average Nusselt number for tube at uniform surface heat flux [4]