CHAPTER 7

FREE CONVECTION

7.1 Introduction

7.2 Features and Parameters of Free Convection

(1) Driving Force.

- Requirements

  (i) Gravitational field
  (ii) Density change with temperature

(2) Governing Parameters.

Two parameters:

(i) Grashof number

$$\text{Grashof number} = Gr_L = \frac{\beta g (T_s - T_\infty) L^3}{\nu^2} \tag{7.1}$$

(ii) Prandtl number

- β is the Coefficient of thermal expansion, also known as compressibility factor. For ideal gases it is given by

$$\beta = \frac{1}{T}, \quad \text{for ideal gas} \tag{2.21}$$

- Rayleigh number

$$Ra_L = Gr_L Pr = \frac{\beta g (T_s - T_\infty) L^3}{\nu^2} Pr = \frac{\beta g (T_s - T_\infty) L^3}{\nu \alpha} \tag{7.2}$$

(3) Boundary Layer.

- Flow: Laminar, turbulent, or mixed.
- Boundary layer approximations are valid for $Ra_x > 10^4$.

(4) Transition from Laminar to Turbulent Flow.

- For vertical plates: transition Rayleigh number, Ra_{x_t}, is

$$Ra_{x_t} \approx 10^9 \tag{7.3}$$

(5) External vs. Enclosure Free Convection.

(i) External free convection: surface is immersed in infinite medium.
(ii) Enclosure free convection. Free convection takes place inside closed volumetric regions.

(6) Analytic Solutions.

- Velocity and temperature fields are coupled.
- Momentum and energy equation must be solved simultaneously.

7.3 Governing Equations

Approximations:

(1) Constant density, except in evaluating gravity forces.
(2) The Boussinesq approximation (relates density change to temperature change).
(3) No dissipation.

- Continuity, momentum, and energy equations are obtained from equations (2.2), (2.29) and (2.19), respectively

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \tag{7.4}$$

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\beta g(T-T_\infty)-\frac{1}{\rho_\infty}\frac{\partial}{\partial x}(p-p_\infty)+\nu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right) \tag{7.5}$$

$$u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}=-\frac{1}{\rho_\infty}\frac{\partial}{\partial y}(p-p_\infty)+\nu\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right) \tag{7.6}$$

$$u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}=\alpha\left(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}\right) \tag{7.7}$$

7.3.1 Boundary Layer Equations

- Continuity equation (7.4) is unchanged
- ***x***-component of the Navier-Stokes equations simplifies to

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\beta g(T-T_\infty)+\nu\frac{\partial^2 u}{\partial y^2} \tag{7.8}$$

- Energy equation (7.7)

$$u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}=\alpha\frac{\partial^2 T}{\partial y^2} \tag{7.9}$$

- (7.4), (7.8), and (7.9) contain three unknowns: u, v, and T.
- Momentum and energy are coupled.

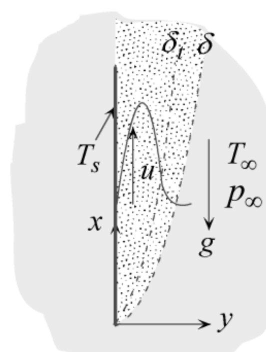

Fig. 7.1

7.4 Laminar Free Convection over a Vertical Plate: Uniform Surface Temperature

- Uniform temperature T_s (Fig. 7.1).
- Infinite fluid at temperature T_∞.
- Determine: velocity and temperature distribution.

7.4.1 Assumptions. Note all assumptions listed in this section.

7.4.2 Governing Equations

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \tag{7.4}$$

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\beta g\left(T-T_\infty\right)+\nu\frac{\partial^2 u}{\partial y^2} \tag{7.8}$$

$$u\frac{\partial \theta}{\partial x}+v\frac{\partial \theta}{\partial y}=\alpha\frac{\partial^2 \theta}{\partial y^2} \tag{7.10}$$

where θ is defined as

$$\theta=\frac{T-T_\infty}{T_s-T_\infty} \tag{7.11}$$

7.4.3 Boundary Conditions.

Velocity:

(1) $u(x,0)=0$
(2) $v(x,0)=0$
(3) $u(x,\infty)=0$
(4) $u(0,y)=0$

Temperature:

(5) $\theta(x,0)=1$
(6) $\theta(x,\infty)=0$
(7) $\theta(0,y)=0$

7.4.4 Similarity Transformation. Introduce the similarity variable η

$$\eta=\left(\frac{Gr_x}{4}\right)^{1/4}\frac{y}{x} \tag{7.14}$$

where

$$Gr_x=\frac{\beta g(T_s-T_\infty)x^3}{\nu^2} \tag{7.15}$$

Let

$$\theta(x,y)=\theta(\eta) \tag{7.16}$$

$$u=2\nu\frac{\sqrt{Gr_x}}{x}\frac{d\xi}{d\eta} \tag{7.20}$$

Continuity gives

$$v=\frac{\nu}{(4)^{1/4}}\frac{(Gr_x)^{1/4}}{x}\left[\eta\frac{d\xi}{d\eta}-3\xi\right] \tag{7.21}$$

(7.20) and (7.21) into (7.8) and (7.10) and using (7.11) and (7.16), gives

$$\frac{d^3\xi}{d\eta^3}+3\xi\frac{d^2\xi}{d\eta^2}-2\left(\frac{d\xi}{d\eta}\right)^2+\theta=0 \tag{7.22}$$

$$\frac{d^2\theta}{d\eta^2} + 3Pr\xi\frac{d\theta}{d\eta} = 0 \qquad (7.23)$$

Transformation of boundary conditions:

Velocity:

(1) $\dfrac{d\xi(0)}{d\eta} = 0$

(2) $\xi(0) = 0$

(3) $\dfrac{d\xi(\infty)}{d\eta} = 0$

(4) $\dfrac{d\xi(\infty)}{d\eta} = 0$

Temperature:

(1) $\theta(0) = 1$

(2) $\theta(\infty) = 0$

(3) $\theta(\infty) = 0$

- The problem is characterized by a single parameter which is the Prandtl number.

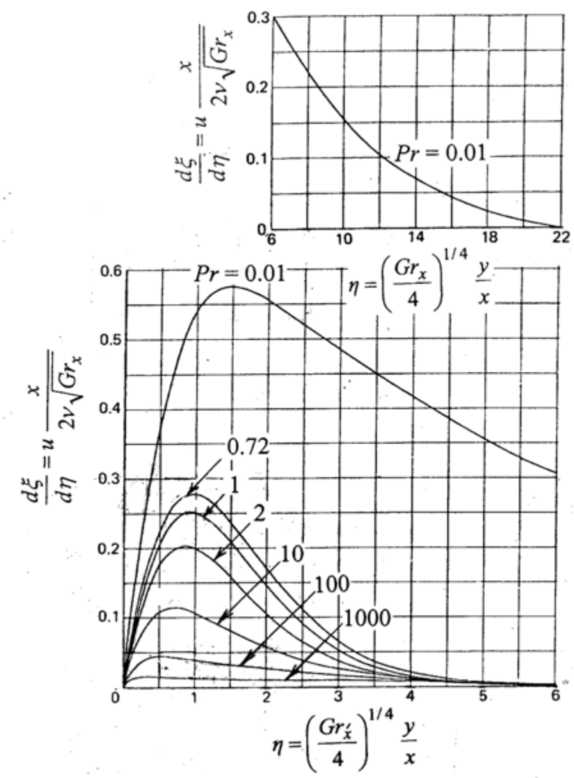

Fig. 7.2 Axial velocity distribution [1]

7.4.5 Solution.

- (7.22) and (7.23) and their five boundary conditions are solved numerically.
- The solution is presented graphically in Figs. 7.2 and 7.3. Fig. 7.2 gives the

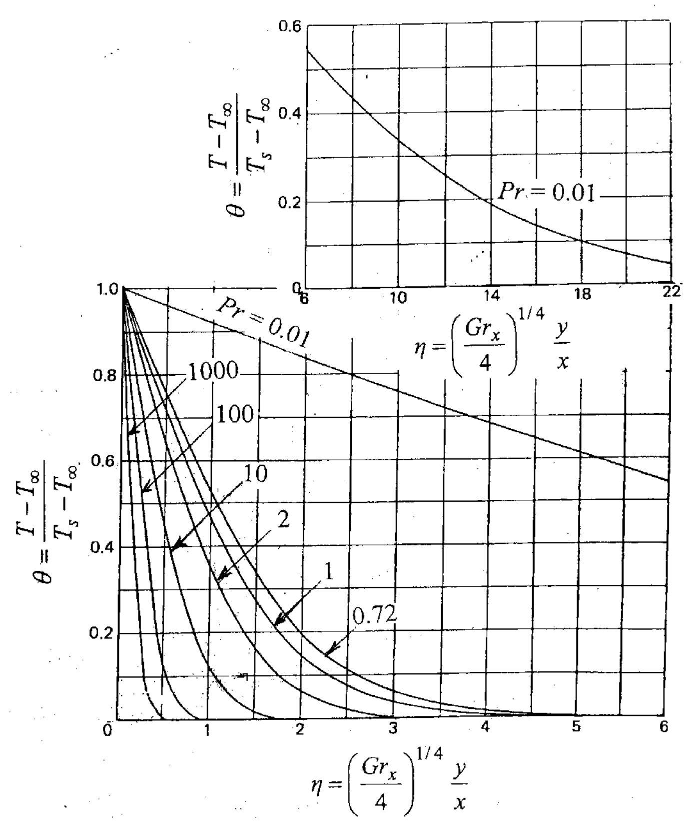

Fig. 7.3 Temperature distribution [1]

7.4.6 Heat Transfer Coefficient and Nusselt Number.

Fourier's law and Newton's law:

$$h = \frac{-k\dfrac{\partial T(x,0)}{\partial y}}{T_s - T_\infty} \tag{7.24}$$

Express in terms of θ and η

$$h = \frac{-k}{T_s - T_\infty}\frac{dT}{d\theta}\frac{d\theta(0)}{d\eta}\frac{\partial\eta}{\partial y}$$

Use(7.11) and (7.14)

$$h = \frac{-k}{x}\left[\frac{Gr_x}{4}\right]^{1/4}\frac{d\theta(0)}{d\eta} \tag{7.25}$$

Local Nusselt number

$$Nu_x = \frac{hx}{k} = -\left[\frac{Gr_x}{4}\right]^{1/4}\frac{d\theta(0)}{d\eta} \tag{7.26}$$

Average h

$$\bar{h} = \frac{1}{L}\int_0^L h(x)dx \tag{2.50}$$

(7.25) into (2.50), and performing the integration

$$\bar{h} = -\frac{4}{3}\frac{k}{L}\left(\frac{Gr_L}{4}\right)^{1/4}\frac{d\theta(0)}{d\eta} \tag{7.27}$$

Average Nusselt number is

$$\overline{Nu}_L = \frac{\bar{h}L}{k} = -\frac{4}{3}\left(\frac{Gr_L}{4}\right)^{1/4}\frac{d\theta(0)}{d\eta} \tag{7.28}$$

- Solution depends on a single parameter which is the Prandtl number.
- Numerical solution gives $\frac{d\theta(0)}{d\eta}$, listed in Table 7.1.

Table 7.1 [1,2]

Pr	$-\frac{d\theta(0)}{d\eta}$	$\frac{d^2\xi(0)}{d\eta^2}$
0.01	0.0806	0.9862
0.03	0.136	
0.09	0.219	
0.5	0.442	
0.72	0.5045	0.676
0.733	0.508	0.6741
1.0	0.5671	0.6421
1.5	0.6515	
2.0	0.7165	0.5713
3.5	0.8558	
5.0	0.954	
7.0	1.0542	
10	1.1649	0.4192
100	2.191	0.2517
1000	3.9660	0.1450

Special Cases

- Very small and very large Prandtl numbers:

$$Nu_x = 0.600\,(PrRa_x)^{1/4}, \quad Pr \to 0 \tag{7.29a}$$

$$Nu_x = 0.503\,(PrGr_x)^{1/4}, \quad Pr \to \infty \tag{7.29b}$$

7.5 Laminar Free Convection over a Vertical Plate: Uniform Surface Heat Flux

- Assumptions: Same as constant temperature plate.

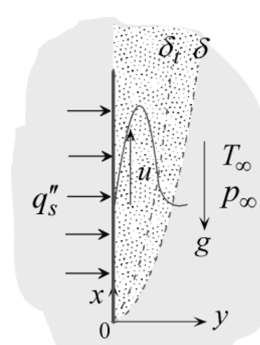

Fig. 7.4

- Surface boundary conditions

$$-k\frac{\partial T(x,0)}{\partial y} = q_s'' \qquad (7.30)$$

  - Surface flux is specified.
  - Determine: Surface temperature $T_s(x)$ and local Nusselt number Nu_x.
  - Solution by similarity transformation.

Solution:

- Surface temperature

$$T_s(x) - T_\infty = -\left[5\frac{\nu^2 (q_s'')^4}{\beta g k^4}x\right]^{1/5} \theta(0) \qquad (7.31)$$

- Local Nusselt number

$$Nu_x = -\left[\frac{\beta g q_s''}{5\nu^2 k}x^4\right]^{1/5} \frac{1}{\theta(0)} \qquad (7.32)$$

- $\theta(0)$ is a dimensionless parameter which depends on the Prandtl number and is given in Table 7.2 [4].
- Correlation equation for $\theta(0)$

$$\theta(0) = -\left[\frac{4 + 9Pr^{1/2} + 10Pr}{5Pr^2}\right]^{1/5}, \quad 0.001 < Pr < 1000 \qquad (7.33)$$

Table 7.2 [4]

Pr	$\theta(0)$
0.1	- 2.7507
1.0	- 1.3574
10	- 0.76746
100	- 0.46566

Properties at the film temperature T_f

$$T_f = [T_\infty + T_s(L/2)]/2 \qquad 7.34)$$

7.6 Inclined Plates

- Vertical plate solutions of Sections 7.4 and 7.5 apply to inclined plates, with g replaced by $g\cos\theta$.
- This approach is recommended for $\theta \le 60^\circ$.

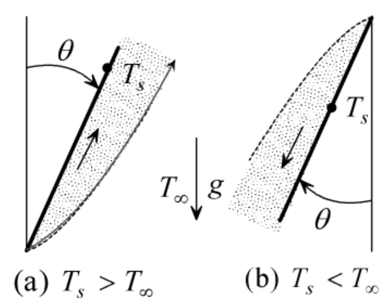

Fig. 7.5

7.7 Integral Method

7.7.1 Integral Formulation of Conservation of Momentum

- Assume:

$$\delta = \delta_t \qquad (a)$$

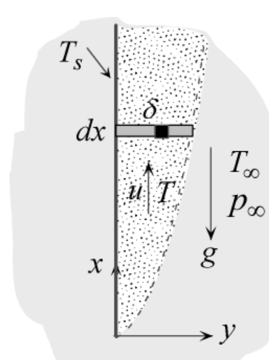

Fig 7.6

- Application of the momentum theorem in the x-direction to the element $\delta \times dx$, Fig. 7.6

$$\sum F_x = M_x(\text{out}) - M_x(\text{in}) \tag{b}$$

$\delta \times dx$ is enlarged in Fig. 7.7

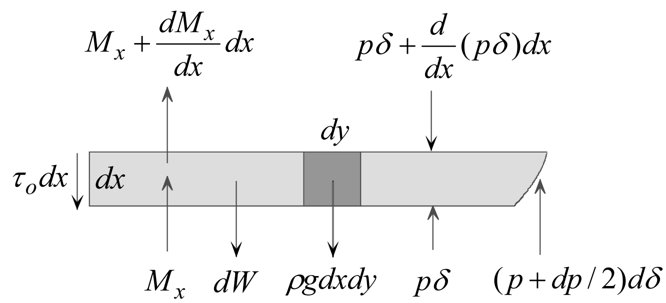

Fig. 7.7

$$p\delta + \left(p + \frac{dp}{2}\right)d\delta - p\delta - \frac{d}{dx}(p\delta)dx - \tau_o dx - = \left(M_x + \frac{dM_x}{dx}dx\right) - M_x \tag{c}$$

Simplify

$$-\delta dp - \tau_o dx - dW = \frac{dM_x}{dx}dx \tag{d}$$

Wall shearing stress

$$\tau_o = \mu \frac{\partial u(x,0)}{\partial y} \tag{e}$$

Weight of element

$$dW = dx \int_0^{\delta} \rho g \, dy \tag{f}$$

The x-momentum of the fluid entering element

$$M_x = \rho \int_0^{\delta(x)} u^2 dy \tag{g}$$

(e), (f) and (g) into (d)

$$-\mu \frac{\partial u(x,0)}{\partial y} - \delta \frac{dp}{dx} - \int_0^{\delta} \rho g dy = \rho \frac{d}{dx} \int_0^{\delta} u^2 dy \tag{h}$$

Combine pressure and gravity terms

$$\frac{dp}{dx} \cong \frac{dp_\infty}{dx} = -\rho_\infty g \tag{i}$$

Multiply by δ and rewrite as integral

$$\delta \frac{dp}{dx} = -\rho_\infty g\delta = -\int_0^\delta \rho_\infty g dy \tag{j}$$

(j) into (h)

$$-\mu \frac{\partial u(x,0)}{\partial y} + g \int_0^\delta (\rho_\infty - \rho) dy = \rho \frac{d}{dx} \int_0^\delta u^2 dy \tag{k}$$

Express density difference in terms of temperature change

$$\rho_\infty - \rho = \rho\beta(T - T_\infty) \tag{2.28}$$

(2.28) into (k)

$$-v \frac{\partial u(x,0)}{\partial y} + \beta g \int_0^\delta (T - T_\infty) dy = \frac{d}{dx} \int_0^\delta u^2 dy \tag{7.35}$$

- (7.35) applies to laminar as well as turbulent flow.

7.7.2 Integral Formulation of Conservation of Energy

Assume:

(1) No changes in kinetic and potential energy
(2) Negligible axial conduction
(3) Negligible dissipation
(4) Properties are constant

- Forced convection formulation of conservation of energy, (5.7), is applicable to free convection

$$-\alpha \frac{\partial T(x,0)}{\partial y} = \frac{d}{dx} \int_0^{\delta(x)} u(T - T_\infty) dy \tag{7.36}$$

7.7.3 Integral Solution

- Vertical plate, Fig. 7.6.
- Uniform surface temperature T_s .
- We assumed $\delta \approx \delta_t$. Thus we have two equations, (7.35) and (7.36) for the determination of a single unknown δ.
- Since both (7.35) and (7.36) must be satisfied, we introduce another unknown as follows:

Assumed Velocity Profile:

$$u(x, y) = a_0(x) + a_1(x)y + a_2(x)y^2 + a_3(x)y^3 \tag{a}$$

- Boundary conditions on the velocity

(1) $u(x,0) = 0$

(2) $u(x,\delta) \approx 0$

(3) $\dfrac{\partial u(x,\delta)}{\partial y} \approx 0$

(4) $\dfrac{\partial^2 u(x,0)}{\partial y^2} = -\dfrac{\beta g}{\nu}(T_s - T_\infty)$

- Applying the four boundary conditions gives a_n. Equation (a) becomes

$$u = \left[\frac{\beta g (T_s - T_\infty)}{4\nu}\delta^2\right]\frac{y}{\delta}\left[1 - \frac{y}{\delta}\right]^2 \tag{b}$$

Let

$$u_o(x) = \left[\frac{\beta g (T_s - T_\infty)}{4\nu}\delta^2\right] \tag{c}$$

(b) becomes

$$u = u_o(x)\,\frac{y}{\delta}\left[1 - \frac{y}{\delta}\right]^2 \tag{7.37}$$

- Treat $u_o(x)$ as the second unknown function, independent of δ.

Assumed Temperature Profile:

$$T(x,y) = b_0(x) + b_1(x)\,y + b_2(x)\,y^2 \tag{d}$$

The boundary conditions are

(1) $T(x,0) = T_s$

(2) $T(x,\delta) \approx T_\infty$

(3) $\dfrac{\partial T(x,\delta)}{\partial y} \approx 0$

Application of the above boundary conditions gives

$$T(x,y) = T_\infty + (T_s - T_\infty)\left[1 - \frac{y}{\delta}\right]^2 \tag{7.38}$$

Heat Transfer Coefficient and Nusselt Number

$$h = \frac{-k\dfrac{\partial T(x,0)}{\partial y}}{T_s - T_\infty} \tag{7.24}$$

(7.38) into (7.24)

$$h = \frac{2k}{\delta(x)} \tag{7.39}$$

Thus the local Nusselt number is

$$Nu_x = \frac{hx}{k} = 2\frac{x}{\delta(x)} \tag{7.40}$$

- Must find and $u_o(x)$ and $\delta(x)$.

Solution

(7.37) and (7.38) into (7.35)

$$-\nu\frac{u_o}{\delta}+\beta g(T_s-T_\infty)\int_0^\delta\left[1-\frac{y}{\delta}\right]^2 dy=\frac{d}{dx}\left\{\frac{u_o^2}{\delta^2}\int_0^\delta y^2\left[1-\frac{y}{\delta}\right]^4 dy\right\} \quad \text{(e)}$$

Evaluate the integrals

$$\frac{1}{105}\frac{d}{dx}\left[u_o^2\delta\right]=\frac{1}{3}\beta g\left(T_s-T_\infty\right)\delta-\nu\frac{u_o}{\delta} \quad (7.41)$$

(7.37) and (7.38) into (7.36)

$$2\alpha(T_s-T_\infty)\frac{1}{\delta}=(T_s-T_\infty)\ \frac{d}{dx}\left\{\frac{u_o}{\delta}\int_0^{\delta(x)} y\left[1-\frac{y}{\delta}\right]^4 dy\right\} \quad \text{(f)}$$

Evaluate the integrals

$$\frac{1}{60}\frac{d}{dx}\left[u_o\delta\right]=\alpha\frac{1}{\delta} \quad (7.42)$$

- (7.41) and (7.42) are two equation for $\delta(x)$ and $u_o(x)$.
- Assume a solution of the form

$$u_o(x)=Ax^m \quad (7.43)$$

$$\delta(x)=Bx^n \quad (7.44)$$

- A, B, m and n are constants.
- substitute (7.43) and (7.44) into (7.41) and (7.42)

$$\frac{2m+n}{105}A^2Bx^{2m+n-1}=\frac{1}{3}\beta g\left(T_o-T_\infty\right)Bx^n-\frac{A}{B}\nu x^{m-n} \quad (7.45)$$

$$\frac{m+n}{210}ABx^{m+n-1}=\alpha\frac{1}{B}x^{-n} \quad (7.46)$$

- Exponents in each equation must be identical. Thus

$$2m+n-1=n=m-n \quad \text{(g)}$$

$$m+n-1=-n \quad \text{(h)}$$

- Solve (g) and (h) for m and n gives

$$m=\frac{1}{2},\quad n=\frac{1}{4} \quad \text{(i)}$$

- (i) into (7.45) and (7.46) gives A and B

$$A=5.17\nu\left[Pr+\frac{20}{21}\right]^{-1/2}\left[\frac{\beta g(T_s-T_\infty)}{\nu^2}\right]^{1/2} \quad \text{(l)}$$

and

$$B = 3.93\, Pr^{-1/2} \left(Pr + \frac{20}{21} \right)^{1/4} \left[\frac{\beta g (T_s - T_\infty)}{\nu^2} \right]^{-1/4} \quad \text{(m)}$$

(i) and (m) into (7.44)

$$\frac{\delta}{x} = 3.93 \left[\frac{20}{21}\frac{1}{Pr} + 1 \right]^{1/4} (Ra_x)^{-1/4} \quad (7.47)$$

(7.47) into (7.40)

$$Nu_x = 0.508 \left[\frac{20}{21}\frac{1}{Pr} + 1 \right]^{-1/4} (Ra_x)^{1/4} \quad (7.48)$$

7.7.4 Comparison with Exact Solution for Nusselt Number

- (7.26) is the exact solution to the local Nusselt number

$$Nu_x = -\left[\frac{Gr_x}{4} \right]^{1/4} \frac{d\theta(0)}{d\eta} \quad (7.26)$$

- Rewrite (7.26) as

$$\left[\frac{Gr_x}{4} \right]^{-1/4} Nu_x = -\frac{d\theta(0)}{d\eta} \quad (7.49)$$

- Rewrite (7.48)

$$\left[\frac{Gr_x}{4} \right]^{-1/4} Nu_x = 0.508 \left[\frac{20}{21}\frac{1}{Pr} + 1 \right]^{-1/4} (4Pr)^{1/4} \quad (7.50)$$

- The right hand side of (7.49) and (7.50) are compared in Table 7.3.
- The exact solution for $Pr \to 0$

$$Nu_x\big|_{\text{exact}} = 0.600 (Pr Ra_x)^{1/4}, \quad Pr \to 0 \quad (7.29a)$$

- Applying integral solution (7.47) to $Pr \to 0$

$$Nu_x\big|_{\text{integral}} = 0.514 (Pr Ra_x)^{1/4}, Pr \to 0 \quad (7.51a)$$

- Exact and integral solutions for $Pr \to \infty$ are

$$Nu_x\big|_{\text{exact}} = 0.503 (Ra_x)^{1/4}, \quad Pr \to \infty \quad (7.29b)$$

$$Nu_x\big|_{\text{integral}} = 0.508 (Ra_x)^{1/4}, \quad Pr \to \infty \quad (7.51b)$$

Table 7.3

Pr	$-\frac{d\theta(0)}{d\eta}$	$0.508\left[\frac{20}{21}\frac{1}{Pr}+1\right]^{-1/4}(4Pr)^{1/4}$
0.01	0.0806	0.0725
0.03	0.136	0.1250
0.09	0.219	0.2133
0.5	0.442	0.4627
0.72	0.5045	0.5361
0.733	0.508	0.5399
1.0	0.5671	0.6078
1.5	0.6515	0.7031
2.0	0.7165	0.7751
3.5	0.8558	0.9253
5.0	0.954	1.0285
7.0	1.0542	1.1319
10	1.1649	1.2488
100	2.191	2.2665
1000	3.9660	4.0390

NOTE: The error ranges from 1% for $Pr \to \infty$ to 14% for $Pr \to 0$.