Chapter 9: Convection in Turbulent Channel Flow

9.1 Introduction

- Laminar channel flow was discussed in Chapter 6; many features of turbulent flow are similar
- Chapter begins with the criteria for fully developed velocity and temperature profiles
- Chapter Focus: Analysis of fully developed flows
- Analysis is limited to the following general boundary conditions:
  - (i) uniform surface temperature
  - (ii) uniform surface heat flux

9.2 Entry Length

- Criteria for entry length was discussed in Chapter 6
- As a rule of thumb, fully developed velocity and temperature profiles exist for

$$\frac{L_h}{D_e} \approx \frac{L_t}{D_e} \approx 10 \tag{6.7}$$

  where D_e is the *hydraulic* or *equivalent* diameter

$$D_e = \frac{4A_f}{P}$$

  where A_f is the flow area and P is the wetted perimeter

- Eq. (6.7) is recommended for $Pr = 1$ fluids
- More elaborate correlations exist, especially for hydrodymanic entry length; the following approximations are recommended:
  - From White

$$\frac{L_h}{D_e} \approx 4.4 Re_{D_e}^{1/6} \tag{9.1}$$

  - From Latzko

$$\frac{L_h}{D_e} \approx 0.623 Re_{D_e}^{1/4} \tag{9.2}$$

- Thermal entry length doesn't lend itself to a simple, universally-applicable equation since the flow is influenced by fluid properties and boundary conditions
- Hydrodynamic entry length is much shorter for turbulent flow than for laminar, so much so that sometimes it's neglected from analysis
- Thermal entry length is often important

- Analysis of heat transfer in the thermal entry length is complicated and is not covered in the text

9.3 Governing Equations

- Figure 9.1 shows a circular pipe with the velocity in the *x*-direction is labeled as u
- Assumptions:
    - Two-dimensional
    - Axisymmetric
    - Incompressible flow

9.3.1 Conservation Equations

- After Reynolds-averaging, conservation of mass reduces to:

$$\frac{\partial \overline{u}}{\partial x} + \frac{1}{r}\frac{\partial}{\partial r}(r\overline{v}_r) = 0 \tag{9.3}$$

- Using the same conditions, the Reynolds-averaged *x*-momentum equation reduces to:

$$\overline{u}\frac{\partial \overline{u}}{\partial x} + \overline{v}_r \frac{\partial \overline{v}_r}{\partial r} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \frac{1}{r}\frac{\partial}{\partial r}\left[r(\nu + \varepsilon_M)\frac{\partial \overline{u}}{\partial r}\right] \tag{9.4}$$

- Conservation of energy becomes:

$$\overline{u}\frac{\partial \overline{T}}{\partial x} + \overline{v}_r \frac{\partial \overline{T}}{\partial r} = \frac{1}{r}\frac{\partial}{\partial r}\left[r(\alpha + \varepsilon_H)\frac{\partial \overline{T}}{\partial r}\right] \tag{9.5}$$

9.3.2 Apparent Shear Stress and Heat Flux

- The apparent shear stress and heat flux are defined similarly to that of the flat plate development:

$$\frac{\tau_{app}}{\rho} = (\nu + \varepsilon_M)\frac{\partial \overline{u}}{\partial r} \tag{9.6}$$

$$\frac{q''_{app}}{\rho c_p} = -(\alpha + \varepsilon_H)\frac{\partial \overline{T}}{\partial r} \tag{9.7}$$

9.3.3 Mean Velocity and Temperature

- Mean velocity and bulk, or mean, temperature are used in correlations for predicting friction and heat transfer in duct flow
- Mean velocity is calculated by evaluating the mass flow rate in the duct

$$m = \rho u_m A = \int_0^{r_o} \rho \bar{u}(2\pi r)dr$$

- Assuming constant density, this becomes

$$u_m = \frac{1}{\pi r_o^2} \int_0^{r_o} \bar{u}(2\pi r)dr = \frac{2}{r_o^2} \int_0^{r_o} \bar{u} r dr \tag{9.8}$$

- The mean temperature in the duct is evaluated by integrating the total energy of the flow:

$$mc_p T_m = \int_0^{r_o} c_p \bar{T}\bar{u}(2\pi r)dr$$

- Substituting (9.8) for the mass flow rate, and assuming constant specific heat,

$$T_m \equiv \frac{\int_0^{r_o} \bar{T}\,\bar{u} r dr}{\int_0^{r_o} \bar{u} r dr}$$

- This can be simplified by substituting the mean velocity, equation (9.8):

$$T_m = \frac{2}{u_m r_o^2} \int_0^{r_o} \bar{T}\bar{u} r dr \tag{9.9}$$

9.4 Universal Velocity Profile

9.4.1 Results from Flat Plate Flow

- It was shown that universal velocity profile in a pipe is very similar to that over a flat plate (see Fig. 8.13), especially when the plate is exposed to a zero or favorable pressure gradient
    - A pipe flow friction factor model was used to analyze flow over a flat plate using the momentum integral method (see Section 8.4.3)
- The characteristics of the flow near the wall of a pipe are not influenced greatly by the curvature of the wall of the radius of the pipe, so, invoking the two-layer model that we used to model flow over a flat plate:
    - Viscous sublayer

$$u^+ = y^+ \tag{8.54}$$

- Law of the Wall

$$u^+ = \frac{1}{\kappa}\ln y^+ + B \tag{8.58}$$

- Continuous wall law models by Spalding and Reichardt were applied to pipe flow and discussed in Section 8.4.2

- For pipe flow, the wall coordinates are a little different than for flat plate flow, so the *y*-coordinate is

$$y = r_o - r \tag{9.10}$$

- This yields the $y+$ wall coordinate:

$$y^+ = r_0^+ - r^+ = \frac{(r_o - r)u^*}{\nu} \tag{9.11}$$

- The velocity wall coordinate is the same as before:

$$u^+ \equiv \frac{\overline{u}}{u^*} \tag{8.49}$$

- The friction velocity is the same

$$u^* \equiv \sqrt{\tau_o / \rho} \tag{8.46}$$

- The friction factor is based on the mean flow velocity instead of the free stream velocity:

$$C_f = \frac{\tau_o}{(1/2)\rho u_m^2} \tag{9.12}$$

- Therefore, the friction velocity can be expressed as:

$$u^* = u_m \sqrt{C_f / 2}$$

9.4.2 Development in Cylindrical Coordinates

- Since the velocity profile data for pipe flow matches that of flat plate flow, it allowed us to develop expressions for universal velocity profiles solely from flat plate (Cartesian) coordinates

- Developing expressions for universal velocity profiles using cylindrical coordinates is developed after revealing important issues and insights

- Assuming fully-developed flow, the left side of the x-momentum equation (9.4) goes to zero, leaving

$$\frac{1}{r}\frac{\partial}{\partial r}\left(\frac{r\tau}{\rho}\right) = \frac{1}{\rho}\frac{\partial \overline{p}}{\partial x} \tag{9.13}$$

- Rearranging and integrating yields an expression for shear stress anywhere in the flow

$$\tau(r) = \frac{r}{2}\frac{\partial \overline{p}}{\partial x} + C \tag{9.14}$$

- The constant C is zero since the velocity gradient (and hence the shear stress) is expected to go to zero at $r = 0$
- Evaluating (9.14) at r and ro and taking the ratio of the two gives

$$\frac{\tau(r)}{\tau_o} = \frac{r}{r_o} \tag{9.15}$$

- Equation (9.15) shows that the local shear is a linear function of radial location and raises the following important issue:
  - Equation (9.15) is a linear shear profile depicted in Fig. 9.2
  - This expression contradicts how the fluid is expected to behave near the wall (recall the Couette Flow assumption led us to the idea that τ is approximately constant in the direction normal to the wall for the flat plate)
  - Yet, in Section 8.4 that the universal velocity profile that resulted from this assumption works well for flat plate flow as well as pipe flow
- This is reconciled as follows
  - The near-wall region over which we make the Couette flow assumption covers a very small distance
  - Assume that, in that small region vary close to the wall of the pipe, the shear is nearly constant: $\tau \approx \tau_0$
  - The Couette assumption approximates the behavior near the pipe wall as

$$(\nu + \varepsilon_M)\frac{\partial \overline{u}}{\partial r} = \frac{\tau_o}{\rho} = \text{constant} \tag{9.16}$$

- Experimental data shows that near the wall, velocity profiles for flat plate and pipe flow are essentially the same and suggests that the near-wall behavior is not influenced by the outer flow, or even the curvature of the wall (see Fig. 8.13)

9.4.3 Velocity Profile for the Entire Pipe

- From the previous development the velocity gradient (and the shear stress) is expected to be zero at the centerline of the pipe; none of the universal velocity profiles developed thus far behave this way
- Reichardt attempted to account for the entire region of the pipe by suggesting the following model for eddy viscosity:

$$\frac{\varepsilon_M}{\nu} = \frac{\kappa y^+}{6}\left(1+\frac{r}{r_o}\right)\left[1+2\left(\frac{r}{r_o}\right)^2\right] \tag{9.17}$$

- This leads to the following expression for the velocity profile

$$u^+ = \frac{1}{\kappa}\ln\left[y^+ \frac{1.5(1+r/r_o)}{1+2(r/r_o)^2}\right] + B \tag{9.18}$$

  where Reichardt used $\kappa = 0.40$ and $B = 5.5$

  - The slope of this equation is zero at $r = 0$, so the behavior matches that at the core of the pipe
  - The viscous sublayer is not accounted for
  - As $r \to r_0$, equation (9.18) reduces to the Law of the Wall form, eq. (8.56)

9.5 Friction Factor for Pipe Flow

9.5.1 Blasius Correlation for Smooth Pipe

- Blasius developed a purely empirical correlation for flow through a smooth pipe using dimensional analysis and experimental data:

$$C_f \approx 0.0791\, Re_D^{-1/4}, \quad \left(4000 < Re_D < 10^5\right) \tag{9.19}$$

  where the friction factor is based on the mean flow velocity $C_f \equiv \tau_o / (1/2)\rho u_m^2$

- Though less accurate and versatile than later correlations, this lead to the 1/7th Power Law velocity profile

9.5.2 The 1/7th Power Law Velocity Profile

- A crude but simple approximation for the velocity profile in a circular pipe was discovered by Prandtl and von Kármán, leading from the Blasius correlation
- **Formulation**
  - Recasting the Blasius correlation in terms of wall shear stress

$$\frac{\tau_o}{\frac{1}{2}\rho u_m^2} = 0.0791\left(\frac{2 r_o u_m}{\nu}\right)^{-1/4}$$

  - Then rearranging:

$$\tau_o = 0.03326\rho u_m^{7/4} r_o^{-1/4} \nu^{1/4} \tag{a}$$

  - Assume a power law can be used to approximate the velocity profile:

$$\frac{\bar{u}}{u_{CL}} = \left(\frac{y}{r_o}\right)^q \tag{b}$$

with u_{CL} representing the centerline velocity

- Assume that the mean velocity in the flow can be related to the centerline velocity as

$$u_{CL} = (const)u_m \tag{c}$$

- Substituting (b) and (c) for the mean velocity in (a) yields

$$\tau_o = (const)\rho\left[\bar{u}\left(\frac{y}{r_o}\right)^{-1/q}\right]^{7/4} r_o^{-1/4}\nu^{1/4}$$

- This is simplified:

$$\tau_o = (const)\rho\bar{u}^{7/4} y^{(-7/4q)} r_o^{(7/4q-1/4)}\nu^{1/4} \tag{d}$$

- Prandtl and von Kármán argued that the wall shear stress is not a function of the size of the pipe, so the exponent on r_0 should be zero
- Setting the exponent to zero, the value of q must be equal to 1/7, leading to the classic $1/7^{th}$ power law velocity profile:

$$\frac{\bar{u}}{u_{CL}} = \left(\frac{y}{r_o}\right)^{1/7} \tag{9.20}$$

- Experimental data shows that this profile adequately models velocity profile through a large portion of the pipe and is frequently used in models for momentum and heat transfer (recall Section 8.4.3)

- **Limitations**
  - Accurate for a narrow range of Reynolds numbers: (10^4 to 10^6)
  - Yields an infinite velocity gradient at the wall
  - Does not yield a gradient of zero at the centerline
- Nikuradse (a student of Prandtl's) measured velocity profiles in smooth pipe over a wide range of Reynolds numbers, and reported that the exponent varied with Reynolds number:

$$\frac{\bar{u}}{u_{CL}} = \left(\frac{y}{r_o}\right)^{1/n} \tag{9.21}$$

- Table 9.1 lists Nikuradse's measurements, including measurement of pipe friction factor of the form

$$C_f = \frac{C}{\mathrm{Re}_D^{1/m}} \tag{9.22}$$

- Nikuradse's results show that the velocity profile becomes fuller as the mean velocity increases

9.5.3 Prandtl's Law for Smooth Pipe

- A more theoretical model for friction factor is developed by employing the universal velocity profile
- **Formulation**
    - Beginning with the Law of the Wall, equation (8.58), substituting the wall coordinates u^+ and y^+, as well as the friction velocity $u^* = \sqrt{\tau_o / \rho} = u_m \sqrt{C_f / 2}$ yields

$$\frac{\bar{u}}{u_m}\sqrt{\frac{2}{C_f}} = \frac{1}{\kappa}\ln\left(\frac{y u_m}{\nu}\sqrt{\frac{C_f}{2}}\right) + B \qquad (9.23)$$

    - Assume that the equation holds at any value of *y*; evaluate the expression at the centerline of the duct, $y = r_o = D/2$, where $\bar{u} = u_{CL}$, by substituting these and the Reynolds number:

$$\frac{u_{CL}}{u_m}\sqrt{\frac{2}{C_f}} = \frac{1}{\kappa}\ln\left(\frac{Re_D}{2}\sqrt{\frac{C_f}{2}}\right) + B \qquad (9.24)$$

    - By looking at the Law of the Wall, one can obtain a functional relationship for the friction factor, though unfortunately a relationship for u_{CL}/u_m is unknown
    - To evaluate the mean velocity, u_m, substitute the velocity profile (8.58) into an expression for the mean velocity, equation (9.10):

$$u_m = \frac{1}{\pi r_o^2}\int_0^{r_o} \bar{u}(2\pi r)dr = \frac{2}{r_o^2}\int_0^{r_o} \bar{u}(r_o - y)dy \qquad (9.25)$$

where $y = r_o - r$

    - Performing the integration, the mean velocity becomes:

$$u_m = u^*\left[\frac{1}{\kappa}\ln\left(\frac{r_o u^*}{\nu}\right) + B - \frac{3}{2\kappa}\right] \qquad (9.26)$$

    - Making substitutions again for u^*

$$u_m = u_m\sqrt{\frac{C_f}{2}}\left[\frac{1}{\kappa}\ln\left(\frac{Re_D}{2}\sqrt{\frac{C_f}{2}}\right) + B - \frac{3}{2\kappa}\right] \qquad (9.27)$$

- u_m cancels out of the equation; however u_{CL} does not appear either, so the above expression can be used directly to find an expression for C_f by rearranging and substituting $\kappa = 0.40$ and $B = 5.0$

$$\frac{1}{\sqrt{C_f/2}} = 2.44\ln\left(Re_D\sqrt{C_f/2}\right) - 0.349$$

- The expression is not yet complete, as it was assumed that the Law of the Wall is accurate everywhere and ignores the presence of a viscous sublayer or wake region
- The constants are adjusted to fit the experimental data, yielding

$$\frac{1}{\sqrt{C_f/2}} = 2.46\ln\left(\mathrm{Re}_D/2\sqrt{C_f/2}\right) + 0.29, \quad \left(Re_D > 4000\right) \qquad (9.28)$$

- This is Prandtl's universal law of friction for smooth pipes; it is also known as the Kármán-Nikuradse equation
- Despite the empiricism of using a curve fit to obtain the constants in (9.28), using a more theoretical basis to develop the function has given the result a wider range of applicability than Blasius's correlation
- Equation (9.28) must be solved iteratively for C_f

- A simpler, empirical relation that closely matches Prandtl's is

$$\frac{C_f}{2} \approx 0.023 Re_D^{-1/5}, \quad \left(3\times 10^4 < Re_D < 10^6\right) \qquad (9.29)$$

  - This equation is also suitable for non-circular ducts with the Reynolds number calculated using the hydraulic diameter

9.5.4 Effect of Surface Roughness

- Roughness shifts the universal velocity profile downward (see Fig. 8.16 and Section 8.4.4)
- The velocity profile in the logarithmic layer can be written as

$$u^+ = \frac{1}{\kappa}\ln y^+ + B - \Delta B$$

  where ΔB is the shift in the curve, which increases with wall roughness k^+
- The behavior also depends on the type of roughness, which ranges from uniform geometries like rivets to random structures like sandblasted metal
- *Equivalent sand-based roughness* model:

$$\frac{1}{f^{1/2}} \approx 2.0\log_{10}\left[\frac{Re_D f^{1/2}}{1 + 0.1(k/D)Re_D f^{1/2}}\right] - 0.8 \qquad (9.30)$$

where it is common to use the Darcy friction factor

$$f = 4C_f \tag{9.31}$$

- If the relative roughness k/D is low enough, it doesn't have much of an effect on the equation
    - Scaling shows that roughness is not important if $(k/D)Re_D < 10$
- If $(k/D)Re_D > 1000$, the roughness term dominates in the denominator, and the Reynolds number cancels
    - Friction is no longer dependent on the Re_D

- Colebrook and White developed the following formula for commercial pipes:

$$\frac{1}{f^{1/2}} = -2.0\log_{10}\left(\frac{k/D}{3.7} + \frac{2.51}{Re_D f^{1/2}}\right) \tag{9.32}$$

    - Representative roughness values presented in Table 9.2
    - This function appears in the classic Moody chart (Fig. 9.3)

9.6 Momentum-Heat Transfer Analogies

- Analogy method is applied to pipe flow to the case of constant heat flux boundary condition
- Though an analogy cannot be made for the case of a constant surface temperature, resulting models approximately hold for this case as well
- **Formulation**
    - For hydrodynamically fully developed flow the *x*-momentum equation (9.4) becomes

$$\frac{1}{\rho}\frac{dp}{dx} = \frac{1}{r}\frac{\partial}{\partial r}\left[r(\nu + \varepsilon_M)\frac{\partial \overline{u}}{\partial r}\right] \tag{9.33a}$$

    - The energy equation reduces to:

$$\overline{u}\frac{\partial \overline{T}}{\partial x} = \frac{1}{r}\frac{\partial}{\partial r}\left[r(\alpha + \varepsilon_H)\frac{\partial \overline{T}}{\partial r}\right] \tag{9.33b}$$

    - Recall that an analogy is possible if the momentum and energy equations are identical, so
        - Note that in pipe flow the pressure gradient is non-zero, although constant with respect to *x*.
        - To ensure an analogy, the left side of (9.33b) must then be constant
        - For thermally fully developed flow and a constant heat flux at the wall, the shape of the temperature profile is constant with respect to *x*, leading to

$$\frac{\partial \bar{T}}{\partial x} = \text{constant}$$

- Analogy appears to be possible; however, boundary conditions must also match
  - Boundary conditions are:

    At $r = 0$:

$$\frac{d\bar{u}(0)}{dr} = 0, \quad \frac{\partial \bar{T}(0)}{\partial r} = 0 \tag{9.34a}$$

At $r = r_0$:

$$\bar{u}(r_o) = 0, \quad \bar{T}(r_o) = T_s(x) \tag{9.34b}$$

$$\mu \frac{d\bar{u}(r_o)}{dr} = \tau_o, \quad k \frac{\partial \bar{T}(r_o)}{\partial r} = q_o'' \tag{9.34c}$$

where q_o'' is assumed to be into the flow (in the negative *r*-direction)

- By normalizing the variables as:

$$U = \frac{\bar{u}}{u_m}, \ \theta = \frac{\bar{T} - T_s}{T_m - T_s}, \ X = \frac{x}{L}, \text{ and } R = \frac{r}{r_o},$$

it can be shown that both governing equations and boundary conditions are identical in form

9.6.1 Reynolds Analogy for Pipe Flow

- Assuming that $\nu = \alpha$ $(Pr = 1)$ and $\varepsilon_H = \varepsilon_M$ $(Pr_t = 1)$, the same assumptions used to develop Reynold's analogy for a flat plate, then the governing equations (9.33a) and (9.33b) are identical
- Following the same procedure as in the original derivation, the Reynolds analogy is essentially identical for pipe flow:

$$St_D \equiv \frac{q_o''}{\rho u_m c_p (T_s - T_m)} = \frac{C_f}{2}, \quad Pr = 1$$

or

$$St_D = \frac{Nu_D}{Re_D Pr} = \frac{C_f}{2} \tag{9.35}$$

- Note that in this case the Stanton number is defined in terms of the mean velocity and bulk temperature, as is the wall shear stress: $\tau_o = \frac{1}{2} C_f \rho u_m^2$

9.6.2 Adapting Flat Plate Analogies to Pipe Flow

- Other flat plate analogies can be adapted to pipe flow, with modifications
- Example: von Kármán analogy
  - For pipe flow, conditions at the edge of the boundary layer are approximated by conditions at the centerline of the pipe:

$$V_\infty \approx u_{CL} \text{ and } T_\infty \approx T_{CL}$$

  - These substitutions affect the friction factor, which translates to:

$$C_f \approx \frac{\tau_o}{\frac{1}{2}\rho u_{CL}^2}$$

  - Following the development exactly as before, the result is almost identical:

$$\frac{q_o''}{\rho \bar{u}_{CL} c_p (T_s - T_{CL})} = \frac{C_f/2}{1 + 5\sqrt{\frac{C_f}{2}}\left\{(Pr-1) + \ln\left[\frac{5Pr+1}{6}\right]\right\}} \qquad (9.36)$$

  - The left side of (9.36) and the friction factor are expressed in terms of centerline variables instead of the more common and convenient mean quantities u_m and T_m
  - This is corrected:

$$\frac{q_o''}{\rho u_m c_p (T_s - T_m)}\left(\frac{u_m (T_s - T_m)}{u_{CL}(T_s - T_{CL})}\right) = \frac{(C_f/2)(u_m/u_{CL})^2}{1 + 5\frac{u_m}{u_{CL}}\sqrt{\frac{C_f}{2}}\left\{(Pr-1) + \ln\left[\frac{5Pr+1}{6}\right]\right\}}$$

    where

    C_f is defined in terms of the mean velocity $C_f = \tau_o / \frac{1}{2}\rho u_m^2$

    The terms $q_o'' / \rho u_m c_p (T_s - T_m)$ collectively are the Stanton number for pipe flow

  - Simplifying yields von Kármán Analogy for pipe flow:

$$\mathrm{St}_D\left(\frac{T_s - T_m}{T_s - T_{CL}}\right) = \frac{(C_f/2)(u_m/u_{CL})}{1 + 5\left(\frac{u_m}{u_{CL}}\right)\sqrt{\frac{C_f}{2}}\left\{(Pr-1) + \ln\left[\frac{5Pr+1}{6}\right]\right\}} \qquad (9.37)$$

- Estimates for the ratios (u_m / u_{CL}) and $(T_s - T_m)/(T_s - T_{CL})$ are found using the definition of mean temperature, equation (9.9):
  - u_m and T_m are estimated using the 1/7th Law profiles, which for a circular pipe are:

$$\frac{\bar{u}}{u_{CL}} = \left(\frac{y}{r_o}\right)^{1/7} \tag{9.20}$$

- Similar to (8.114) for a flat plate:

$$\frac{\bar{T} - T_s}{T_{CL} - T_s} = \left(\frac{y}{r_o}\right)^{1/7} \tag{9.38}$$

- Substituting these models into (9.8) and (9.9), it can be shown that

$$\frac{u_m}{\bar{u}_{CL}} = 0.817 \tag{9.39}$$

$$\frac{T_m - T_s}{T_{CL} - T_s} = 0.833 \tag{9.40}$$

9.6.3 Other Analogy-Based Correlations

- A simple correlation for turbulent flow in a duct is based on the Colburn analogy:
  - Starting equation (8.98), and using equation (9.29) for the friction factor, one finds

$$St_D = 0.023 Re_D^{-1/5} Pr^{-2/3}$$

$$Nu_D = 0.023 Re_D^{4/5} Pr^{1/3} \tag{9.41}$$

- The Dittus-Boelter correlation is an empirical correlation based on the Colburn analogy:

$$Nu_D = 0.023 Re_D^{4/5} Pr^n \tag{9.42}$$

  where $n = 0.4$ for heating ($T_s > T_m$) and $n = 0.3$ for cooling

  - Simplicity and the fact that this analogy compares well with experimental data make it popular
  - In recent years its accuracy, along with that of the Colburn analogy, have been challenged
  - Models by Petukhov and the Gnielinski correlation (see Section 9.8) are preferred for their improved accuracy and range of applicability
- Analogies remain a common way to model heat transfer in pipes; there are models developed specifically for pipe flows

9.7 Algebraic Method Using Uniform Temperature Profile

- As was done for the flat plate, the universal temperature and velocity profiles can be used to estimate the heat transfer in a circular duct
- **Formulation**
  - The Nusselt number for flow in a duct is defined as

$$Nu_D \equiv \frac{hD}{k} = \frac{q_o'' D}{(T_s - T_m)k} \tag{9.43}$$

- To invoke the universal temperature profile, the definition of T^+, equation 8.105, is used to define the mean temperature as

$$T_m^+ = (T_s - T_m)\frac{\rho c_p u^*}{q_o''} = (T_s - T_m)\frac{\rho c_p u_m \sqrt{C_f / 2}}{q_o''} \tag{9.44}$$

- For duct flow, the friction velocity u^* is defined in terms of the mean velocity, so substituting the above into (9.43) for q_o'' and invoking the definitions of the Reynolds and Prandtl numbers:

$$Nu_D = \frac{Re_D Pr \sqrt{C_f / 2}}{T_m^+} \tag{9.45}$$

- There are several ways to proceed from here:
  - One approach: evaluate T_m^+ using a dimensionless version of (9.9):

$$T_m^+ = \frac{2}{u_m^+ r_o^{+2}} \int_0^{r_o^+} T^+ u^+ (r_o^+ - y^+) dy^+ \tag{9.46}$$

    - Substituting appropriate universal temperature and velocity profiles into (9.46) and integrating must be done numerically
  - A second approach yields a simpler closed-form solution
    - Rewrite the original Nusselt number relation (9.43) as follows:

$$Nu_D = \frac{q_o'' D}{(T_s - T_m)k} \frac{(T_s - T_{CL})}{(T_s - T_{CL})}$$

      where T_{CL} is the centerline temperature
    - Substituting the definition of T^+ given by Eqn. (8.108) for the centerline temperature in the denominator:

$$Nu_D = \frac{Re_D Pr \sqrt{C_f / 2}}{T_{CL}^+} \frac{(T_s - T_{CL})}{(T_s - T_m)} \tag{9.47}$$

    - The universal temperature profile, equation (8.118), is used to evaluate T_{CL}^+:

$$T_{CL}^+ = \frac{Pr_t}{\kappa} \ln y_{CL}^+ + 13 Pr^{2/3} - 7 \tag{9.48}$$

- As was done for the flat plate, substitute the Law of the Wall velocity profile (8.59) for $\ln y_{CL}^{+}$:

$$u_{CL}^{+} = \frac{1}{\kappa} \ln y_{CL}^{+} + B \tag{9.49}$$

- Substituting these into the Nusselt number relation:

$$Nu_D = \frac{Re_D Pr \sqrt{C_f / 2}}{\left[Pr_t (u_{CL}^{+} - B) + 13 Pr^{2/3} - 7\right]} \frac{(T_s - T_{CL})}{(T_s - T_m)} \tag{9.50}$$

- Expressions are needed for u_{CL}^{+} and $(T_s - T_{CL})/(T_s - T_m)$:

- For the centerline velocity, use the definition of u^{+} for pipe flow

$$u_{CL}^{+} = \frac{u_{CL}}{u^{*}} = \frac{u_{CL}}{u_m} \sqrt{\frac{2}{C_f}} \tag{9.51}$$

- Mean velocity and temperature are needed, so, the 1/7th power law is used to avoid the complexity of the logarithmic velocity and temperature profiles

- From the previous section, the 1/7th power law yields:

$$\frac{u_m}{\bar{u}_{CL}} = 0.817 \;\text{ and }\; \frac{T_m - T_s}{T_{CL} - T_s} = 0.833$$

- Using the definition of Stanton number, $St_D = Nu_D / (Re_D Pr)$, and selecting $Pr_t = 0.9$ and $B = 5.0$, equation (9.50) is rearranged:

$$St_D = \frac{C_f / 2}{0.92 + 10.8\left(Pr^{2/3} - 0.89\right)\sqrt{C_f / 2}} \tag{9.52}$$

- A first approximation suggests this model be limited to $Re_D < 1 \times 10^5$
- The ultimate test is to compare this model to experimental data

9.8 Other Correlations for Smooth Pipes

- Petukhov evoked Reichardt's model for eddy diffusivity and velocity profile (9.17, 9.18) to obtain:

$$St_D = \frac{C_f / 2}{1.07 + 12.7\left(Pr^{2/3} - 1\right)\sqrt{C_f / 2}}, \begin{pmatrix} 0.5 \le Pr \le 2000 \\ 10^4 < Re_D < 5 \times 10^6 \end{pmatrix} \tag{9.53}$$

- This compares well to experimental data over a wide range of Prandtl and Reynolds numbers
- The following model was used for the friction factor:

$$\frac{C_f}{2} = (2.236 \ln Re_D - 4.639)^{-2} \quad (9.54)$$

- Note the similarity between Petukhov's relation (9.53) and the algebraic result, equation (9.52)
- In 1976, Gnielinski modified Petukhov's model slightly, extending the model to include lower Reynolds numbers:

$$Nu_D = \frac{(Re_D - 1000) Pr C_f / 2}{1 + 12.7\left(Pr^{2/3} - 1\right)\sqrt{C_f / 2}}, \begin{pmatrix} 0.5 \le Pr \le 2000 \\ 3 \times 10^3 < Re_D < 5 \times 10^6 \end{pmatrix} \quad (9.55)$$

- Pethkhov's friction model can be used in (9.55) for the friction factor
- For the above models, properties should be evaluated at the film temperature
- These correlations are reasonable for channels with constant surface temperature as well as constant heat flux; the flows are relatively insensitive to boundary conditions

9.9 Heat Transfer in Rough Pipes

- The effects of roughness on the heat transfer from flat plates was discussed in Section 8.5.6, and much of the same physical intuition applies to flow in channels
- Norris presents the following empirical correlation for flow through circular tubes:

$$\frac{Nu}{Nu_{\text{smooth}}} = \left(\frac{C_f}{C_{f,smooth}}\right)^n, \quad \left(\frac{C_f}{C_{f,smooth}} < 4\right) \quad (9.56)$$

  where $n = 0.68 Pr^{0.215}$

- A correlation like Colebrook's (9.32) could be used to determine the rough-pipe friction factor
- The behavior of this relation reflects what is expected physically:
  - The Prandtl number influences the effect of roughness
  - For very low Prandtl fluids the roughness plays little role in heat transfer
- The influence of roughness size is limited
  - The effect of increasing roughness vanishes beyond $(C_f / C_{f,smooth}) \approx 4$, hence a maximum is reached
- Though roughness enhances heat transfer, it increases friction, which increases pumping costs, though the increase of friction due to roughness also reaches a limiting value
  - The application of roughness to increase heat transfer requires benefits to be weighed against increasing costs