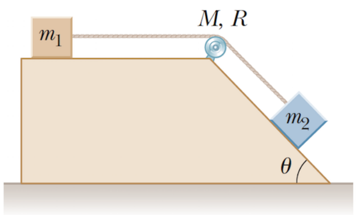
Pulley with and without moment of inertia considered

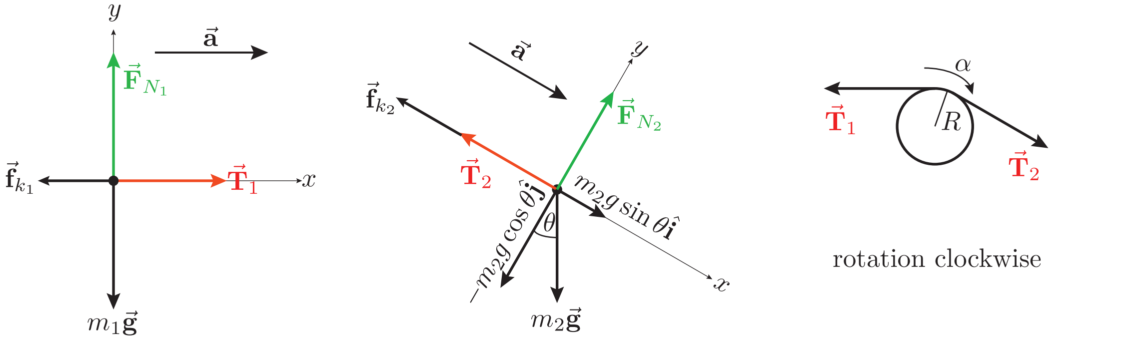
A block of mass $m_1 = 1kg$ and a block of mass $m_2 = 4kg$ are connected by a massless string over a pulley in the shape of a solid disk having radius $R = 0.25m$ and mass $M = 10.0kg$. The fixed, wedge-shaped ramp makes an angle of $\theta = 30.0°$ as shown in the figure. The coefficient of kinetic friction is $\mu_k = 0.25$ for both blocks. (Moment of inertia of solid cylinder $I = \frac{1}{2}MR^2$)



1. Draw the free body diagrams of both blocks and the pulley.

   Solution:

   free body diagram for m_1 free body diagram for m_2 free body diagram for the pulley



2. Determine the linear acceleration of the two blocks.

   Solution:

$$\sum \vec{\mathbf{F}}_1 = (T_1 - f_{k_1})\,\hat{\mathbf{i}} + (F_{N_1} - m_1 g)\,\hat{\mathbf{j}} = m_1 a\hat{\mathbf{i}} \quad (1)$$
$$\sum \vec{\mathbf{F}}_2 = (-f_{k_2} - T_2 + m_2 g\sin\theta)\,\hat{\mathbf{i}} + (F_{N_2} - m_2 g\cos\theta)\,\hat{\mathbf{j}} = m_2 a\hat{\mathbf{i}} \quad (2)$$
$$\sum \tau = T_1 R - T_2 R = -I\alpha \quad (3)$$

From equation (1) we get:

$$T_1 - f_{k_1} = m_1 a \quad (4)$$
$$F_{N_1} - m_1 g = 0 \Longrightarrow F_{N_1} = m_1 g \Longrightarrow f_{k_1} = \mu_k m_1 g \quad (5)$$

From equation (2) we get:

$$-f_{k_2} - T_2 + m_2 g\sin\theta = m_2 a \quad (6)$$
$$F_{N_2} - m_2 g\cos\theta = 0 \Longrightarrow F_{N_2} = m_2 g\cos\theta \Longrightarrow f_{k_2} = \mu_k m_2 g\cos\theta \quad (7)$$

With $\alpha = a/R$ we get from equation (3)

$$T_1R - T_2R = -I\frac{a}{R} \tag{8}$$

So (4),(6), (8)

$$\begin{aligned} T_1 - \mu_k m_1 g &= m_1 a & (9) \\ -\mu_k m_2 g\cos\theta - T_2 + m_2 g\sin\theta &= m_2 a & (10) \\ T_1R - T_2R = -I\frac{a}{R} & & (11) \end{aligned}$$

have to be solved for a, T_1, T_2.

From (11) we get

$$T_1 = T_2 - I\frac{a}{R^2} \tag{12}$$

Substitution of (12) in (14) gives

$$T_2 - I\frac{a}{R^2} - \mu_k m_1 g = m_1 a \tag{13}$$

(10)+(13) gives

$$-\mu_k m_1 g - \mu_k m_2 g\cos\theta + m_2 g\sin\theta - I\frac{a}{R^2} = m_1 a + m_2 a \tag{14}$$

Solving (14) for a gives

$$\begin{aligned} a &= \frac{-\mu_k(m_1 + m_2\cos\theta) + m_2\sin\theta}{m_1 + m_2 + \frac{I}{R^2}} g = \frac{-\mu_k(m_1 + m_2\cos\theta) + m_2\sin\theta}{m_1 + m_2 + \frac{\frac{1}{2}MR^2}{R^2}} g = \\ &= \frac{-0.25(1kg + 4kg\cos 30^\circ) + 4kg\sin 30^\circ}{1kg + 4kg + 5kg} \times 9.8\frac{m}{s^2} = 0.87\frac{m}{s^2} \end{aligned} \tag{15}$$

3. Determine the tensions in the string on both sides of the pulley.

   From (1) we get:

$$T_1 = m_1(a + \mu_k g) = 1kg\left(0.87\frac{m}{s^2} + 0.25 \times 9.8\frac{m}{s^2}\right) = 3.32N \tag{16}$$

   From (11) we get:

$$T_2 = T_1 + \frac{Ia}{R^2} = T_1 + \frac{\frac{1}{2}MR^2a}{R^2} = T_1 + \frac{1}{2}Ma = 3.32N + \frac{1}{2}10kg\,0.87\frac{m}{s^2} = 7.67N \tag{17}$$

If the pulley would be frictionless, i.e. the string slides over the pulley without turning the pulley, then the results can be obtained by setting $I = 0$ and become:

$$\begin{aligned} a &= \frac{-\mu_k(m_1 + m_2\cos\theta) + m_2\sin\theta}{m_1 + m_2} g \\ &= \frac{-0.25(1kg + 4kg\cos 30^\circ) + 4kg\sin 30^\circ}{1kg + 4kg} \times 9.8\frac{m}{s^2} = 1.73\frac{m}{s^2} \\ T &= T_1 = T_2 = \mu_k m_1 g + m_1 a = 0.25 \cdot 1kg \cdot 9.8\frac{m}{s^2} + 1kg \cdot 1.73\frac{m}{s^2} = 4.18N \end{aligned}$$

