Questions:

1. The position vector $\vec{r}(t)$ of a particle moving in the $xy$-plane is given by $\vec{r}(t) = (-t^2 + 4t) \hat{i} + (3t - 6) \hat{j}$ with $\vec{r}(t)$ in meters and $t$ in seconds.

(a) Calculate the position vectors at $t = 0$ and $t = 2s$. (2 P)

Solution:

$\vec{r}(0) = (-6\hat{j})m$, $\vec{r}(2) = ((-2^2 + 4 \cdot 2)\hat{i} + (3 \cdot 2 - 6)\hat{j})m = (4\hat{i})m$

(b) Calculate the displacement and average velocity of the particle between $t = 0$ and $t = 2s$. (2 P)

Solution:

$\Delta \vec{r} = \vec{r}(2s) - \vec{r}(0) = (4\hat{i})m - (-6\hat{j})m = (4\hat{i} + 6\hat{j})m$

$\vec{v}_{avg} = \frac{\Delta \vec{r}}{\Delta t} = \frac{(4\hat{i} + 6\hat{j}) m}{2s - 0} = (2\hat{i} + 3\hat{j}) \frac{m}{s}$

(c) Calculate the velocity and the acceleration of the particle at any time $t$. (2 P)

Solution:

$\vec{v}(t) = ((-2t + 4)\hat{i} + 3\hat{j}) \frac{m}{s}$

$\vec{a}(t) = (-2\hat{i}) \frac{m}{s^2}$

(d) Calculate the speed (magnitude of the velocity) at $t = 4s$. (2 P)

Solution:

$\vec{v}(4s) = ((-2 \cdot 4 + 4)\hat{i} + 3\hat{j}) \frac{m}{s} = (-4\hat{i} + 3\hat{j}) \frac{m}{s}$

$|\vec{v}(4s)| = \sqrt{(-4 \frac{m}{s})^2 + (3 \frac{m}{s})^2} = 5 \frac{m}{s}$
2. A man throws a stone into a small dish as shown in the figure. The dish is on a shelf at a height \( h \) and at a horizontal distance of \( d = 2.1 \text{ m} \) from the point the stone is thrown. If the stone is thrown at a speed \( v_0 = 6.4 \text{ m/s} \) at an angle \( \theta = 60^\circ \) above the horizontal, the stone lands on the dish. (Ignore air resistance)

(a) What is the time of flight of the stone? (3 P)

(b) What is the height \( h \)? (3 P)

(c) What is the velocity of the stone just before it lands on the dish? (2 P)

(d) Did the stone pass its maximum height, when it lands on the dish? Explain your answer shortly. (2 P)

Solution:

Given: \( \vec{r}_0 = 0, \vec{v}_0 = v_0 \cos \theta \hat{i} + v_0 \sin \theta \hat{j}, \vec{a} = -g \hat{j}, \vec{r}_L = d \hat{i} + h \hat{j} \).

(a) The stone should land on the dish means \( \vec{r}(T) = \vec{r}_L \). So we get:

\[
\vec{r}(T) = \vec{r}_0 + \vec{v}_0 T + \frac{1}{2} \vec{a} T^2 = (v_0 \cos \theta \hat{i} + v_0 \sin \theta \hat{j}) T - \frac{1}{2} g T^2 \hat{j} = d \hat{i} + h \hat{j}.
\]

Decomposing this equation into its components gives

\[
\begin{align*}
v_0 \cos \theta T &= d \quad \text{(1)} \\
v_0 \sin \theta T - \frac{1}{2} g T^2 &= h \quad \text{(2)}
\end{align*}
\]

Solving (1) for \( T \) gives

\[
T = \frac{d}{v_0 \cos \theta} = \frac{2.1 \text{ m}}{6.4 \text{ m/s} \cos 60^\circ} = \frac{21}{32} \text{ s} = 0.65625 \text{ s}
\]

(b) Substitution of \( T = 0.65625 \text{ s} \) in (2) gives the height of the shelf

\[
h = v_0 \sin \theta T - \frac{1}{2} g T^2 = 6.4 \text{ m/s} \sin 60^\circ \frac{21}{32} \text{ s} - \frac{1}{2} \left( 9.8 \text{ m/s}^2 \right) \left( \frac{21}{32} \text{ s} \right)^2 = 1.53 \text{ m}.
\]

(c) The velocity of the stone is given as:

\[
\vec{v}(t) = \vec{v}_0 + \vec{a} t = (v_0 \cos \theta \hat{i} + v_0 \sin \theta \hat{j}) - g t \hat{j}.
\]

Then we get for the velocity of the stone, in the instant just before the stone hits the dish at \( T \)

\[
\vec{v}(T) = v_0 (\cos \theta \hat{i} + \sin \theta \hat{j}) - g T \hat{j} = 6.4 \text{ m/s} (\cos 60^\circ \hat{i} + \sin 60^\circ \hat{j}) - 9.8 \text{ m/s}^2 \frac{21}{32} \hat{j} = (3.2 \hat{i} - 0.89 \hat{j}) \frac{m}{s}
\]

(d) The stone passed its maximum height, as the \( y \)-component of the velocity is negative, i.e. directed downward.
3. The seat of a swing is suspended from a cable of length $\ell = 5.00 \text{m}$ rotating horizontally about the central axis. (Canonical pendulum). The seat moves uniformly on a circle, where the cable makes an angle $\theta = 30^\circ$ with the vertical rotation axis. Assume that the mass of the seat and the person is $m = 60 \text{kg}$.

(a) Draw the free body diagram for the seat. (3 P)

(b) Calculate the speed of the seat. (3 P)

(c) Calculate the tension in the cable. (3 P)

(d) Calculate the magnitude of the centripetal force (force in radial direction). (1 P)

Solution:

(a) The free body diagram for the swing and the person is given as:

(b) From the free body diagram above we get by application of Newton’s laws.

$$\sum \vec{F} = (-T \sin \theta) \hat{i} + (T \cos \theta - mg) \hat{j} = -\frac{mv^2}{r} \hat{i}$$

Decomposing the equation in component form gives

$$-T \sin \theta = -\frac{mv^2}{r} \quad (3)$$
$$T \cos \theta = mg \quad (4)$$

Division of (3) by (4) gives

$$\tan \theta = \frac{v^2}{rg} \implies v = \sqrt{rg \tan \theta} = \sqrt{5 \text{m} \sin 30^\circ \cdot 9.8 \frac{m}{s^2} \tan 30^\circ} = 3.76 \frac{m}{s}$$

(c) From (4) we get for $T$:

$$T = \frac{mg}{\cos \theta} = \frac{60 \text{kg} \cdot 9.8 \frac{m}{s^2}}{\cos 30^\circ} = 679 \text{N}$$

(d) We can calculate the magnitude of the centripetal force in two ways:

Either

$$F_c = \frac{mv^2}{R} = \frac{mRg \tan \theta}{R} = mg \tan \theta = 60 \text{kg} \cdot 9.8 \frac{m}{s^2} \tan 30^\circ = 339 \text{N}$$

or

$$F_c = T \sin \theta = 679 \text{N} \sin 30^\circ = 339 \text{N}.$$
4. Two blocks with mass \( m_A = 5 \text{ kg} \) and \( m_B = 2 \text{ kg} \) are connected by a light string over a frictionless pulley of negligible mass. The block with the mass \( m_A \) is on an incline at an angle \( \theta = 37^\circ \), whereas the block with mass \( m_B \) is on a horizontal plane. The coefficient of kinetic friction \( \mu_k \) is the same for the horizontal and the inclined plane. Both blocks slide with constant speed.

(a) Draw the free body diagrams for both blocks. (5 P)

(b) Calculate the coefficient of kinetic friction between the blocks and the surfaces. (4 P)

(c) Calculate the tension in the cord. (3 P)

**Solution:**

(a) Free body diagrams for the masses \( m_A \) and \( m_B \) are:

- **Free body Diagram for \( m_A \)**
  - \( T \)
  - \( m_A g \sin \theta \hat{i} \)
  - \( -m_A g \cos \theta \hat{j} \)

- **Free body Diagram for \( m_B \)**
  - \( T \)
  - \( m_B g \)

(b) The application of Newton’s Laws gives:

\[
\begin{align*}
\sum \vec{F}_A &= (m_A g \sin \theta - T - f_{kA}) \hat{i} + (F_{N_A} - m_A g \cos \theta) \hat{j} = 0 \quad (5) \\
\sum \vec{F}_B &= (T - f_{kB}) \hat{i} + (F_{N_B} - m_B g) \hat{j} = 0 \quad (6)
\end{align*}
\]

From (5) we get:

\[
m_A g \sin \theta - T - \mu_k F_{N_A} = 0 \quad (7)
\]

\[
F_{N_A} = m_A g \cos \theta \quad (8)
\]

Substituting (8) in (7) we get:

\[
m_A g \sin \theta - T - \mu_k m_A g \cos \theta = 0 \quad (9)
\]

From (6) we get

\[
T - \mu_k F_{N_B} = 0 \quad (10)
\]

\[
F_{N_B} = m_B g \quad (11)
\]

Substituting (11) in (10) we get:

\[
T - \mu_k m_B g = 0 \quad (12)
\]

(12) + (9) gives:

\[
m_A g \sin \theta - \mu_k (m_A g \cos \theta + m_B g) = 0 \implies \mu_k = \frac{m_A \sin \theta}{m_A \cos \theta + m_B} = \frac{5 \text{ kg} \sin 37^\circ}{5 \text{ kg} \cos 37^\circ + 2 \text{ kg}} = \frac{1}{2}
\]

(c) From (12) we get

\[
T = \mu_k m_B g = \frac{1}{2} \cdot 2 \text{ kg} \cdot 9.8 \text{ m/s}^2 = 9.8 \text{ N}
\]