Chapter 5

VELOCITY KINEMATICS – THE MANIPULATOR JACOBIAN

In the previous chapters we derived the forward and inverse position equations relating joint positions and end-effector positions and orientations. In this chapter we derive the velocity relationships, relating the linear and angular velocities of the end-effector (or any other point on the manipulator) to the joint velocities. The end-effector frame contains information concerning both the orientation of the frame and the position of the origin of the frame; in this chapter we will derive representations for the velocities associated with each of these quantities. In particular, we will derive the angular velocity of the end-effector frame (which gives the rate of rotation of the frame) and the linear velocity of the origin. We will then relate these velocities to the joint velocities, $\dot{q}_i$.

Mathematically, the forward kinematic equations define a function between the space of cartesian positions and orientations and the space of joint positions. The velocity relationships are then determined by the **Jacobian** of this function. The Jacobian is a matrix-valued function and can be thought of as the vector version of the ordinary derivative of a scalar function. This Jacobian or Jacobian matrix is one of the most important quantities in the analysis and control of robot motion. It arises in virtually every aspect of robotic manipulation: in the planning and execution of smooth trajectories, in the determination of singular configurations, in the execution of coordinated anthropomorphic motion, in the derivation of the dynamic equations of motion, and in the transformation of forces and torques from the end-effector to the manipulator joints.

Since the Jacobian matrix encodes relationships between velocities, we begin this chapter with an investigation of velocities, and how to represent them. We first consider angular velocity about a fixed axis, and then generalize this with the aid of skew symmetric matrices. Equipped with this general representation of angular velocities, we are able to derive equations for both the angular velocity, and the linear velocity for the origin, of a moving frame.

We then proceed to the derivation of the manipulator Jacobian. For an n-link manipulator we first derive the Jacobian representing the instantaneous transformation between the n-vector of joint velocities and the 6-vector consisting of the linear and angular velocities of the end-effector. This Jacobian is then a $6 \times n$ matrix. The same approach is used to determine the transformation between the joint velocities and the linear and angular velocity of any point on the manipulator. This will be important when we discuss the derivation of the dynamic equations of motion in Chapter 6. We then discuss the notion of **singular configurations**. These are configurations in which the manipulator loses one or more degrees-of-freedom. We show how the singular configurations are determined geometrically and give several examples. Following this, we briefly discuss the inverse problems of determining joint velocities and accelerations for specified end-effector velocities and accelerations. We end the chapter by considering redundant manipulators. This includes discussions of the inverse velocity problem, singular value decomposition and manipulability.

5.1 Angular Velocity: The Fixed Axis Case

When a rigid body moves in a pure rotation about a fixed axis, every point of the body moves in a circle. The centers of these circles lie on the axis of rotation. As the body rotates, a perpendicular from any point of the body to the axis sweeps out an angle θ, and this angle is the same for every point of the body. If $\boldsymbol{k}$ is a unit vector in the direction of the axis of rotation, then the angular velocity is given by

$$\omega = \dot{\theta}\boldsymbol{k} \tag{5.1}$$

in which $\dot{\theta}$ is the time derivative of θ.

Given the angular velocity of the body, one learns in introductory dynamics courses that the linear velocity of any point on the body is given by the equation

$$\boldsymbol{v} = \omega \times \boldsymbol{r} \tag{5.2}$$

in which $\boldsymbol{r}$ is a vector from the origin (which in this case is assumed to lie on the axis of rotation) to the point. In fact, the computation of this velocity $\boldsymbol{v}$ is normally the goal in introductory dynamics courses, and therefore, the main role of an angular velocity is to induce linear velocities of points in a rigid body. In our applications, we are interested in describing the motion of a moving frame, including the motion of the origin of the frame through space and also the rotational motion of the frame's axes. Therefore, for our purposes, the angular velocity will hold equal status with linear velocity.

As in previous chapters, in order to specify the orientation of a rigid object, we rigidly attach a coordinate frame to the object, and then specify the orientation of the coordinate frame. Since every point on the object experiences the same angular velocity (each point sweeps out the same angle θ in a given time interval), and since each point of the body is in a fixed geometric relationship to the body-attached frame, we see that the angular velocity is a property of the attached coordinate frame itself. Angular velocity is not a property of individual points. Individual points may experience a *linear velocity* that is induced by an angular velocity, but it makes no sense to speak of a point itself rotating. Thus, in equation (5.2) $\boldsymbol{v}$ corresponds to the linear velocity of a point, while ω corresponds to the angular velocity associated with a rotating coordinate frame.

In this fixed axis case, the problem of specifying angular displacements is really a planar problem, since each point traces out a circle, and since every circle lies in a plane. Therefore, it is tempting to use $\dot{\theta}$ to represent the angular velocity. However, as we have already seen in Chapter 2, this choice does not generalize to the three-dimensional case, either when the axis of rotation is not fixed, or when the angular velocity is the result of multiple rotations about distinct axes. For this reason, we will develop a more general representation for angular velocities. This is analogous to our development of rotation matrices in Chapter 2 to represent orientation in three dimensions. The key tool that we will need to develop this representation is the skew symmetric matrix, which is the topic of the next section.

5.2 Skew Symmetric Matrices

In the Section 5.3 we will derive properties of rotation matrices that can be used to computing relative velocity transformations between coordinate frames. Such transformations involve computing derivatives of rotation matrices. By introducing the notion of a skew symmetric matrix it is possible to simplify many of the computations involved.

Definition 2 *A matrix S is said to be* **skew symmetric** *if and only if*

$$S^T + S = 0. \tag{5.3}$$

We denote the set of all 3×3 skew symmetric matrices by $SS(3)$. If $S \in SS(3)$ has components s_{ij}, $i, j = 1, 2, 3$ then (5.3) is equivalent to the nine equations

$$s_{ij} + s_{ji} = 0 \qquad i, j = 1, 2, 3. \tag{5.4}$$

From (5.4) we see that $s_{ii} = 0$; that is, the diagonal terms of S are zero and the off diagonal terms s_{ij}, $i \neq j$ satisfy $s_{ij} = -s_{ji}$. Thus S contains only three independent entries and every 3×3 skew symmetric matrix has the form

$$S = \begin{bmatrix} 0 & -s_3 & s_2 \\ s_3 & 0 & -s_1 \\ -s_2 & s_1 & 0 \end{bmatrix}. \tag{5.5}$$

If $\boldsymbol{a} = (a_x, a_y, a_z)^T$ is a 3-vector, we define the skew symmetric matrix $S(\boldsymbol{a})$ as

$$S(\boldsymbol{a}) = \begin{bmatrix} 0 & -a_z & a_y \\ a_z & 0 & -a_x \\ -a_y & a_x & 0 \end{bmatrix}. \tag{5.6}$$

Example 5.1 *We denote by $\boldsymbol{i}$, $\boldsymbol{j}$ and $\boldsymbol{k}$ the three unit basis coordinate vectors,*

$$\begin{aligned} \boldsymbol{i} &= (1,0,0)^T \\ \boldsymbol{j} &= (0,1,0)^T \\ \boldsymbol{k} &= (0,0,1)^T. \end{aligned}$$

The skew symmetric matrices $S(\boldsymbol{i})$, $S(\boldsymbol{j})$, and $S(\boldsymbol{k})$ are given by

$$S(\boldsymbol{i}) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}; \quad S(\boldsymbol{j}) = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}; \quad S(\boldsymbol{k}) = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{5.7}$$

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An important property possessed by the matrix $S(\boldsymbol{a})$ is linearity. For any vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ belonging to $\mathbf{R}^3$ and scalars α and β we have

$$S(\alpha\boldsymbol{a} + \beta\boldsymbol{b}) = \alpha S(\boldsymbol{a}) + \beta S(\boldsymbol{b}). \tag{5.8}$$

Another important property of $S(\boldsymbol{a})$ is that for any vector $\boldsymbol{p} = (p_x, p_y, p_z)^T$

$$S(\boldsymbol{a})\boldsymbol{p} = \boldsymbol{a} \times \boldsymbol{p} \tag{5.9}$$

where $\boldsymbol{a} \times \boldsymbol{p}$ denotes the vector cross product defined in Appendix A. Equation (5.9) can be verified by direct calculation.

If $R \in SO(3)$ and $\boldsymbol{a}, \boldsymbol{b}$ are vectors in $\mathbf{R}^3$ it can also be shown by direct calculation that

$$R(\boldsymbol{a} \times \boldsymbol{b}) = R\boldsymbol{a} \times R\boldsymbol{b}. \tag{5.10}$$

Equation (5.10) is **not** true in general unless R is orthogonal. Equation (5.10) says that if we first rotate the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ using the rotation transformation R and then form the cross product of the rotated vectors $R\boldsymbol{a}$ and $R\boldsymbol{b}$, the result is the same as that obtained by first forming the cross product $\boldsymbol{a} \times \boldsymbol{b}$ and then rotating to obtain $R(\boldsymbol{a} \times \boldsymbol{b})$.

For any $R \in SO(3)$ and any $\boldsymbol{b} \in \mathbf{R}^3$, it follows from (5.9) and (5.10) that

$$RS(\boldsymbol{a})R^T\boldsymbol{b} = R(\boldsymbol{a} \times R^T\boldsymbol{b}) \tag{5.11}$$

$$= (R\boldsymbol{a}) \times (RR^T\boldsymbol{b}) \tag{5.12}$$

$$= (R\boldsymbol{a}) \times \boldsymbol{b} \tag{5.13}$$

$$= S(R\boldsymbol{a})\boldsymbol{b}. \tag{5.14}$$

Here, (5.12) follows because of (5.10), and (5.13) follows since R is orthogonal. Since this equality holds for all $\boldsymbol{b} \in \mathbf{R}^3$, we have shown the useful fact that

$$RS(\boldsymbol{a})R^T = S(R\boldsymbol{a}) \tag{5.15}$$

for $R \in SO(3)$, $\boldsymbol{a} \in \mathbf{R}^3$. As we will see, (5.15) is one of the most useful expressions that we will derive. The left hand side of Equation (5.15) represents a similarity transformation of the matrix $S(\boldsymbol{a})$. The equation says therefore that the matrix representation of $S(\boldsymbol{a})$ in a coordinate frame rotated by R is the same as the skew symmetric matrix $S(R\boldsymbol{a})$ corresponding to the vector $\boldsymbol{a}$ rotated by R.

Suppose now that a rotation matrix R is a function of the single variable θ. Hence $R = R(\theta) \in SO(3)$ for every θ. Since R is orthogonal for all θ it follows that

$$R(\theta)R(\theta)^T = I. \tag{5.16}$$

Differentiating both sides of (5.16) with respect to θ using the product rule gives

$$\frac{dR}{d\theta}R(\theta)^T + R(\theta)\frac{dR^T}{d\theta} = 0. \tag{5.17}$$

Let us define the matrix

$$S := \frac{dR}{d\theta}R(\theta)^T. \tag{5.18}$$

Then the transpose of S is

$$S^T = \left(\frac{dR}{d\theta}R(\theta)^T\right)^T = R(\theta)\frac{dR^T}{d\theta}. \tag{5.19}$$

Equation (5.17) says therefore that

$$S + S^T = 0. \tag{5.20}$$

In other words, the matrix S defined by (5.18) is skew symmetric. Multiplying both sides of (5.18) on the right by R and using the fact that $R^TR = I$ yields

$$\frac{dR}{d\theta} = SR(\theta). \tag{5.21}$$

Equation (5.21) is very important. It says that computing the derivative of the rotation matrix R is equivalent to a matrix multiplication by a skew symmetric matrix S. The most commonly encountered situation is the case where R is a basic rotation matrix or a product of basic rotation matrices.

Example 5.2 *If $R = R_{x,\theta}$, the basic rotation matrix given by (2.19), then direct computation shows that*

$$S = \frac{dR}{d\theta}R^T = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -\sin\theta & -\cos\theta \\ 0 & \cos\theta & -\sin\theta \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix} \tag{5.22}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} = S(\boldsymbol{i}). \tag{5.23}$$

Thus we have shown that

$$\frac{dR_{x,\theta}}{d\theta} = S(\boldsymbol{i})R_{x,\theta}. \tag{5.24}$$

Similar computations show that

$$\frac{dR_{y,\theta}}{d\theta} = S(\boldsymbol{j})R_{y,\theta}; \qquad \frac{dR_{z,\theta}}{d\theta} = S(\boldsymbol{k})R_{z,\theta}. \tag{5.25}$$

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Example 5.3 *Let $R_{k,\theta}$ be a rotation about the axis defined by $\boldsymbol{k}$ as in (2.71). Note that in this example $\boldsymbol{k}$ is not the unit coordinate vector $(0,0,1)^T$. It is easy to check that $S(\boldsymbol{k})^3 = -S(\boldsymbol{k})$. Using this fact together with Problem 12 it follows that*

$$\frac{dR_{k,\theta}}{d\theta} = S(\boldsymbol{k})R_{k,\theta}. \tag{5.26}$$

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5.3 Angular Velocity: The General Case

We now consider the general case of angular velocity about an arbitrary, possibly moving, axis. Suppose that a rotation matrix R is time varying, so that $R = R(t) \in SO(3)$ for every $t \in \mathbf{R}$. An argument identical to the one in the previous section shows that the time derivative $\dot{R}(t)$ of $R(t)$ is given by

$$\dot{R}(t) = S(t)R(t) \tag{5.27}$$

where the matrix $S(t)$ is skew symmetric. Now, since $S(t)$ is skew symmetric, it can be represented as $S(\boldsymbol{\omega}(t))$ for a unique vector $\boldsymbol{\omega}(t)$. This vector $\boldsymbol{\omega}(t)$ is the **angular velocity** of the rotating frame with respect to the fixed frame at time t. Thus, the time derivative $\dot{R}(t)$ is given by

$$\dot{R}(t) = S(\boldsymbol{\omega}(t))R(t) \tag{5.28}$$

in which $\boldsymbol{\omega}(t)$ is the angular velocity.

Example 5.4 *Suppose that $R(t) = R_{x,\theta(t)}$. Then $\dot{R}(t) = \frac{dR}{dt}$ is computed using the chain rule as*

$$\dot{R} = \frac{dR}{d\theta}\frac{d\theta}{dt} = \dot{\theta}S(\boldsymbol{i})R(t) = S(\boldsymbol{\omega}(t))R(t) \tag{5.29}$$

where $\boldsymbol{\omega} = \boldsymbol{i}\dot{\theta}$ is the **angular velocity**. *Note, here $\boldsymbol{i} = (1,0,0)^T$.*

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5.4 Addition of Angular Velocities

We are often interested in finding the resultant angular velocity due to the relative rotation of several coordinate frames. We now derive the expressions for the composition of angular velocities of two moving frames $o_1x_1y_1z_1$ and $o_2x_2y_2z_2$ relative to the fixed frame $o_0x_0y_0z_0$. For now, we assume that the three frames share a common origin. Let the relative orientations of the frames $o_1x_1y_1z_1$ and $o_2x_2y_2z_2$ be given by the rotation matrices $R_1^0(t)$ and $R_2^1(t)$ (both time varying). As in Chapter 2,

$$R_2^0(t) \quad = \quad R_1^0(t)R_2^1(t). \tag{5.30}$$

Taking derivatives of both sides of (5.30) with respect to time yields

$$\dot{R}_2^0 \quad = \quad \dot{R}_1^0R_2^1 + R_1^0\dot{R}_2^1. \tag{5.31}$$

Using (5.28), the term $\dot{R}_2^0$ on the left-hand side of (5.31) can be written

$$\dot{R}_2^0 \quad = \quad S(\omega_2^0)R_2^0. \tag{5.32}$$

In this expression, ω_2^0 denotes the total angular velocity experienced by frame $o_2x_2y_2z_2$. This angular velocity results from the combined rotations expressed by R_1^0 and R_2^1.

The first term on the right-hand side of (5.31) is simply

$$\dot{R}_1^0R_2^1 \quad = \quad S(\omega_a^0)R_1^0R_2^1 = S(\omega_a^0)R_2^0. \tag{5.33}$$

Note that in this equation, ω_a^0 denotes the angular velocity of frame $o_1x_1y_1z_1$ that results from the changing R_1^0, and this angular velocity vector is expressed relative to the coordinate system $o_0x_0y_0z_0$.

Let us examine the second term on the right hand side of (5.31). Using the expression (5.15) we have

$$\begin{aligned} R_1^0\dot{R}_2^1 \quad &= \quad R_1^0S(\omega_b^1)R_2^1 &\quad (5.34)\\ &= \quad R_1^0S(\omega_b^1){R_1^0}^TR_1^0R_2^1 = S(R_1^0\omega_b^1)R_1^0R_2^1 \\ &= \quad S(R_1^0\omega_b^1)R_2^0. &\quad (5.35)\end{aligned}$$

Note that in this equation, ω_b^1 denotes the angular velocity of frame $o_2x_2y_2z_2$ that corresponds to the changing R_2^1, expressed relative to the coordinate system $o_1x_1y_1z_1$. Thus, the product $R_1^0\omega_b^1$ expresses this angular velocity relative to the coordinate system $o_0x_0y_0z_0$.

Now, combining the above expressions we have shown that

$$S(\omega_2^0)R_2^0 = \{S(\omega_a^0) + S(R_1^0\omega_b^1)\}R_2^0. \quad (5.36)$$

Since $S(\boldsymbol{a}) + S(\boldsymbol{b}) = S(\boldsymbol{a} + \boldsymbol{b})$, we see that

$$\omega_2^0 = \omega_a^0 + R_1^0\omega_b^1. \quad (5.37)$$

In other words, the angular velocities can be added once they are expressed relative to the same coordinate frame, in this case $o_0x_0y_0z_0$.

The above reasoning can be extended to any number of coordinate systems. In particular, suppose that we are given

$$R_n^0 = R_1^0R_2^1 \cdots R_n^{n-1}. \quad (5.38)$$

Although it is a slight abuse of notation, let us represent by ω_i^{i-1} the angular velocity due to the rotation given by R_i^{i-1}, expressed relative to frame $o_{i-1}x_{i-1}y_{i-1}z_{i-1}$. Extending the above reasoning we obtain

$$\dot{R}_n^0 = S(\omega_n^0)R_n^0 \quad (5.39)$$

where

$$\omega_n^0 = \omega_1^0 + R_1^0\omega_2^1 + R_2^0\omega_3^2 + R_3^0\omega_4^3 + \cdots + R_{n-1}^0\omega_n^{n-1}. \quad (5.40)$$

5.5 Linear Velocity of a Point Attached to a Moving Frame

We now consider the linear velocity of a point that is rigidly attached to a moving frame. Suppose the point p is rigidly attached to the frame $o_1x_1y_1z_1$, and that $o_1x_1y_1z_1$ is rotating relative to the frame $o_0x_0y_0z_0$. Then the coordinates of p with respect to the frame $o_0x_0y_0z_0$ are given by

$$p^0 = R_1^0(t)p^1. \quad (5.41)$$

The velocity $\dot{p}^0$ is then given as

$$\begin{aligned} \dot{p}^0 &= \dot{R}_1^0(t)p^1 + R_1^0(t)\dot{p}^1 & (5.42)\\ &= S(\omega^0)R_1^0(t)p^1 & (5.43)\\ &= S(\omega^0)p^0 = \omega^0 \times p^0 & \end{aligned}$$

which is the familiar expression for the velocity in terms of the vector cross product. Note that (5.43) follows from that fact that p is rigidly attached to frame $o_1x_1y_1z_1$, and therefore its coordinates relative to frame $o_1x_1y_1z_1$ do not change, giving $\dot{p}^1 = 0$.

Now suppose that the motion of the frame $o_1x_1y_1z_1$ relative to $o_0x_0y_0z_0$ is more general. Suppose that the homogeneous transformation relating the two frames is time-dependent, so that

$$H_1^0(t) = \begin{bmatrix} R_1^0(t) & O_1^0(t) \\ \mathbf{0} & 1 \end{bmatrix}. \tag{5.44}$$

For simplicity we omit the argument t and the subscripts and superscripts on R_1^0 and O_1^0, and write

$$p^0 = Rp^1 + O. \tag{5.45}$$

Differentiating the above expression using the product rule gives

$$\begin{aligned} \dot{p}^0 &= \dot{R}p^1 + \dot{O} \\ &= S(\boldsymbol{\omega})Rp^1 + \dot{O} \\ &= \boldsymbol{\omega} \times \boldsymbol{r} + \boldsymbol{v} \end{aligned} \tag{5.46}$$

where $\boldsymbol{r} = Rp^1$ is the vector from O_1 to p expressed in the orientation of the frame $o_0x_0y_0z_0$, and $\boldsymbol{v}$ is the rate at which the origin O_1 is moving.

If the point p is moving relative to the frame $o_1x_1y_1z_1$, then we must add to the term $\boldsymbol{v}$ the term $R(t)\dot{p}^1$, which is the rate of change of the coordinates p^1 expressed in the frame $o_0x_0y_0z_0$.

5.6 Derivation of the Jacobian

Consider an n-link manipulator with joint variables $q_1, \ldots, q_n$. Let

$$T_n^0(\boldsymbol{q}) = \begin{bmatrix} R_n^0(\boldsymbol{q}) & O_n^0(\boldsymbol{q}) \\ \mathbf{0} & 1 \end{bmatrix} \tag{5.47}$$

denote the transformation from the end-effector frame to the base frame, where $\boldsymbol{q} = (q_1, \ldots, q_n)^T$ is the vector of joint variables. As the robot moves about, both the joint variables q_i and the end-effector position O_n^0 and orientation R_n^0 will be functions of time. The objective of this section is to relate the linear and angular velocity of the end-effector to the vector of joint velocities $\dot{\boldsymbol{q}}(t)$. Let

$$S(\omega_n^0) = \dot{R}_n^0(R_n^0)^T \tag{5.48}$$

define the angular velocity vector ω_n^0 of the end-effector, and let

$$v_n^0 = \dot{O}_n^0 \tag{5.49}$$

denote the linear velocity of the end effector. We seek expressions of the form

$$v_n^0 = J_{\boldsymbol{v}}\dot{\boldsymbol{q}} \tag{5.50}$$
$$\omega_n^0 = J_\omega\dot{\boldsymbol{q}} \tag{5.51}$$

where $J_{\boldsymbol{v}}$ and $J_{\boldsymbol{\omega}}$ are $3 \times n$ matrices. We may write (5.50) and (5.51) together as

$$\begin{bmatrix} v_n^0 \\ \omega_n^0 \end{bmatrix} = J_n^0\dot{\boldsymbol{q}} \tag{5.52}$$

where J_n^0 is given by

$$J_n^0 = \left[\frac{J_{\boldsymbol{v}}}{J_\omega}\right]. \tag{5.53}$$

The matrix J_n^0 is called the **Manipulator Jacobian** or **Jacobian** for short. Note that J_n^0 is a $6 \times n$ matrix where n is the number of links. We next derive a simple expression for the Jacobian of any manipulator.

5.6.1 Angular Velocity

Recall from Equation (5.40) that angular velocities can be added vectorially provided that they are expressed relative to a common coordinate frame. Thus we can determine the angular velocity of the end-effector relative to the base by expressing the angular velocity contributed by each joint in the orientation of the base frame and then summing these.

If the i-th joint is revolute, then the i-th joint variable q_i equals θ_i and the axis of rotation is z_{i-1}. Following the convention that we introduced above, let ω_i^{i-1} represent the angular velocity of link i that is imparted by the rotation of joint i, expressed relative to frame $o_{i-1}x_{i-1}y_{i-1}z_{i-1}$. This angular velocity is expressed in the frame $i-1$ by

$$\omega_i^{i-1} = \dot{q}_i z_{i-1}^{i-1} = \dot{q}_i\boldsymbol{k} \tag{5.54}$$

in which, as above, $\boldsymbol{k}$ is the unit coordinate vector $(0, 0, 1)^T$.

If the i-th joint is prismatic, then the motion of frame i relative to frame $i-1$ is a translation and

$$\omega_i^{i-1} = 0. \tag{5.55}$$

Thus, if joint i is prismatic, the angular velocity of the end-effector does not depend on q_i, which now equals d_i.

Therefore, the overall angular velocity of the end-effector, ω_n^0, in the base frame is determined by Equation (5.40) as

$$\begin{aligned} \omega_n^0 &= \rho_1 \dot{q}_1 \boldsymbol{k} + \rho_2 \dot{q}_2 R_1^0 \boldsymbol{k} + \cdots + \rho_n \dot{q}_n R_{n-1}^0 \boldsymbol{k} \\ &= \sum_{i-1}^{n} \rho_i \dot{q}_i z_{i-1}^0 \end{aligned} \tag{5.56}$$

in which ρ_i is equal to 1 if joint i is revolute and 0 if joint i is prismatic, since

$$z_{i-1}^0 = R_{i-1}^0 \boldsymbol{k}. \tag{5.57}$$

Of course $z_0^0 = \boldsymbol{k} = (0,0,1)^T$.

The lower half of the Jacobian J_ω, in (5.53) is thus given as

$$J_\omega = [\rho_1 z_0 \cdots \rho_n z_{n-1}]. \tag{5.58}$$

Note that in this equation, we have omitted the superscripts for the unit vectors along the z-axes, since these are all referenced to the world frame. In the remainder of the chapter we will follow this convention when there is no ambiguity concerning the reference frame.

5.6.2 Linear Velocity

The linear velocity of the end-effector is just $\dot{O}_n^0$. By the chain rule for differentiation

$$\dot{O}_n^0 = \sum_{i=1}^{n} \frac{\partial O_n^0}{\partial q_i} \dot{q}_i. \tag{5.59}$$

Thus we see that the i-th column of $J_{\boldsymbol{v}}$, which we denote as $J_{\boldsymbol{v}_i}$ is given by

$$J_{\boldsymbol{v}_i} = \frac{\partial O_n^0}{\partial q_i}. \tag{5.60}$$

Furthermore this expression is just the linear velocity of the end-effector that would result if $\dot{q}_i$ were equal to one and the other $\dot{q}_j$ were zero. In other words, the i-th column of the Jacobian can be generated by holding all joints fixed but the i-th and actuating the i-th at unit velocity. We now consider the two cases (prismatic and revolute joints) separately.

(i) Case 1: Prismatic Joints

If joint i is prismatic, then it imparts a pure translation to the end-effector. From our study of the DH convention in Chapter 3, we can write the T_n^0 as the product of three transformations as follows

$$\begin{aligned}
\begin{bmatrix} R_n^0 & O_n^0 \\ \mathbf{0} & 1 \end{bmatrix} &= T_n^0 && (5.61)\\
&= T_{i-1}^0 T_i^{i-1} T_n^i && (5.62)\\
&= \begin{bmatrix} R_{i-1}^0 & O_{i-1}^0 \\ \mathbf{0} & 1 \end{bmatrix} \begin{bmatrix} R_i^{i-1} & O_i^{i-1} \\ \mathbf{0} & 1 \end{bmatrix} \begin{bmatrix} R_n^i & O_n^i \\ \mathbf{0} & 1 \end{bmatrix} && (5.63)\\
&= \begin{bmatrix} R_n^0 & R_i^0 O_n^i + R_{i-1}^0 O_i^{i-1} + O_{i-1}^0 \\ \mathbf{0} & 1 \end{bmatrix}, && (5.64)
\end{aligned}$$

which gives

$$O_n^0 = R_i^0 O_n^i + R_{i-1}^0 O_i^{i-1} + O_{i-1}^0. \tag{5.65}$$

If only joint i is allowed to move, then both of O_n^i and O_{i-1}^0 are constant. Furthermore, if joint i is prismatic, then the rotation matrix R_{i-1}^0 is also constant (again, assuming that only joint i is allowed to move). Finally, recall from Chapter 3 that, by the DH convention, $O_i^{i-1} = (a_i c_i, a_i s_i, d_i)^T$. Thus, differentiation of O_n^0 gives

$$\begin{aligned}
\frac{\partial O_n^0}{\partial q_i} &= \frac{\partial}{\partial d_i} R_{i-1}^0 O_i^{i-1} && (5.66)\\
&= R_{i-1}^0 \frac{\partial}{\partial d_i} \begin{bmatrix} a_i c_i \\ a_i s_i \\ d_i \end{bmatrix} && (5.67)\\
&= \dot{d}_i R_{i-1}^0 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} && (5.68)\\
&= \dot{d}_i z_{i-1}^0, && (5.69)
\end{aligned}$$

in which d_i is the joint variable for prismatic joint i. Thus, (again, dropping the zero superscript on the z-axis) for the case of prismatic joints we have

$$J_{\boldsymbol{v}_i} = z_{i-1}. \tag{5.70}$$

(ii) Case 2: Revolute Joints

If joint i is revolute, then we have $q_i = \theta_i$. Starting with (5.65), and letting $q_i = \theta_i$, since R_i^0 is not constant with respect to θ_i, we obtain

$$\begin{aligned}
\frac{\partial}{\partial \theta_i} O_n^0 &= \frac{\partial}{\partial \theta_i}\left[R_i^0 O_n^i + R_{i-1}^0 O_i^{i-1}\right] && (5.71)\\
&= \frac{\partial}{\partial \theta_i} R_i^0 O_n^i + R_{i-1}^0 \frac{\partial}{\partial \theta_i} O_i^{i-1} && (5.72)\\
&= \dot{\theta}_i S(z_{i-1}^0) R_i^0 O_n^i + \dot{\theta}_i S(z_{i-1}^0) R_{i-1}^0 O_i^{i-1} && (5.73)\\
&= \dot{\theta}_i S(z_{i-1}^0)\left[R_i^0 O_n^i + R_{i-1}^0 O_i^{i-1}\right] && (5.74)\\
&= \dot{\theta}_i S(z_{i-1}^0)(O_n^0 - O_{i-1}^0) && (5.75)\\
&= \dot{\theta}_i z_{i-1}^0 \times (O_n^0 - O_{i-1}^0). && (5.76)
\end{aligned}$$

The second term in (5.73) is derived as follows:

$$\begin{aligned}
R_{i-1}^0 \frac{\partial}{\partial \theta_i}\begin{bmatrix} a_i c_i \\ a_i s_i \\ d_i \end{bmatrix} &= R_{i-1}^0 \begin{bmatrix} -a_i s_i \\ a_i c_i \\ 0 \end{bmatrix} \dot{\theta}_i && (5.77)\\
&= R_{i-1}^0 S(\boldsymbol{k}\dot{\theta}_i) O_i^{i-1} && (5.78)\\
&= R_{i-1}^0 S(\boldsymbol{k}\dot{\theta}_i)\left(R_{i-1}^0\right)^T R_{i-1}^0 O_i^{i-1} && (5.79)\\
&= S(R_{i-1}^0 \boldsymbol{k}\dot{\theta}_i) R_{i-1}^0 O_i^{i-1} && (5.80)\\
&= \dot{\theta}_i S(z_{i-1}^0) R_{i-1}^0 O_i^{i-1}. && (5.81)
\end{aligned}$$

Equation (5.78) follows by straightforward computation. Thus

$$J_{\boldsymbol{v}_i} = z_{i-1} \times (O_n - O_{i-1}), \quad (5.82)$$

in which we have, following our convention, omitted the zero superscripts. Figure 5.1 illustrates a second interpretation of (5.82). As can be seen in the figure, $O_n - O_{i-1} = \boldsymbol{r}$ and $z_{i-1} = \omega$ in the familiar expression $\boldsymbol{v} = \omega \times \boldsymbol{r}$.

Combining the Angular and Linear Jacobians

As we have seen in the preceding section, the upper half of the Jacobian $J_{\boldsymbol{v}}$ is given as

$$J_{\boldsymbol{v}} = [J_{\boldsymbol{v}_1} \cdots J_{\boldsymbol{v}_n}] \quad (5.83)$$

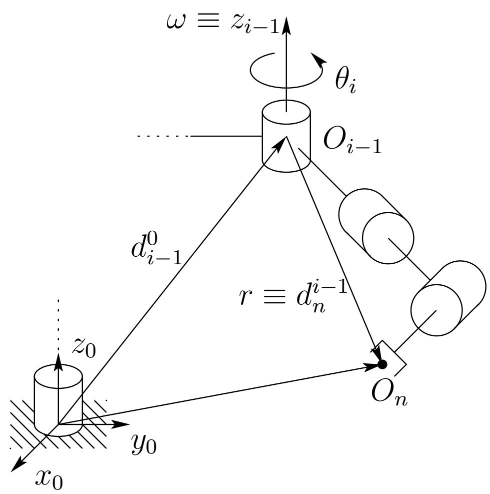

Figure 5.1: Motion of the end-effector due to link i.

where the i-th column $J_{\boldsymbol{v}_i}$ is

$$J_{\boldsymbol{v}_i} = z_{i-1} \times (O_n - O_{i-1}) \tag{5.84}$$

if joint i is revolute and

$$J_{\boldsymbol{v}_i} = z_{i-1} \tag{5.85}$$

if joint i is prismatic.

The lower half of the Jacobian is given as

$$J_\omega = [J_{\omega_1} \cdots J_{\omega_n}] \tag{5.86}$$

where the i-th column J_{ω_i} is

$$J_{\omega_i} = z_{i-1} \tag{5.87}$$

if joint i is revolute and

$$J_{\omega_i} = 0 \tag{5.88}$$

if joint i is prismatic.

Now putting the upper and lower halves of the Jacobian together we have shown that the Jacobian for an n-link manipulator is of the form

$$J = [J_1 J_2 \cdots J_n] \tag{5.89}$$

where the i-th column J_i is given by

$$J_i = \begin{bmatrix} z_{i-1} \times (O_n - O_{i-1}) \\ z_{i-1} \end{bmatrix} \tag{5.90}$$

if joint i is revolute and

$$J_i = \begin{bmatrix} z_{i-1} \\ 0 \end{bmatrix} \tag{5.91}$$

if joint i is prismatic.

The above formulas make the determination of the Jacobian of any manipulator simple since all of the quantities needed are available once the forward kinematics are worked out. Indeed the only quantities needed to compute the Jacobian are the unit vectors z_i and the coordinates of the origins $O_1, \ldots, O_n$. A moment's reflection shows that the coordinates for z_i w.r.t. the base frame are given by the first three elements in the third column of T_i^0 while O_i is given by the first three elements of the fourth column of T_i^0. Thus only the third and fourth columns of the T matrices are needed in order to evaluate the Jacobian according to the above formulas.

The above procedure works not only for computing the velocity of the end-effector but also for computing the velocity of any point on the manipulator. This will be important in Chapter 6 when we will need to compute the velocity of the center of mass of the various links in order to derive the dynamic equations of motion.

Example 5.5 *Consider the three-link planar manipulator of Figure 5.2. Suppose we wish to compute the linear velocity* $\boldsymbol{v}$ *and the angular velocity* $\boldsymbol{\omega}$

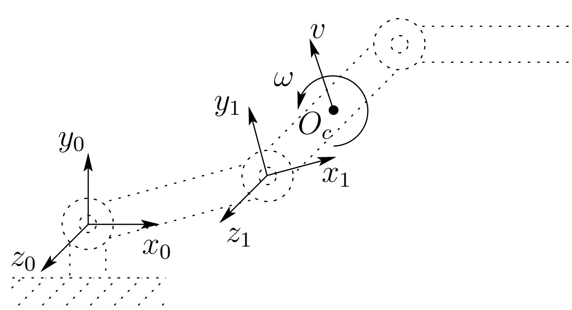

Figure 5.2: Finding the velocity of link 2 of a 3-link planar robot.

of the center of link 2 as shown. In this case we have that

$$\begin{bmatrix} \boldsymbol{v} \\ \boldsymbol{\omega} \end{bmatrix} = [J_1 \ J_2 \ J_3]\dot{q} \tag{5.92}$$

where the columns of the Jacobian are determined using the above formula with O_c in place of O_n. Thus we have

$$\begin{aligned} J_1 &= z_0 \times (O_c - O_0) \\ J_2 &= z_1 \times (O_c - O_1) \end{aligned} \tag{5.93}$$

and

$$J_3 = 0$$

since the velocity of the second link is unaffected by motion of link 3[1]*. Note that in this case the vector O_c must be computed as it is not given directly by the T matrices (Problem 5-1).*

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5.7 Examples

Example 5.6 *Consider the two-link planar manipulator of Example 3.1. Since both joints are revolute the Jacobian matrix, which in this case is 6×2, is of the form*

$$J(q) = \begin{bmatrix} z_0 \times (O_2 - O_0) & z_1 \times (O_2 - O_1) \\ z_0 & z_1 \end{bmatrix}. \tag{5.94}$$

The various quantities above are easily seen to be

$$O_0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad O_1 = \begin{bmatrix} a_1c_1 \\ a_1s_1 \\ 0 \end{bmatrix} \quad O_2 = \begin{bmatrix} a_1c_1 + a_2c_{12} \\ a_1s_1 + a_2s_{12} \\ 0 \end{bmatrix} \tag{5.95}$$

$$z_0 = z_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}. \tag{5.96}$$

[1]Note that we are treating only kinematic effects here. Reaction forces on link 2 due to the motion of link 3 will influence the motion of link 2. These dynamic effects are treated by the methods of Chapter 6.

Performing the required calculations then yields

$$J = \begin{bmatrix} -a_1s_1 - a_2s_{12} & -a_2s_{12} \\ a_1c_1 + a_2c_{12} & a_2c_{12} \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 1 \end{bmatrix}. \tag{5.97}$$

It is easy to see how the above Jacobian compares with the expression (1.1) derived in Chapter 1. The first two rows of (5.96) are exactly the 2×2 *Jacobian of Chapter 1 and give the linear velocity of the origin* O_2 *relative to the base. The third row in (5.97) is the linear velocity in the direction of* z_0*, which is of course always zero in this case. The last three rows represent the angular velocity of the final frame, which is simply a rotation about the vertical axis at the rate* $\dot{\theta}_1 + \dot{\theta}_2$*.*

◇

Example 5.7 Stanford Manipulator *Consider the Stanford manipulator of Example 3.3.5 with its associated Denavit-Hartenberg coordinate frames. Note that joint 3 is prismatic and that* $O_3 = O_4 = O_5$ *as a consequence of the spherical wrist and the frame assignment. Denoting this common origin by* O *we see that the Jacobian is of the form*

$$J = \begin{bmatrix} z_0 \times (O_6 - O_0) & z_1 \times (O_6 - O_1) & z_2 & z_3 \times (O_6 - O) & z_4 \times (O_6 - O) & z_5 \times (O_6 - O) \\ z_0 & z_1 & \mathbf{0} & z_3 & z_4 & z_5 \end{bmatrix}.$$

Now, using the A-matrices given by the expressions (3.35)-(3.40) and the T*-matrices formed as products of the A-matrices, these quantities are easily computed as follows: First,* O_j *is given by the first three entries of the last column of* $T_j^0 = A_1 \cdots A_j$*, with* $O_0 = (0,0,0)^T = O_1$*. The vector* z_j *is given as*

$$z_j = R_j^0 \boldsymbol{k} \tag{5.98}$$

where R_j^0 *is the rotational part of* T_j^0*. Thus it is only necessary to compute the matrices* T_j^0 *to calculate the Jacobian. Carrying out these calculations one obtains the following expressions for the Stanford manipulator:*

$$O_6 = (d_x, d_y, d_z)^T = \begin{bmatrix} c_1s_2d_3 - s_1d_2 + d_6(c_1c_2c_4s_5 + c_1c_5s_2 - s_1s_4s_5) \\ s_1s_2d_3 - c_1d_2 + d_6(c_1s_4s_5 + c_2c_4s_1s_5 + c_5s_1s_2) \\ c_2d_3 + d_6(c_2c_5 - c_4s_2s_5) \end{bmatrix} \tag{5.99}$$

$$O_3 = \begin{bmatrix} c_1s_2d_3 - s_1d_2 \\ s_1s_2d_3 + c_1d_2 \\ c_2d_3 \end{bmatrix}. \tag{5.100}$$

The z_i are given as

$$z_0 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \qquad z_1 = \begin{bmatrix} -s_1 \\ c_1 \\ 0 \end{bmatrix} \tag{5.101}$$

$$z_2 = \begin{bmatrix} c_1 s_2 \\ s_1 s_2 \\ c_2 \end{bmatrix} \qquad z_3 = \begin{bmatrix} c_1 s_2 \\ s_1 s_2 \\ c_2 \end{bmatrix} \tag{5.102}$$

$$z_4 = \begin{bmatrix} -c_1 c_2 s_4 - s_1 c_4 \\ -s_1 c_2 s_4 + c_1 c_4 \\ s_2 s_4 \end{bmatrix} \tag{5.103}$$

$$z_5 = \begin{bmatrix} c_1 c_2 c_4 s_5 - s_1 s_4 s_5 + c_1 s_2 c_5 \\ s_1 c_2 c_4 s_5 + c_1 s_4 s_5 + s_1 s_2 c_5 \\ -s_2 c_4 s_5 + c_2 c_5 \end{bmatrix}. \tag{5.104}$$

The Jacobian of the Stanford Manipulator is now given by combining these expressions according to the given formulae (Problem 7).

◇

Example 5.8 SCARA Manipulator *We will now derive the Jacobian of the SCARA manipulator of Example 3.3.6. This Jacobian is a 6×4 matrix since the SCARA has only four degrees-of-freedom. As before we need only compute the matrices $T_j^0 = A_1 \dots A_j$, where the A-matrices are given by (3.45)-(3.48).*

Since joints 1,2, and 4 are revolute and joint 3 is prismatic, and since $O_4 - O_3$ is parallel to z_3 (and thus, $z_3 \times (O_4 - O_3) = 0$), the Jacobian is of the form

$$J = \begin{bmatrix} z_0 \times (O_4 - O_0) & z_1 \times (O_4 - O_1) & z_2 & 0 \\ z_0 & z_1 & 0 & z_3 \end{bmatrix}. \tag{5.105}$$

Performing the indicated calculations, one obtains

$$O_1 = \begin{bmatrix} a_1 c_1 \\ a_1 s_1 \\ 0 \end{bmatrix} \qquad O_2 = \begin{bmatrix} a_1 c_1 + a_2 c_{12} \\ a_1 s_1 + a_2 s_{12} \\ 0 \end{bmatrix} \tag{5.106}$$

$$O_4 = \begin{bmatrix} a_1 c_1 + a_2 c_{12} \\ a_1 s_2 + a_2 s_{12} \\ d_3 - d_4 \end{bmatrix}. \tag{5.107}$$

Similarly $z_0 = z_1 = \boldsymbol{k}$, and $z_2 = z_3 = -\boldsymbol{k}$. Therefore the Jacobian of the SCARA Manipulator is

$$J = \begin{bmatrix} -a_1 s_1 - a_2 s_{12} & -a_2 s_{12} & 0 & 0 \\ a_1 c_1 + a_2 c_{12} & a_2 c_{12} & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & -1 \end{bmatrix}. \tag{5.108}$$

◇

5.8 Singularities

The $6 \times n$ Jacobian $J(\boldsymbol{q})$ defines a mapping

$$\dot{\boldsymbol{X}} = J(\boldsymbol{q})\dot{\boldsymbol{q}} \tag{5.109}$$

between the vector $\dot{\boldsymbol{q}}$ of joint velocities and the vector $\dot{\boldsymbol{X}} = (\boldsymbol{v}, \boldsymbol{\omega})^T$ of end-effector velocities. Infinitesimally this defines a linear transformation

$$\boldsymbol{dX} = J(\boldsymbol{q})\boldsymbol{dq} \tag{5.110}$$

between the differentials $\boldsymbol{dq}$ and $\boldsymbol{dX}$. These differentials may be thought of as defining directions in $\mathbf{R}^6$, and $\mathbf{R}^n$, respectively.

Since the Jacobian is a function of the configuration $\boldsymbol{q}$, those configurations for which the rank of J decreases are of special significance. Such configurations are called **singularities** or **singular configurations**. Identifying manipulator singularities is important for several reasons.

1. Singularities represent configurations from which certain directions of motion may be unattainable.

2. At singularities, bounded end-effector velocities may correspond to unbounded joint velocities.

3. At singularities, bounded end-effector forces and torques may correspond to unbounded joint torques. (We will see this in Chapter **??**).

4. Singularities usually (but not always) correspond to points on the boundary of the manipulator workspace, that is, to points of maximum reach of the manipulator.

5. Singularities correspond to points in the manipulator workspace that may be unreachable under small perturbations of the link parameters, such as length, offset, etc.

6. Near singularities there will not exist a unique solution to the inverse kinematics problem. In such cases there may be no solution or there may be infinitely many solutions.

Example 5.9 *Consider the two-dimensional system of equations*

$$\boldsymbol{dX} = J\boldsymbol{dq} \quad = \quad \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \boldsymbol{dq} \tag{5.111}$$

that corresponds to the two equations

$$dx \quad = \quad dq_1 + dq_2 \tag{5.112}$$

$$dy \quad = \quad 0. \tag{5.113}$$

In this case the rank of J is one and we see that for any values of the variables dq_1 and dq_2 there is no change in the variable dy. Thus any vector $\boldsymbol{dX}$ having a nonzero second component represents an unattainable direction of instantaneous motion.

◇

5.8.1 Decoupling of Singularities

We saw in Chapter 3 that a set of forward kinematic equations can be derived for any manipulator by attaching a coordinate frame rigidly to each link in any manner that we choose, computing a set of homogeneous transformations relating the coordinate frames, and multiplying them together as needed. The D-H convention is merely a systematic way to do this. Although the resulting equations are dependent on the coordinate frames chosen, the manipulator configurations themselves are geometric quantities, independent of the frames used to describe them. Recognizing this fact allows us to decouple the determination of singular configurations, for those manipulators with spherical wrists, into two simpler problems. The first is to determine so-called **arm singularities**, that is, singularities resulting from motion of the arm, which consists of the first three or more links, while the second is to determine the **wrist singularities** resulting from motion of the spherical wrist.

For the sake of argument, suppose that $n = 6$, that is, the manipulator consists of a 3-DOF arm with a 3-DOF spherical wrist. In this case the Jacobian is a 6×6 matrix and a configuration $\boldsymbol{q}$ is singular if and only if

$$\det J(\boldsymbol{q}) \quad = \quad 0. \tag{5.114}$$

If we now partition the Jacobian J into 3×3 blocks as

$$J = [J_P \mid J_O] \quad = \quad \left[\begin{array}{c|c} J_{11} & J_{12} \\ \hline J_{21} & J_{22} \end{array} \right] \tag{5.115}$$

then, since the final three joints are always revolute

$$J_O \quad = \quad \left[\begin{array}{ccc} z_3 \times (O_6 - O_3) & z_4 \times (O_6 - O_4) & z_5 \times (O_6 - O_5) \\ z_3 & z_4 & z_5 \end{array} \right] \tag{5.116}$$

Since the wrist axes intersect at a common point O, if we choose the coordinate frames so that $O_3 = O_4 = O_5 = O_6 = O$, then J_O becomes

$$J_O \quad = \quad \left[\begin{array}{ccc} 0 & 0 & 0 \\ z_3 & z_4 & z_5 \end{array} \right] \tag{5.117}$$

and the i-th column J_i of J_p is

$$J_i \quad = \quad \left[\begin{array}{c} z_{i-1} \times (O - O_{i-1}) \\ z_{i-1} \end{array} \right] \tag{5.118}$$

if joint i is revolute and

$$J_i \quad = \quad \left[\begin{array}{c} z_{i-1} \\ 0 \end{array} \right] \tag{5.119}$$

if joint i is prismatic. In this case the Jacobian matrix has the block triangular form

$$J \quad = \quad \left[\begin{array}{cc} J_{11} & 0 \\ J_{21} & J_{22} \end{array} \right] \tag{5.120}$$

with determinant

$$\det J \quad = \quad \det J_{11} \det J_{22} \tag{5.121}$$

where J_{11} and J_{22} are each 3×3 matrices. J_{11} has i-th column $z_{i-1} \times (O - O_{i-1})$ if joint i is revolute, and z_{i-1} if joint i is prismatic, while

$$J_{22} = [z_3 \; z_4 \; z_5]. \tag{5.122}$$

Therefore the set of singular configurations of the manipulator is the union of the set of arm configurations satisfying $\det J_{11} = 0$ and the set of wrist configurations satisfying $\det J_{22} = 0$. *Note that this form of the Jacobian does not necessarily give the correct relation between the velocity of the end-effector and the joint velocities.* It is intended only to simplify the determination of singularities.

5.8.2 Wrist Singularities

We can now see from (5.122) that a spherical wrist is in a singular configuration whenever the vectors z_3, z_4 and z_5 are linearly dependent. Referring to Figure 5.3 we see that this happens when the joint axes z_3 and z_5 are

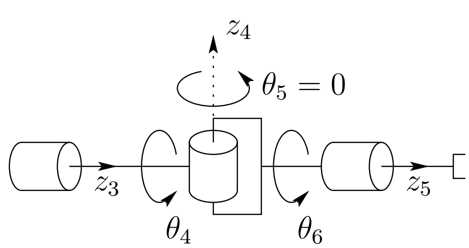

Figure 5.3: Spherical wrist singularity.

collinear. In fact, whenever two revolute joint axes anywhere are collinear, a singularity results since an equal and opposite rotation about the axes results in no net motion of the end-effector. This is the only singularity of the spherical wrist, and is unavoidable without imposing mechanical limits on the wrist design to restrict its motion in such a way that z_3 and z_5 are prevented from lining up.

5.8.3 Arm Singularities

In order to investigate arm singularities we need only to compute J_{11} according to (5.118) and (5.119), which is the same formula derived previously with the wrist center O in place of O_6.

Example 5.10 Elbow Manipulator Singularities *Consider the three-link articulated manipulator with coordinate frames attached as shown in Figure 5.4. It is left as an exercise (Problem 2) to show that*

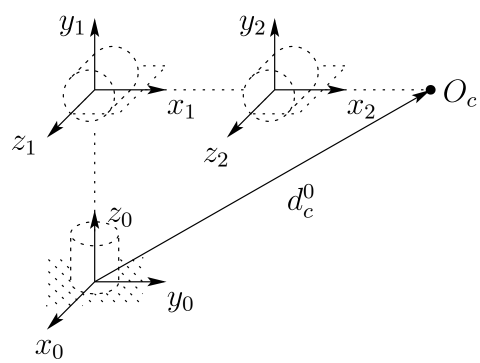

Figure 5.4: Elbow manipulator.

$$J_{11} = \begin{bmatrix} -a_2s_1c_2 - a_3s_1c_{23} & -a_2s_2c_1 - a_3s_{23}c_1 & -a_3c_1s_{23} \\ a_2c_1c_2 + a_3c_1c_{23} & -a_2s_1s_2 - a_3s_1s_{23} & -a_3s_1s_{23} \\ 0 & a_2c_2 + a_3c_{23} & a_3c_{23} \end{bmatrix} \tag{5.123}$$

and that the determinant of J_{11} is

$$\det J_{11} = a_2a_3s_3(a_2c_2 + a_3c_{23}). \tag{5.124}$$

We see from (5.124) that the elbow manipulator is in a singular configuration whenever

$$s_3 = 0, \quad \text{that is, } \theta_3 = 0 \text{ or } \pi \tag{5.125}$$

and whenever

$$a_2c_2 + a_3c_{23} = 0. \tag{5.126}$$

The situation of (5.125) is shown in Figure 5.5 and arises when the elbow is fully extended or fully retracted as shown. The second situation (5.126) is shown in Figure 5.6. This configuration occurs when the wrist center intersects the axis of the base rotation, z_0. As we saw in Chapter 4, there are infinitely many singular configurations and infinitely many solutions to the inverse position kinematics when the wrist center is along this axis. For

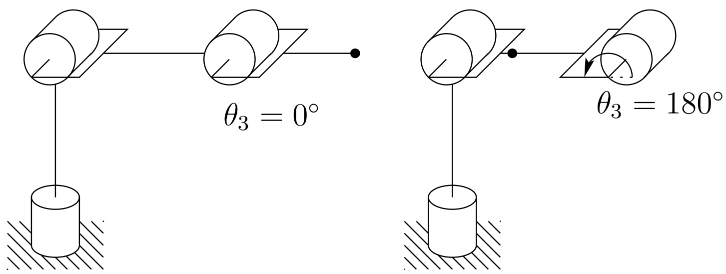

Figure 5.5: Elbow singularities of the elbow manipulator.

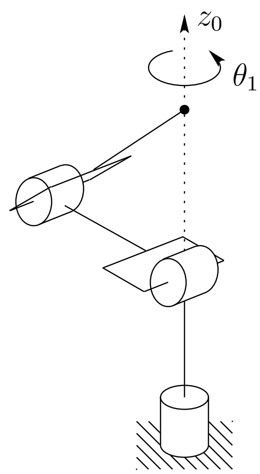

Figure 5.6: Singularity of the elbow manipulator with no offsets.

an elbow manipulator with an offset, as shown in Figure 5.7, the wrist center cannot intersect z_0, which corroborates our earlier statement that points reachable at singular configurations may not be reachable under arbitrarily small perturbations of the manipulator parameters, in this case an offset in either the elbow or the shoulder.

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Example 5.11 Spherical Manipulator *Consider the spherical arm of Figure 5.8. This manipulator is in a singular configuration when the wrist center intersects z_0 as shown since, as before, any rotation about the base leaves this point fixed.*

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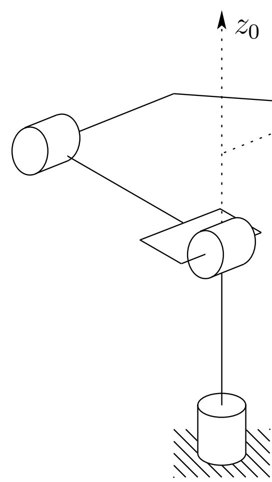

Figure 5.7: Elbow manipulator with shoulder offset.

Example 5.12 SCARA Manipulator *We have already derived the complete Jacobian for the the SCARA manipulator. This Jacobian is simple enough to be used directly rather than deriving the modified Jacobian from this section. Referring to Figure 5.9 we can see geometrically that the only singularity of the SCARA arm is when the elbow is fully extended or fully retracted. Indeed, since the portion of the Jacobian of the SCARA governing arm singularities is given as*

$$J_{11} = \begin{bmatrix} \alpha_1 & \alpha_3 & 0 \\ \alpha_2 & \alpha_4 & 0 \\ 0 & 0 & -1 \end{bmatrix} \tag{5.127}$$

where

$$\begin{aligned} \alpha_1 &= -a_1 s_1 - a_2 s_{12} \quad (5.128) \\ \alpha_2 &= a_1 c_1 + a_2 c_{12} \\ \alpha_3 &= -a_1 s_{12} \\ \alpha_4 &= a_1 c_{12} \quad (5.129) \end{aligned}$$

we see that the rank of J_{11} will be less than three precisely whenever $\alpha_1 \alpha_4 - \alpha_2 \alpha_3 = 0$. It is easy to compute this quantity and show that it is equivalent to (Problem 4)

$$s_2 = 0, \quad \textit{which implies} \quad \theta_2 = 0, \pi. \tag{5.130}$$

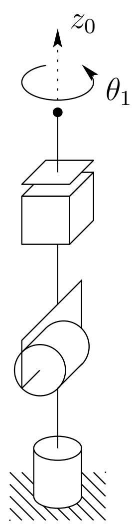

Figure 5.8: Singularity of spherical manipulator with no offsets.

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5.9 Inverse Velocity and Acceleration

It is perhaps a bit surprising that the inverse velocity and acceleration relationships are conceptually simpler than inverse position. Recall from (5.109) that the joint velocities and the end-effector velocities are related by the Jacobian as

$$\dot{\boldsymbol{X}} = J(\boldsymbol{q})\dot{\boldsymbol{q}}. \tag{5.131}$$

Thus the inverse velocity problem becomes one of solving the system of linear equations (5.131), which is conceptually simple.

Differentiating (5.131) yields the acceleration equations

$$\ddot{\boldsymbol{X}} = J(\boldsymbol{q})\ddot{\boldsymbol{q}} + \left(\frac{d}{dt}J(\boldsymbol{q})\right)\dot{\boldsymbol{q}}. \tag{5.132}$$

Thus, given a vector $\ddot{\boldsymbol{X}}$ of end-effector accelerations, the instantaneous joint acceleration vector q is given as a solution of

$$\boldsymbol{b} = J(\boldsymbol{q})\ddot{\boldsymbol{q}} \tag{5.133}$$

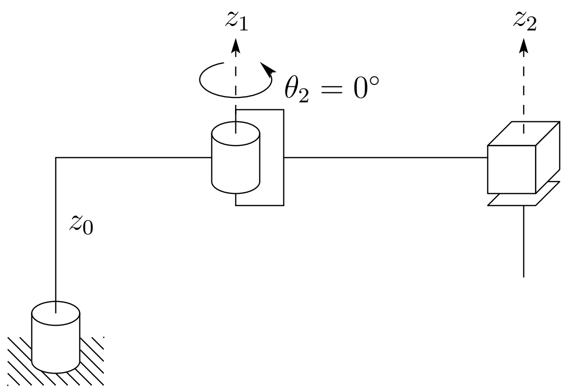

Figure 5.9: SCARA manipulator singularity.

where

$$\boldsymbol{b} = \ddot{\boldsymbol{X}} - \frac{d}{dt}J(\boldsymbol{q})\dot{\boldsymbol{q}} \tag{5.134}$$

For 6-DOF manipulators the inverse velocity and acceleration equations can therefore be written as

$$\dot{q} = J(\boldsymbol{q})^{-1}\dot{\boldsymbol{X}} \tag{5.135}$$

and

$$\ddot{\boldsymbol{q}} = J(\boldsymbol{q})^{-1}\boldsymbol{b} \tag{5.136}$$

provided $\det J(\boldsymbol{q}) \neq 0$. In the next section, we address the case of manipulators with more than 6-DOF.

5.10 Redundant Robots and Manipulability

In this section we briefly address the topic of *redundant manipulators*. Informally, a redundant manipulator is one that is equipped with more internal degrees of freedom than are required to perform a specified task. For example, a three link planar arm is redundant for the task of positioning in the plane. As we have briefly seen in Chapter 4, in such cases there in no unique solution for the inverse kinematics problem. Further, the Jacobian matrix for a redundant manipulator is not square, and thus cannot be inverted to solve the inverse velocity problem.

In this section, we begin by giving a brief and general introduction to the subject of redundant manipulators. We then turn our attention to the inverse velocity problem. To address this problem, we will introduce the concept of a pseudoinverse and the Singular Value Decomposition. We end the section by introducing *manipulability*, a measure that can be used to quantify the quality of the internal configuration of a manipulator, and can therefore be used in an optimization framework to aid in the solution for the inverse kinematics problem.

5.10.1 Redundant Manipulators

A precise definition of what is meant by the term *redundant* requires that we specify a task, and the number of degrees of freedom required to perform that task. In previous chapters, we have dealt primarily with positioning tasks. In these cases, the task was determined by specifying the position, orientation or both for the end effector or some tool mounted at the end effector. For these kinds of positioning tasks, the number of degrees of freedom for the task is equal to the number of parameters required to specify the position and orientation information. For example, if the task involves positioning the end effector in a 3D workspace, then the task can be specified by an element of $\Re^3 \times SO(3)$. As we have seen in Chapter 2, $\Re^3 \times SO(3)$ can be parameterized by $(x, y, z, \phi, \theta, \psi)$, i.e., using six parameters. Thus, for this task, the task space is six-dimensional. A manipulator is said to be redundant when its number of internal degrees of freedom (or joints) is greater than the dimension of the task space. Thus, for the 3D position and orientation task, any manipulator with more than six joints would be redundant.

A simpler example is a three-link planar arm performing the task of positioning the end effector in the plane. Here, the task can be specified by $(x, y) \in \Re^2$, and therefore the task space is two-dimensional. The forward kinematic equations for this robot are given by

$$\begin{aligned} x &= a_1C_1 + a_2C_{12} + a_3C_{123} \\ y &= a_1S_1 + a_2S_{12} + a_3S_{123}. \end{aligned}$$

Clearly, since there are three variables $(\theta_1, \theta_2, \theta_3)$ and only two equations, it is not possible to solve uniquely for $\theta_1, \theta_2, \theta_3$ given a specific (x, y).

The Jacobian for this manipulator is given by

$$\mathbf{J} = \begin{bmatrix} -a_1S_1 - a_2S_{12} & -a_2S_{12} & -a_3S_{123} \\ a_1C_1 + a_2C_{12} & a_2C_{12} & a_3C_{123} \end{bmatrix}. \tag{5.137}$$

When using the relationship $\dot{\boldsymbol{x}} = \mathbf{J}\dot{\boldsymbol{q}}$ to for $\dot{\boldsymbol{q}}$, we have a system of two linear equations in three unknowns. Thus there are also infinitely many solutions to this system, and the inverse velocity problem cannot be solved uniquely. We now turn our attention to the specifics of dealing with these inverse problems.

5.10.2 The Inverse Velocity Problem for Redundant Manipulators

We have seen in Section 5.9 that the inverse velocity problem is easily solved when the Jacobian is square with nonzero determinant. However, when the Jacobian is not square, as is the case for redundant manipulators, the method of Section 5.9 cannot be used, since a nonsquare matrix cannot be inverted. To deal with the case when $m < n$, we use the following result from linear algebra.

Proposition: For $\mathbf{J} \in \Re^{m\times n}$, if $m < n$ and rank $\mathbf{J} = m$, then $(\mathbf{J}\mathbf{J}^T)^{-1}$ exists.

In this case $(\mathbf{J}\mathbf{J}^T) \in \Re^{m\times m}$, and has rank m. Using this result, we can regroup terms to obtain

$$\begin{aligned} (\mathbf{J}\mathbf{J}^T)(\mathbf{J}\mathbf{J}^T)^{-1} &= \mathbf{I} \\ \mathbf{J}\left[\mathbf{J}^T(\mathbf{J}\mathbf{J}^T)^{-1}\right] &= \mathbf{I} \\ \mathbf{J}\mathbf{J}^+ &= \mathbf{I}. \end{aligned}$$

Here, $\mathbf{J}^+ = \mathbf{J}^T(\mathbf{J}\mathbf{J}^T)^{-1}$ is called a right pseudoinverse of $\mathbf{J}$, since $\mathbf{J}\mathbf{J}^+ = \mathbf{I}$. Note that, $\mathbf{J}^+\mathbf{J} \in \Re^{n\times n}$, and that in general, $\mathbf{J}^+\mathbf{J} \neq \mathbf{I}$ (recall that matrix multiplication is not commutative).

It is now easy to demonstrate that a solution to (5.131) is given by

$$\dot{\boldsymbol{q}} = \mathbf{J}^+\dot{\boldsymbol{x}} + (\mathbf{I} - \mathbf{J}^+\mathbf{J})\boldsymbol{b} \tag{5.138}$$

in which $\boldsymbol{b} \in \Re^n$ is an arbitrary vector. To see this, multiply this solution by $\mathbf{J}$:

$$\begin{aligned} \mathbf{J}\dot{\boldsymbol{q}} &= \mathbf{J}\left[\mathbf{J}^+\dot{\boldsymbol{x}} + (\mathbf{I} - \mathbf{J}^+\mathbf{J})\boldsymbol{b}\right] \\ &= \mathbf{J}\mathbf{J}^+\dot{\boldsymbol{x}} + \mathbf{J}(\mathbf{I} - \mathbf{J}^+\mathbf{J})\boldsymbol{b} \end{aligned}$$

$$\begin{aligned} &= \mathbf{J}\mathbf{J}^{+}\dot{\boldsymbol{x}} + (\mathbf{J} - \mathbf{J}\mathbf{J}^{+}\mathbf{J})\boldsymbol{b} \\ &= \dot{\boldsymbol{x}} + (\mathbf{J} - \mathbf{J})\boldsymbol{b} \\ &= \dot{\boldsymbol{x}}. \end{aligned}$$

In general, for $m < n$, $(\mathbf{I} - \mathbf{J}^{+}\mathbf{J}) \neq 0$, and all vectors of the form $(\mathbf{I} - \mathbf{J}^{+}\mathbf{J})\mathbf{b}$ lie in the null space of $\mathbf{J}$, i.e., if $\dot{\boldsymbol{q}}_n$ is a joint velocity vector such that $\dot{\boldsymbol{q}}_n = (\mathbf{I} - \mathbf{J}^{+}\mathbf{J})\mathbf{b}$, then when the joints move with velocity $\dot{\boldsymbol{q}}_n$, the end effector will remain fixed since $\mathbf{J}\dot{\boldsymbol{q}}_n = \mathbf{0}$. Thus, if $\dot{\boldsymbol{q}}$ is a solution to (5.131), then so is $\dot{\boldsymbol{q}} + \dot{\boldsymbol{q}}_n$ with $\dot{\boldsymbol{q}}_n = (\mathbf{I} - \mathbf{J}^{+}\mathbf{J})\mathbf{b}$, for any value of $\mathbf{b}$. If the goal is to minimize the resulting joint velocities, we choose $\mathbf{b} = \mathbf{0}$. To see this, apply the triangle inequality to obtain

$$\begin{aligned} ||\, \dot{\boldsymbol{q}} \,|| &= ||\, \mathbf{J}^{+}\dot{\boldsymbol{x}} + (\mathbf{I} - \mathbf{J}^{+}\mathbf{J})\boldsymbol{b} \,|| \\ &\leq ||\, \mathbf{J}^{+}\dot{\boldsymbol{x}} \,|| + ||\, (\mathbf{I} - \mathbf{J}^{+}\mathbf{J})\boldsymbol{b} \,||. \end{aligned}$$

5.10.3 Singular Value Decomposition (SVD)

For robots that are not redundant, the Jacobian matrix is square, and we can use tools such as the determinant, eigenvalues and eigenvectors to analyze its properties. However, for redundant robots, the Jacobian matrix is not square, and these tools simply do not apply. Their generalizations are captured by the Singular Value Decomposition (SVD) of a matrix, which we now introduce.

As we described above, for $\mathbf{J} \in \Re^{m \times n}$, we have $\mathbf{J}\mathbf{J}^{T} \in \Re^{m \times m}$. This square matrix has eigenvalues and eigenvectors that satisfy

$$\mathbf{J}\mathbf{J}^{T}\boldsymbol{u}_i = \lambda_i \boldsymbol{u}_i \tag{5.139}$$

in which λ_i and $\boldsymbol{u}_i$ are corresponding eigenvalue and eigenvector pairs for $\mathbf{J}\mathbf{J}^{T}$. We can rewrite this equation to obtain

$$\begin{aligned} \mathbf{J}\mathbf{J}^{T}\boldsymbol{u}_i - \lambda_i \boldsymbol{u}_i &= \mathbf{0} \\ (\mathbf{J}\mathbf{J}^{T} - \lambda_i \mathbf{I})\boldsymbol{u}_i &= \mathbf{0}. \end{aligned} \tag{5.140}$$

The latter equation implies that the matrix $(\mathbf{J}\mathbf{J}^{T} - \lambda_i \mathbf{I})$ is singular, and we can express this in terms of its determinant as

$$\det(\mathbf{J}\mathbf{J}^{T} - \lambda_i \mathbf{I}) = 0. \tag{5.141}$$

We can use (5.141) to find the eigenvalues $\lambda_1 \geq \lambda_2 \cdots \geq \lambda_m \geq 0$ for $\mathbf{J}\mathbf{J}^{T}$. The *singular values* for the Jacobian matrix $\mathbf{J}$ are given by the square roots of the eigenvalues of $\mathbf{J}\mathbf{J}^{T}$,

$$\sigma_i = \sqrt{\lambda_i}. \tag{5.142}$$

The singular value decomposition of the matrix $\mathbf{J}$ is then given by

$$\mathbf{J} = \mathbf{U}\Sigma\mathbf{V}^T, \tag{5.143}$$

in which

$$\mathbf{U} = [\mathbf{u}_1\mathbf{u}_2\ldots\mathbf{u}_m], \qquad \mathbf{V} = [\mathbf{v}_1\mathbf{v}_2\ldots\mathbf{v}_n] \tag{5.144}$$

are orthogonal matrices, and $\Sigma \in \mathbf{R}^{m\times n}$.

$$\Sigma = \left[\begin{array}{ccccc|c} \sigma_1 & & & & & \\ & \sigma_2 & & & & \\ & & \cdot & & & 0 \\ & & & \cdot & & \\ & & & & \sigma_m & \end{array}\right]. \tag{5.145}$$

We can compute the SVD of $\mathbf{J}$ as follows. We begin by finding the singular values, σ_i, of $\mathbf{J}$ using (5.141) and (5.142). These singular values can then be used to find the eigenvectors $\mathbf{u}_1, \cdots \mathbf{u}_m$ that satisfy

$$\mathbf{J}\mathbf{J}^T\mathbf{u}_i = \sigma_i^2\mathbf{u}_i. \tag{5.146}$$

These eigenvectors comprise the matrix $\mathbf{U} = [\mathbf{u}_1\mathbf{u}_2\ldots\mathbf{u}_m]$. The system of equations (5.146) can be written as

$$\mathbf{J}\mathbf{J}^T\mathbf{U} = \mathbf{U}\Sigma_m^2 \tag{5.147}$$

if we define the matrix Σ_m as

$$\Sigma_m = \begin{bmatrix} \sigma_1 & & & & \\ & \sigma_2 & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & \sigma_m \end{bmatrix}.$$

Now, define

$$\mathbf{V}_m = \mathbf{J}^T\mathbf{U}\Sigma_m^{-1} \tag{5.148}$$

and let $\mathbf{V}$ be any orthogonal matrix that satisfies $\mathbf{V} = [\mathbf{V}_m \mid \mathbf{V}_{n-m}]$ (note that here $\mathbf{V}_{n-m}$ contains just enough columns so that the matrix $\mathbf{V}$ is an $n \times n$ matrix). It is a simple matter to combine the above equations to verify

(5.143):

$$\begin{aligned}
\mathbf{U}\Sigma\mathbf{V}^T &= \mathbf{U}\left[\Sigma_m \mid \mathbf{0}\right]\begin{bmatrix} \mathbf{V}_m^T \\ \mathbf{V}_{n-m}^T \end{bmatrix} && (5.149)\\
&= \mathbf{U}\Sigma_m\mathbf{V}_m^T && (5.150)\\
&= \mathbf{U}\Sigma_m\left(\mathbf{J}^T\mathbf{U}\Sigma_m^{-1}\right)^T && (5.151)\\
&= \mathbf{U}\Sigma_m(\Sigma_m^{-1})^T\mathbf{U}^T\mathbf{J} && (5.152)\\
&= \mathbf{U}\Sigma_m\Sigma_m^{-1}\mathbf{U}^T\mathbf{J} && (5.153)\\
&= \mathbf{U}\mathbf{U}^T\mathbf{J} && (5.154)\\
&= \mathbf{J}. && (5.155)
\end{aligned}$$

Here, (5.149) follows immediately from our construction of the matrices $\mathbf{U}$, $\mathbf{V}$ and Σ_m. Equation (5.151) is obtained by substituting (5.148) into (5.150). Equation (5.153) follows because Σ_m^{-1} is a diagonal matrix, and thus symmetric. Finally, (5.155) is obtained using the fact that $\mathbf{U}^T = \mathbf{U}^{-1}$, since $\mathbf{U}$ is orthogonal.

It is a simple matter construct the right pseudoinverse of $\mathbf{J}$ using the SVD,

$$\mathbf{J}^+ = \mathbf{V}\Sigma^+\mathbf{U}^T$$

in which

$$\Sigma^+ = \left[\begin{array}{ccccc|c} \sigma_1^{-1} & & & & & \\ & \sigma_2^{-1} & & & & \\ & & \cdot & & & 0 \\ & & & \cdot & & \\ & & & & \sigma_m^{-1} & \end{array}\right]^T .$$

5.10.4 Manipulability

For a specific value of $\boldsymbol{q}$, the Jacobian relationship defines the linear system given by $\dot{\boldsymbol{x}} = \mathbf{J}\dot{\boldsymbol{q}}$. We can think of $\mathbf{J}$ a scaling the input, $\dot{\boldsymbol{q}}$, to produce the output, $\dot{\boldsymbol{x}}$. It is often useful to characterize quantitatively the effects of this scaling. Often, in systems with a single input and a single output, this kind of characterization is given in terms of the so called impulse response of a system, which essentially characterizes how the system responds to a unit input. In this multidimensional case, the analogous concept is to characterize the output in terms of an input that has unit norm. Consider the set of all robot tool velocities $\dot{\boldsymbol{q}}$ such that

$$\|\dot{\boldsymbol{q}}\| = (\dot{q}_1^2 + \dot{q}_2^2 + \ldots \dot{q}_m^2)^{1/2} \leq 1. \tag{5.156}$$

If we use the minimum norm solution $\dot{\boldsymbol{q}} = \mathbf{J}^{+}\dot{\boldsymbol{x}}$, we obtain

$$\begin{aligned} \|\dot{\boldsymbol{q}}\| &= \dot{\boldsymbol{q}}^T\dot{\boldsymbol{q}} \\ &= (\mathbf{J}^{+}\dot{\boldsymbol{x}})^T\mathbf{J}^{+}\dot{\boldsymbol{x}} \\ &= \dot{\boldsymbol{x}}^T(\mathbf{J}^{+})^T\mathbf{J}^{+}\dot{\boldsymbol{x}} \\ &= \dot{\boldsymbol{x}}^T(\mathbf{J}^T(\mathbf{J}\mathbf{J}^T)^{-1})^T\mathbf{J}^T(\mathbf{J}\mathbf{J}^T)^{-1}\dot{\boldsymbol{x}} \\ &= \dot{\boldsymbol{x}}^T[(\mathbf{J}\mathbf{J}^T)^{-1}]^T\mathbf{J}\mathbf{J}^T(\mathbf{J}\mathbf{J}^T)^{-1}\dot{\boldsymbol{x}} \\ &= \dot{\boldsymbol{x}}^T(\mathbf{J}\mathbf{J}^T)^{-1}\dot{\boldsymbol{x}} \leq 1. \end{aligned} \tag{5.157}$$

This final inequality gives us a quantitative characterization of the scaling that is effected by the Jacobian. In particular, if the manipulator Jacobian is full rank, i.e., rank $\mathbf{J} = m$, then (5.157) defines an m-dimensional ellipsoid that is known as the *manipulability ellipsoid*. If the input (i.e., joint velocity) vector has unit norm, then the output (i.e., task space velocity) will lie within the ellipsoid given by (5.157). We can more easily see that (5.157) defines an ellipsoid by replacing $\mathbf{J}$ by its SVD to obtain

$$\begin{aligned} \dot{\boldsymbol{x}}^T(\mathbf{J}\mathbf{J}^T)^{-1}\dot{\boldsymbol{x}}^T &= \dot{\boldsymbol{x}}^T[\mathbf{U}\Sigma\mathbf{V}^T(\mathbf{U}\Sigma\mathbf{V}^T)^T]^{-1}\dot{\boldsymbol{x}} \\ &= \dot{\boldsymbol{x}}^T[\mathbf{U}\Sigma\mathbf{V}^T\mathbf{V}\Sigma^T\mathbf{U}^T]^{-1}\dot{\boldsymbol{x}} \\ &= \dot{\boldsymbol{x}}^T[\mathbf{U}\Sigma\Sigma^T\mathbf{U}^T]^{-1}\dot{\boldsymbol{x}} \\ &= \dot{\boldsymbol{x}}^T[\mathbf{U}\Sigma_m^2\mathbf{U}^T]^{-1}\dot{\boldsymbol{x}} \\ &= \dot{\boldsymbol{x}}^T[\mathbf{U}\Sigma_m^{-2}\mathbf{U}^T]\dot{\boldsymbol{x}} \\ &= (\dot{\boldsymbol{x}}^T\mathbf{U})\Sigma_m^{-2}(\mathbf{U}^T\dot{\boldsymbol{x}}) \\ &= (\mathbf{U}^T\dot{\boldsymbol{x}})^T\Sigma_m^{-2}(\mathbf{U}^T\dot{\boldsymbol{x}}) \end{aligned} \tag{5.158}$$

in which

$$\Sigma_m^{-2} = \begin{bmatrix} \sigma_1^{-2} & & & & \\ & \sigma_2^{-2} & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & \sigma_m^{-2} \end{bmatrix}.$$

If we make the substation $\mathbf{w} = \mathbf{U}^T\dot{\boldsymbol{x}}$, then (5.158) can be written as

$$\mathbf{w}^T\Sigma_m^{-2}\mathbf{w} = \sum \sigma_i^{-2}w_i^2 \leq 1 \tag{5.159}$$

and it is clear that this is the equation for an axis-aligned ellipse in a new coordinate system that is obtained by rotation according to the orthogonal matrix $\mathbf{U}$. In the original coordinate system, the axes of the ellipsoid are given by the vectors $\sigma_i\mathbf{u_i}$. The volume of the ellipsoid is given by

$$\text{volume} = K\sigma_1\sigma_2\cdots\sigma_m,$$

in which K is a constant that depends only on the dimension, m, of the ellipsoid. The manipulability measure, as defined by Yoshikawa [**?**], is given by

$$\omega = \sigma_1 \sigma_2 \cdots \sigma_m. \tag{5.160}$$

Note that the constant K is not included in the definition of manipulability, since it is fixed once the task has been defined (i.e., once the dimension of the task space has been fixed).

Now, consider the special case that the robot is not redundant, i.e., $\mathbf{J} \in \Re^{m \times m}$. Recall that the determinant of a product is equal to the product of the determinants, and that a matrix and its transpose have the same determinant. Thus, we have

$$\begin{aligned} \det \mathbf{J}\mathbf{J}^T &= \det \mathbf{J} \;\det \mathbf{J}^T \\ &= \det \mathbf{J} \;\det \mathbf{J} \\ &= (\lambda_1 \lambda_2 \cdots \lambda_m)(\lambda_1 \lambda_2 \cdots \lambda_m) \\ &= \lambda_1^2 \lambda_2^2 \cdots \lambda_m^2 \end{aligned} \tag{5.161}$$

in which $\lambda_1 \geq \lambda_2 \cdots \leq \lambda_m$ are the eigenvalues of $\mathbf{J}$. This leads to

$$\omega = \sqrt{\det \mathbf{J}\mathbf{J}^T} = |\lambda_1 \lambda_2 \cdots \lambda_m| = |\det \mathbf{J}|. \tag{5.162}$$

The manipulability, ω, has the following properties.

- In general, $\omega = 0$ holds if and only if $\mathrm{rank}(\mathbf{J}) < m$, (i.e., when J is not full rank).
- Suppose that there is some error in the measured velocity, $\Delta \dot{\boldsymbol{x}}$. We can bound the corresponding error in the computed joint velocity, $\Delta \dot{\boldsymbol{q}}$, by

$$(\sigma_1)^{-1} \leq \frac{||\Delta \dot{\boldsymbol{q}}||}{||\Delta \dot{\boldsymbol{x}}||} \leq (\sigma_m)^{-1}. \tag{5.163}$$

Example 5.13 Two-link Planar Arm. *We can use manipulability to determine the optimal configurations in which to perform certain tasks. In some cases it is desirable to perform a task in the configuration for which the end effector has the maximum dexterity. We can use manipulability as a measure of dexterity. Consider the two-link planar arm and the task of positioning in the plane. For the two link arm, the Jacobian is given by*

$$J = \begin{bmatrix} -a_1 S_1 - a_2 S_{12} & -a_2 S_{12} \\ a_1 C_1 + a_2 C_{12} & a_2 C_{12} \end{bmatrix}. \tag{5.164}$$

and the manipulability is given by

$$\omega = |det\ \mathbf{J}| = a_1 a_2 |S_2|$$

Thus, for the two-link arm, the maximum manipulability is obtained for $\theta_2 = \pm\pi/2$.

Manipulability can also be used to aid in the design of manipulators. For example, suppose that we wish to design a two-link planar arm whose total link length, $a_1 + a_2$, *is fixed. What values should be chosen for* a_1 *and* a_2*? If we design the robot to maximize the maximum manipulability, the we need to maximize* $\omega = a_1 a_2 |S_2|$. *We have already seen that the maximum is obtained when* $\theta_2 = \pm\pi/2$, *so we need only find* a_1 *and* a_2 *to maximize the product* $a_1 a_2$. *This is achieved when* $a_1 = a_2$. *Thus, to maximize manipulability, the link lengths should be chosen to be equal.*
◇

5.11 Problems

1. For the three-link planar manipulator of Example 5.5, compute the vector O_c and derive the Jacobian (5.92).

2. Compute the Jacobian J_{11} for the 3-link elbow manipulator of Example 5.10 and show that it agrees with (5.123). Show that the determinant of this matrix agrees with (5.124).

3. Compute the Jacobian J_{11} for the three-link spherical manipulator of Example 5.11.

4. Show from (5.128) that the singularities of the SCARA manipulator are given by (5.130).

5. Find the 6×3 Jacobian for the three links of the cylindrical manipulator of Figure 3.7. Show that there are no singular configurations for this arm. Thus the only singularities for the cylindrical manipulator must come from the wrist.

6. Repeat Problem 5 for the cartesian manipulator of Figure 3.17.

7. Complete the derivation of the Jacobian for the Stanford manipulator from Example 5.7.

8. Verify Equation (5.9) by direct calculation.

9. Prove assertion (5.10) that $R(\boldsymbol{a} \times \boldsymbol{b}) = R\boldsymbol{a} \times R\boldsymbol{b}$, for $R \in S0(3)$.

10. Suppose that $\boldsymbol{a} = (1, -1, 2)^T$ and that $R = R_{x,90}$. Show by direct calculation that

$$RS(\boldsymbol{a})R^T \quad = \quad S(R\boldsymbol{a}).$$

11. Given $R_1^0 = R_{x,\theta}R_{y,\phi}$, compute $\frac{\partial R_1^0}{\partial \phi}$. Evaluate $\frac{\partial R_1^0}{\partial \phi}$ at $\theta = \frac{\pi}{2}$, $\phi = \frac{\phi}{2}$.

12. Use Equation (2.71) to show that

$$R_{\boldsymbol{k},\theta} \quad = \quad I + S(\boldsymbol{k})\sin(\theta) + S^2(\boldsymbol{k}) \operatorname{vers}(\theta).$$

13. Verify (5.25) by direct calculation.

14. Show that $S(\boldsymbol{k})^3 = -S(\boldsymbol{k})$. Use this and Problem 12 to verify Equation (5.26).

15. Given any square matrix A, the exponential of A is a matrix defined as

$$e^A \quad = \quad I + A + \frac{1}{2}A^2 + \frac{1}{3!}A^3 + \cdot$$

Given $S \in SS(3)$ show that $e^S \in SO(3)$.

[Hint: Verify the facts that $e^A e^B = e^{A+B}$ provided that A and B commute, that is, $AB = BA$, and also that $\det(e^A) = e^{Tr(A)}$.]

16. Show that $R_{\boldsymbol{k},\theta} = e^{S(\boldsymbol{k})\theta}$.

[Hint: Use the series expansion for the matrix exponential together with Problems 12 and 14. Alternatively use the fact that $R_{\boldsymbol{k},\theta}$ satisfies the differential equation

$$\frac{dR}{d\theta} \quad = \quad S(\boldsymbol{k})R.$$

17. Use Problem 16 to show the converse of Problem 15, that is, if $R \in SO(3)$ then there exists $S \in SS(3)$ such that $R = e^S$.

18. Given the Euler angle transformation

$$R \quad = \quad R_{z,\psi}R_{y,\theta}R_{z,\phi}$$

show that $\frac{d}{dt}R = S(\boldsymbol{\omega})R$ where

$$\boldsymbol{\omega} = \{c_\psi s_\theta \dot{\phi} - s_\psi \dot{\theta}\}\boldsymbol{i} + \{s_\psi s_\theta \dot{\phi} + c_\psi \dot{\theta}\}\boldsymbol{j} + \{\dot{[si} + c_\theta \dot{\phi}\}\boldsymbol{k}.$$

The components of $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$, respectively, are called the **nutation**, **spin**, and **precession**.

19. Repeat Problem 18 for the Roll-Pitch-Yaw transformation. In other words, find an explicit expression for $\boldsymbol{\omega}$ such that $\frac{d}{dt}R = S(\boldsymbol{\omega})R$, where R is given by (2.65).

20. Two frames $o_0x_0y_0z_0$ and $o_1x_1y_1z_1$ are related by the homogeneous transformation

$$H = \begin{bmatrix} 0 & -1 & 0 & 1 \\ 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

A particle has velocity $\boldsymbol{v}_1(t) = (3, 1, 0)^T$ relative to frame $o_1x_1y_1z_1$. What is the velocity of the particle in frame $o_0x_0y_0z_0$?

21. Three frames $o_0x_0y_0z_0$ and $o_1x_1y_1z_1$, and $o_2x_2y_2z_2$ are given below. If the angular velocities ω_1^0 and ω_2^1 are given as

$$\omega_1^0 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}; \qquad \omega_2^1 = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$$

what is the angular velocity ω_2^0 at the instant when

$$R_1^0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}.$$