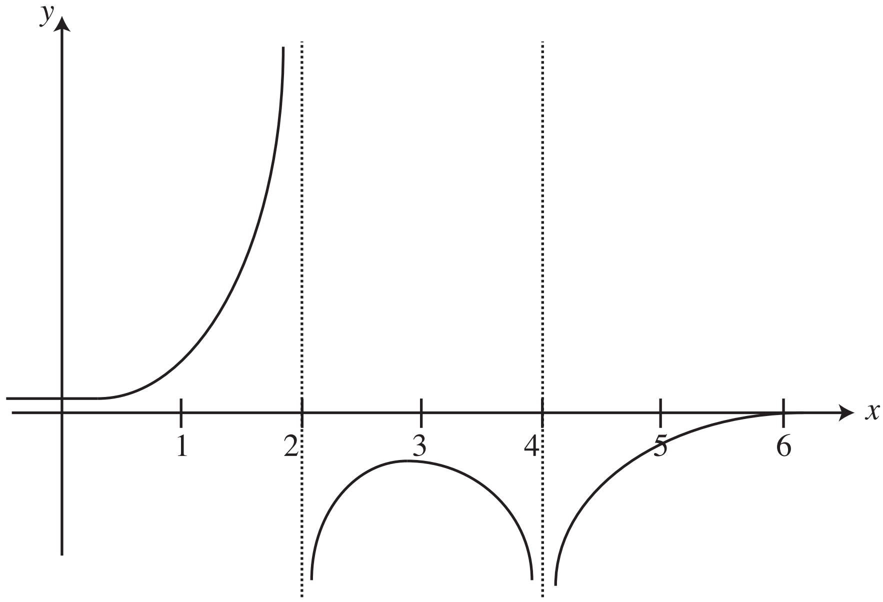
Questions related to SECTION 2.4

1. The graph of g in the figure has vertical asymptotes at $x = 2$ and $x = 4$. Find the following limits if possible.



(a) $\lim_{x\to 2^-} g(x) = \infty$

(b) $\lim_{x\to 2^+} g(x) = -\infty$

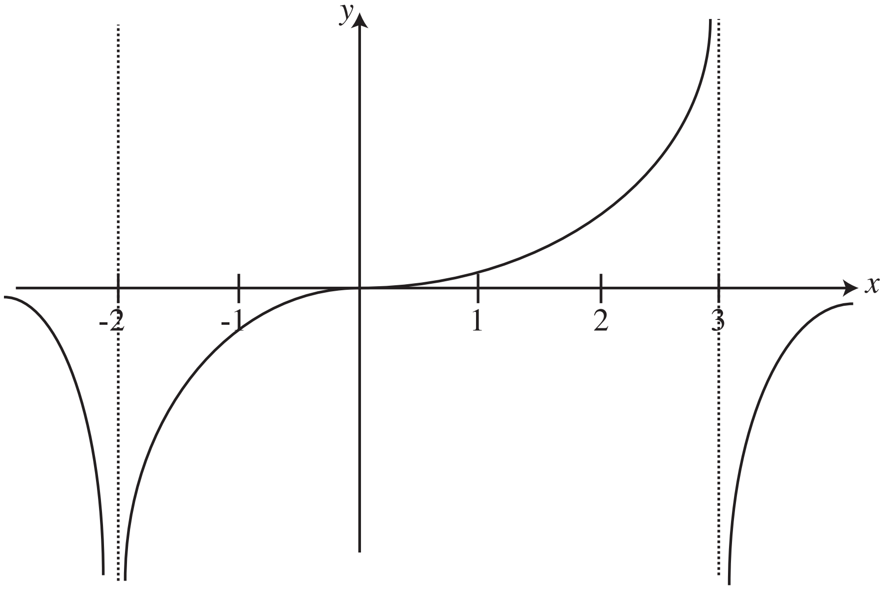
(c) $\lim_{x\to 2} g(x)$ does not exist

(d) $\lim_{x\to 4^-} g(x) = -\infty$

(e) $\lim_{x\to 4^+} g(x) = -\infty$

(f) $\lim_{x\to 4} g(x) = -\infty$

2. The graph of h in the figure has vertical asymptotes at $x = -2$ and $x = 3$. Find the following limits, if possible.



(a) $\lim_{x \to -2^-} h(x) = -\infty$

(b) $\lim_{x \to -2^+} h(x) = -\infty$

(c) $\lim_{x \to -2} h(x) = -\infty$

(d) $\lim_{x \to 3^-} h(x) = \infty$

(e) $\lim_{x \to 3^+} h(x) = -\infty$

(f) $\lim_{x \to 3} h(x)$ does not exist

3. Evaluate the following limits, or state that they do not exist.

(a) $\lim_{t \to 5} \frac{4t^2 - 100}{t - 5} = \lim_{t \to 5} \frac{4(t^2 - 25)}{t - 5} = \lim_{t \to 5} \frac{4(t-5)(t+5)}{t - 5} = \lim_{t \to 5} 4(t+5) = 40$

(b) $\lim_{x \to 1^+} \frac{x^2 - 5x + 6}{x - 1} = \lim_{x \to 1^+} \frac{(x-3)(x-2)}{x - 1} = \frac{2}{0} = \infty$

4. Find all vertical asymptotes, $x = a$, of the following functions. For each value of a , evaluate $\lim_{x\to a^+} f(x)$, $\lim_{x\to a^-} f(x)$ and $\lim_{x\to a} f(x)$

   (a) $f(x) = \dfrac{x^2 - 9x + 14}{x^2 - 5x + 6} = \dfrac{(x-2)(x-7)}{(x-3)(x-2)} = \dfrac{x-7}{x-3}$

   $x = 3$ makes the given function $f(x)$ undefined. Thus $x = 3$ is a vertical asymptote.

   $$\lim_{x\to 3^-} f(x) = \infty, \quad \lim_{x\to 3^+} f(x) = -\infty, \quad \Longrightarrow \lim_{x\to 3} f(x) \quad \text{does not exist}$$

   (b) $f(x) = \dfrac{x+1}{x^3 - 4x^2 + 4x} = \dfrac{x+1}{x(x^2 - 4x + 4)} = \dfrac{x+1}{x(x-2)^2}$

   $x = 0$ and $x = 2$ make the given function $f(x)$ undefined. Thus $x = 0$ and $x = 2$ are vertical asymptotes.

   $$\lim_{x\to 0^-} f(x) = -\infty, \quad \lim_{x\to 0^+} f(x) = \infty, \quad \Longrightarrow \lim_{x\to 0} f(x) \text{ does not exist}$$
   $$\lim_{x\to 2^-} f(x) = \infty, \quad \lim_{x\to 2^+} f(x) = \infty, \quad \Longrightarrow \lim_{x\to 2} f(x) = \infty$$

Questions related to SECTION 2.5

1. Evaluate $\lim_{x\to\infty} f(x)$ and $\lim_{x\to -\infty} f(x)$ for the following rational functions. Then give the horizontal asymptotes of f, (if any).

   (a) $f(x) = \dfrac{2x+1}{3x^4 - 2}$

   $$\lim_{x\to\infty} \frac{2x+1}{3x^4-2} = \lim_{x\to\infty} \frac{x^4\left(\frac{2}{x^3} + \frac{1}{x^4}\right)}{x^4\left(3 - \frac{2}{x^4}\right)} = \frac{0}{3} = 0$$

   Similarly

   $$\lim_{x\to -\infty} \frac{2x+1}{3x^4-2} = 0$$

   Thus the line $y = 0$ is a horizontal asymptote.

   (b) $f(x) = \dfrac{12x^8 - 3}{3x^8 - 2x^7}$

$$\lim_{x\to\infty}\frac{12x^8-3}{3x^8-2x^7}=\lim_{x\to\infty}\frac{x^8\left(12-\frac{3}{x^8}\right)}{x^8\left(3-\frac{2}{x}\right)}=\frac{12}{3}=4$$

Similarly

$$\lim_{x\to-\infty}\frac{12x^8-3}{3x^8-2x^7}=4$$

Thus the line $y=4$ is a horizontal asymptote.

(c) $f(x)=\dfrac{-x^3+1}{2x+8}$

$$\lim_{x\to\infty}\frac{-x^3+1}{2x+8}=\lim_{x\to\infty}\frac{x\left(-x^2+\frac{1}{x}\right)}{x\left(2+\frac{8}{x}\right)}=-\infty$$

Similarly

$$\lim_{x\to-\infty}\frac{-x^3+1}{2x+8}=-\infty$$

Thus there are no horizontal asymptotes.

(d) $f(x)=\dfrac{4x^3}{2x^3+\sqrt{9x^6+15x^4}}$

First of all note that

$$\sqrt{x^6}=\begin{cases}x^3, & \text{if } x>0\\ -x^3, & \text{if } x<0\end{cases}$$

Then we have

$$\lim_{x\to\infty}\frac{4x^3}{2x^3+\sqrt{9x^6+15x^4}}=\lim_{x\to\infty}\frac{4x^3}{2x^3+x^3\sqrt{9+\frac{15}{x^2}}}=\lim_{x\to\infty}\frac{4x^3}{x^3\left(2+\sqrt{9+\frac{15}{x^2}}\right)}$$

$$=\frac{4}{2+\sqrt{9}}=\frac{4}{5}$$

On the other hand

$$\lim_{x\to-\infty}\frac{4x^3}{2x^3+\sqrt{9x^6+15x^4}}=\lim_{x\to-\infty}\frac{4x^3}{2x^3-x^3\sqrt{9+\frac{15}{x^2}}}=\lim_{x\to-\infty}\frac{4x^3}{x^3\left(2-\sqrt{9+\frac{15}{x^2}}\right)}$$

$$=\frac{4}{2-\sqrt{9}}=-4$$

Thus $y = \dfrac{4}{5}$ is a horizontal asymptote as $x \to \infty$ and $y = -4$ is a horizontal asymptote as $x \to -\infty$

2. (a) Evaluate $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$ and then identify the horizontal asymptotes of f, (if any).

   (b) Find the vertical asymptotes. For each vertical asymptote $x = a$, evaluate $\lim_{x\to a^-} f(x)$ and $\lim_{x\to a^+} f(x)$.

i. $f(x) = 16x^2\left(4x^2 - \sqrt{16x^4+1}\right)$

First of all let us write the given function $f(x)$ as a rational function.

$$f(x) = 16x^2\left(4x^2 - \sqrt{16x^4+1}\right) \times \frac{4x^2+\sqrt{16x^4+1}}{4x^2+\sqrt{16x^4+1}} = \frac{-16x^2}{4x^2+\sqrt{16x^4+1}}$$

Then we have

$$\lim_{x\to\infty} \frac{-16x^2}{4x^2+\sqrt{16x^4+1}} = \lim_{x\to\infty} \frac{-16x^2}{4x^2+x^2\sqrt{16+\frac{1}{x^4}}} = \lim_{x\to\infty} \frac{-16x^2}{x^2\left(4+\sqrt{16+\frac{1}{x^4}}\right)} =$$
$$\frac{-16}{4+\sqrt{16}} = -2$$

Similarly

$$\lim_{x\to-\infty} f(x) = -2$$

Thus $y = -2$ is a horizontal asymptote.

Since $f(x)$ is defined for all values of x, $f(x)$ has no vertical asymptotes.

ii. $f(x) = \frac{x^2-9}{x(x-3)}$

$$\lim_{x\to\infty} \frac{x^2-9}{x(x-3)} = \lim_{x\to\infty} \frac{x+3}{x} = \lim_{x\to\infty} \frac{x\left(1+\frac{3}{x}\right)}{x} = 1$$

Similarly

$$\lim_{x\to-\infty} f(x) = 1$$

Thus $y = 1$ is a horizontal asymptote.

In order to find the vertical asymptotes we need to find the values of x that makes the function $f(x)$ undefined. For the given function $f(x)$ $x = 0$ is a point that makes the function undefined. Thus $x = 0$ is a vertical asymptote.Let us find $\lim_{x \to 0^-} f(x)$ and $\lim_{x \to 0^+} f(x)$.

$$\lim_{x \to 0^-} \frac{x+3}{x} = -\infty$$

$$\lim_{x \to 0^+} \frac{x+3}{x} = \infty$$

