Questions related to SECTION 2.6

1. Determine whether the following functions are continuous at a.

(a) $f(x) = \begin{cases} \frac{x^2-1}{x-1}, & \text{if } x \neq 1 \\ 3, & \text{if } x = 1 \end{cases} \qquad a = 1$

Any function $f(x)$ is continuous at a given point a iff

$$\lim_{x\to a} f(x) = f(a)$$

Thus firstly we need to find if $\lim_{x\to a} f(x)$ exists and then if the limit exists we will check if the condition for continuity is satisfied or not.

$$\lim_{x\to 1} \frac{x^2-1}{x-1} = \lim_{x\to 1}(x+1) = 2 \qquad \text{and} \qquad f(1) = 3$$

Since $\lim_{x\to 1} f(x) \neq f(1)$ we say that the given function is NOT continuous at $a = 1$.

(b) $f(x) = \begin{cases} \frac{x^2-4x+3}{x-3}, & \text{if } x \neq 3 \\ 2, & \text{if } x = 3 \end{cases} \qquad a = 3$

$$\lim_{x\to 3} \frac{x^2-4x+3}{x-3} = \lim_{x\to 3} \frac{(x-3)(x-1)}{x-3} = 2 \qquad \text{and} \qquad f(3) = 2$$

Since $\lim_{x\to 3} f(x) = f(3)$, $f(x)$ is continuous at $a = 3$.

2. Let

$$f(x) = \begin{cases} x^3+4x+1, & \text{if } x \leq 0 \\ 2x^3, & \text{if } x > 0 \end{cases}$$

(a) Use a continuity checklist to show that f is not continuous at 0.

From the given piece-wise defined function it is easy to see that $f(0) = 1$. Then Let us find $\lim_{x\to 0} f(x)$. Since the given function is defined for the values greater and less than 0 we need to find $\lim_{x\to 0^-} f(x)$ and $\lim_{x\to 0^+} f(x)$

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} (x^3 + 4x + 1) = 1$$

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} 2x^3 = 0$$

Since $\lim_{x\to 0^-} f(x) \neq \lim_{x\to 0^+} f(x)$ we say that the limit does not exist at $x = 0$. Thus the given function is NOT continuous at $x = 0$.

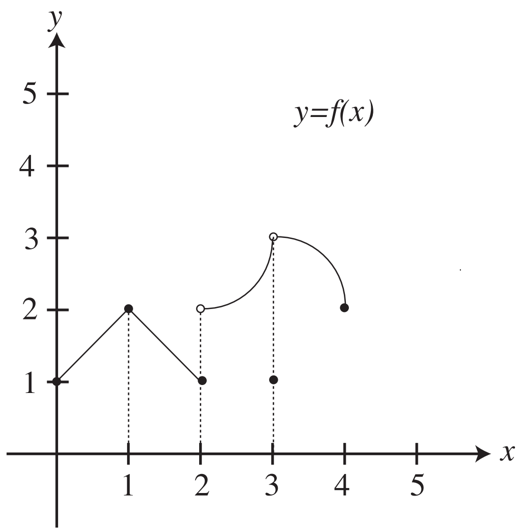
(b) Is f continuous from the left or right at 0?

Since $\lim_{x\to 0^-} f(x) = f(0) = 1$ we say that f is continuous from the LEFT at 0.

(c) State the interval(s) of continuity.

f is continuous on $(-\infty, 0]$ and $(0, \infty)$.

3. Determine the points at which the following function f has continuities and discontinuities.



$\lim_{x\to 1^-} f(x) = 2$ and $\lim_{x\to 1^+} f(x) = 2.$

Thus $\lim_{x\to 1} f(x) = 2.$ Also $f(1) = 2.$

Since $\lim_{x\to 1} f(x) = f(1) = 2$, f is continuous at $x = 1$

$\lim_{x\to 2^-} f(x) = 1$ and $\lim_{x\to 2^+} f(x) = 2.$

Since $\lim_{x\to 2^-} f(x) \neq \lim_{x\to 2^+} f(x)$ we say that $\lim_{x\to 2} f(x)$ does not exist.

Thus f has a discontinuity at $x = 2$

$\lim_{x\to 3^-} f(x) = 3$ and $\lim_{x\to 3^+} f(x) = 3.$

Thus $\lim_{x\to 3} f(x) = 3$. But on the other hand $f(3) = 1$.

Since $\lim_{x\to 3} f(x) \neq f(3)$, f is discontinuous at $x = 3$

Questions related to SECTION 3.1

1. (a) Find the slope of the line tangent to the graph of f at P.

   (b) Determine an equation of the tangent line at P.

   (c) Plot the graph of f and the tangent line at P.

   i. $f(x) = x^2 - 5$ at the point $P(3, 4)$

   $m_{tan} = \lim_{x\to x_0} \frac{y - y_0}{x - x_0} = \lim_{x\to 3} \frac{x^2 - 5 - 4}{x - 3} = \lim_{x\to 3} \frac{(x + 3)(x - 3)}{x - 3}$
   $= \lim_{x\to 3}(x + 3) = 6$

   Thus $m_{tan} = 6$.

   On the other hand we know that the equation of a line has the form

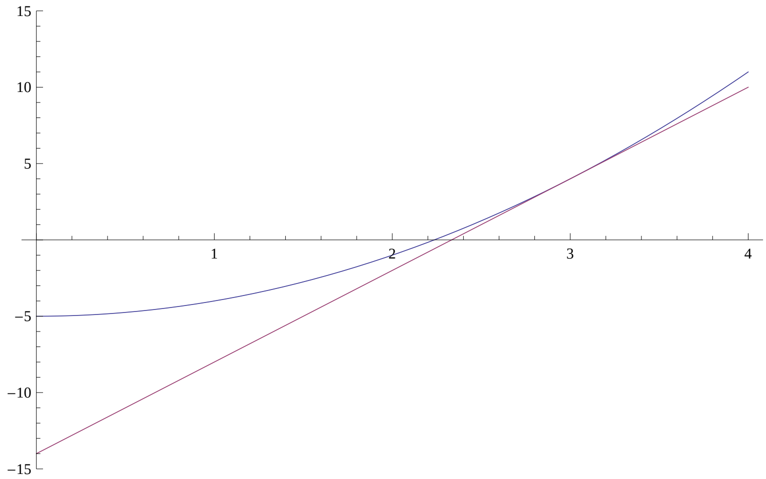
   $$y - y_0 = m(x - x_0)$$

   Then by substituting the values $m_{tan} = 6$, $x_0 = 3$ and $y_0 = 4$ to the equation, we get

   $$y - 4 = 6(x - 3)$$

   After rearranging we get the equation of the tangent line as

   $$y = 6x - 14$$


15
10
5
−5
−10
−15
1
2
3
4

ii. $f(x) = \frac{1}{x}$ P(-1,-1)

$m_{tan} = \lim_{x \to x_0} \frac{y - y_0}{x - x_0} = \lim_{x \to -1} \frac{\frac{1}{x} + 1}{x + 1} = \lim_{x \to -1} \frac{\frac{1+x}{x}}{x + 1}$
$= \lim_{x \to -1} \left(\frac{1+x}{x} \frac{1}{x+1} \right) = -1$

Thus $m_{tan} = -1$.

Since we know that the equation of a line has the form

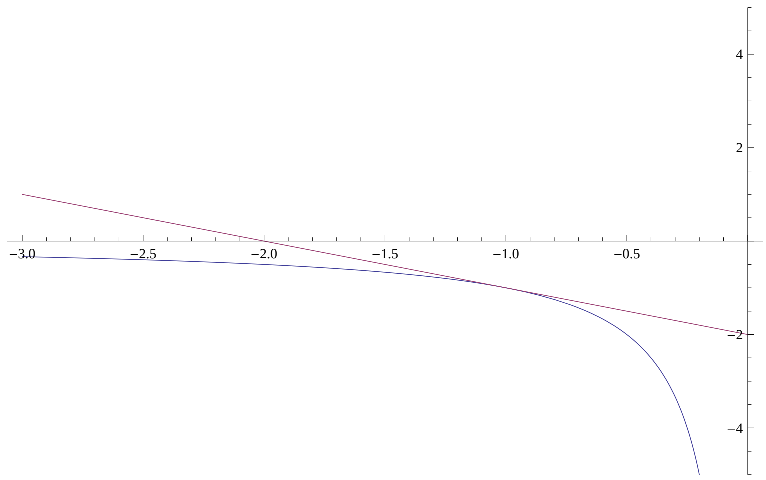
$$y - y_0 = m(x - x_0)$$

Then by substituting the values $m_{tan} = -1$, $x_0 = -1$ and $y_0 = -1$ to the equation we get

$$y + 1 = -1(x + 1)$$

After rearranging we get the equation of the tangent line as

$$y = -x - 2$$



2. (a) For the functions and points, find $f'(a)$.

(b) Determine an equation of the line tangent to the graph of f at $(a, f(a))$ for the given value of a.

i. $f(x) = \frac{1}{\sqrt{x}}$ $a = \frac{1}{4}$

First of all, let us write the limit definition of the derivative.

$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

Thus applying this to the given function we get

$$f'(\tfrac{1}{4}) = \lim_{h\to 0} \frac{\frac{1}{\sqrt{\frac{1}{4}+h}} - 2}{h} = \lim_{h\to 0} \frac{1-2\sqrt{\frac{1}{4}+h}}{h\sqrt{\frac{1}{4}+h}}$$

$$= \lim_{h\to 0} \left(\frac{1-2\sqrt{\frac{1}{4}+h}}{h\sqrt{\frac{1}{4}+h}} \times \frac{1+2\sqrt{\frac{1}{4}+h}}{1+2\sqrt{\frac{1}{4}+h}} \right) = \lim_{h\to 0} \frac{1-4\left(\frac{1}{4}+h\right)}{h\sqrt{\frac{1}{4}+h}\left(1+2\sqrt{\frac{1}{4}+h}\right)}$$

$$= \lim_{h\to 0} \frac{-4}{\sqrt{\frac{1}{4}+h}\left(1+2\sqrt{\frac{1}{4}+h}\right)} = \frac{-4}{\frac{1}{2}(1+1)} = -4$$

Thus we get that $f'\left(\dfrac{1}{4}\right) = -4$

Since we know that $m = f'(\frac{1}{4}) = -4$, from the equation of the line which is $y - y_0 = m(x - x_0)$ we have $y - 2 = -4(x - \frac{1}{4})$. Then by rearranging the equation we get $y = -4x + 3$

ii. $f(x) = \frac{1}{x^2}$ $\qquad a = 1$

Applying the limit definition of derivative to the given function we get

$$f'(1) = \lim_{h\to 0} \frac{\frac{1}{(1+h)^2} - 1}{h} = \lim_{h\to 0} \frac{1-(1+h)^2}{h(1+h)^2} = \lim_{h\to 0} \frac{1-1-2h-h^2}{h(1+h)^2}$$

$$= \lim_{h\to 0} \frac{-2-h}{(1+h)^2} = -2$$

Thus we get that $f'(1) = -2$

Since we know that $m = f'(1) = -2$, from the equation of the line which is $y - y_0 = m(x - x_0)$ we have $y - 1 = -2(x - 1)$. Then by rearranging the equation we get $y = -2x + 3$

Questions related to SECTION 3.2

1. Find the derivative of the following functions.

   (a) $f(x) = 10x^4 - 32x + e^2$
   $f'(x) = 40x^3 - 32$

(b) $f(t) = 6\sqrt{t} - 4t^3 + 9$
$f'(t) = \frac{3}{\sqrt{t}} - 12t^2$

2. Find the derivative of the following function by simplifying the expression.

$$h(x) = \frac{x^3 - 6x^2 + 8x}{x^2 - 2x}$$

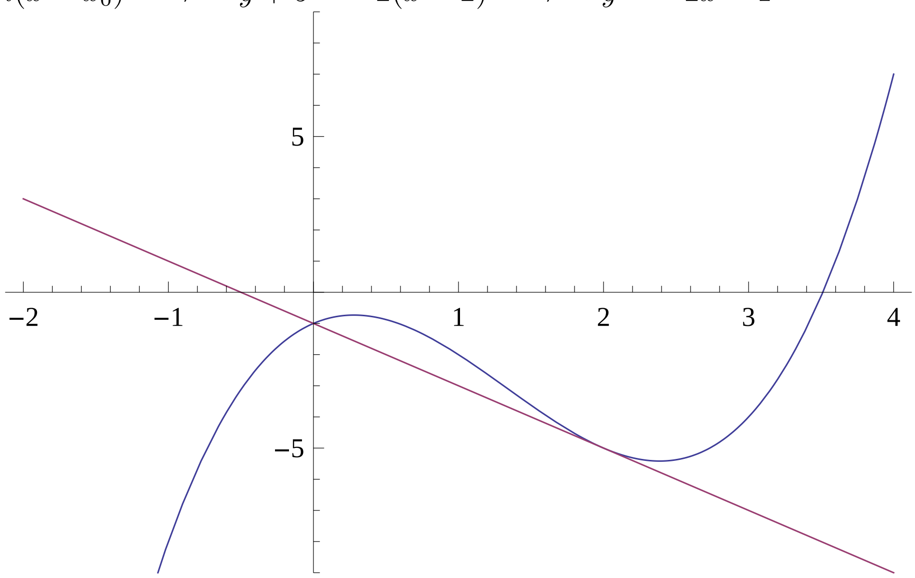
$$h(x) = \frac{x(x^2 - 6x + 8)}{x(x-2)} = \frac{(x-2)(x-4)}{x-2} = x - 4$$

$$h'(x) = 1$$

3. (a) Find the equation of the tangent line at $x = a$.
   (b) Graph the curve and the tangent line on the same set of axes.

$$y = x^3 - 4x^2 + 2x - 1 \qquad a = 2$$

$y' = 3x^2 - 8x + 2$
Slope of the tangent line at $a = 2$ is :
$m_{tan} = y'(2) = 3(2)^2 - 8(2) + 2 = -2$
Equation of the tangent line at $(2, -5)$ is:
$y - y_0 = m(x - x_0) \quad \Rightarrow \quad y + 5 = -2(x-2) \quad \Rightarrow \quad y = -2x - 1$



Questions related to SECTION 3.3

1. Find the derivative of the following functions.

   (a) $f(x) = 3x^4(2x^2 - 1)$

   $$f'(x) = 12x^3(2x^2 - 1) + 4x(3x^4) = 24x^5 - 12x^3 + 12x^5 = 36x^5 - 12x^3$$

   (b) $f(x) = \left(1 + \frac{1}{x^2}\right)(x^2 + 1)$

   $$f'(x) = -2x^{-3}(x^2 + 1) + 2x\left(1 + \frac{1}{x^2}\right) = -\frac{2}{x} - \frac{2}{x^3} + 2x + \frac{2}{x} = 2x - \frac{2}{x^3}$$

   (c) $s(t) = 4e^t\sqrt{t}$

   $$s'(t) = 4e^t t^{\frac{1}{2}} + \frac{1}{2}t^{\frac{-1}{2}}4e^t = 4e^t\sqrt{t} + \frac{2e^t}{\sqrt{t}} = e^t\left(4\sqrt{t} + \frac{2}{\sqrt{t}}\right)$$

   (d) $f(x) = \dfrac{x}{x+1}$

   $$f'(x) = \frac{(x+1) - x}{(x+1)^2} = \frac{1}{(x+1)^2}$$

   (e) $g(x) = \dfrac{e^x}{x^2 - 1}$

   $$g'(x) = \frac{e^x(x^2 - 1) - 2xe^x}{(x^2 - 1)^2} = \frac{e^x(x^2 - 2x - 1)}{(x^2 - 1)^2}$$

   (f) $p(x) = \dfrac{4x^3 + 3x + 1}{2x^5}$

   $$p'(x) = \frac{(12x^2 + 3)2x^5 - 10x^4(4x^3 + 3x + 1)}{(2x^5)^2} = \frac{24x^7 + 6x^5 - 40x^7 - 30x^5 - 10x^4}{4x^{10}} =$$

   $$= -\frac{4}{x^3} - \frac{6}{x^5} - \frac{5}{2x^6}$$

