Questions related to SECTION 3.4

1. Evaluate the following limits.

   Note that:

$$\lim_{x\to 0}\frac{\sin(x)}{x}=1 \quad and \quad \lim_{x\to 0}\frac{\cos(x)-1}{x}=0$$

   (a) $\lim_{x\to 0}\frac{\sin(5x)}{3x}$

$$\lim_{x\to 0}\frac{\sin(5x)}{3x}=\frac{1}{3}\lim_{x\to 0}\frac{\sin(5x)}{x}=\frac{5}{3}\lim_{x\to 0}\frac{\sin(5x)}{5x}=\frac{5}{3}\cdot 1=\frac{5}{3}$$

   (b) $\lim_{\theta\to 0}\frac{\cos^2\theta-1}{\theta}$

$$\lim_{\theta\to 0}\frac{\cos^2\theta-1}{\theta}=\lim_{\theta\to 0}\left[\frac{(\cos\theta-1)(\cos\theta+1)}{\theta}\right]=$$

$$\lim_{\theta\to 0}(\cos\theta+1)\lim_{\theta\to 0}\frac{\cos\theta-1}{\theta}=2\cdot 0=0$$

   (c) $\lim_{x\to 0}\frac{\tan(7x)}{\sin(x)}$

$$\lim_{x\to 0}\frac{\tan(7x)}{\sin(x)}=\lim_{x\to 0}\left[\frac{\sin(7x)}{\cos(7x)\sin(x)}\right]=\lim_{x\to 0}\left(\frac{1}{\cos(7x)}\frac{x}{\sin(x)}\frac{7\sin(7x)}{7x}\right)=$$

$$=7\lim_{x\to 0}\frac{1}{\cos(7x)}\cdot\lim_{x\to 0}\frac{x}{\sin(x)}\cdot\lim_{x\to 0}\frac{\sin(7x)}{7x}=7\cdot 1\cdot 1\cdot 1=7$$

2. Calculate the following derivatives by using the product rule.

   (a) $\frac{d}{dx}(\sin^2 x)=$

$$\frac{d}{dx}(\sin^2 x)=\frac{d}{dx}(\sin(x)\cdot\sin(x))=\cos(x)\sin(x)+\cos(x)\sin(x)=2\sin(x)\cos(x)$$

   (b) $\frac{d}{dx}(\sin^3 x)=$

$$\frac{d}{dx}(\sin^3 x)=\frac{d}{dx}(\sin^2(x)\cdot\sin(x))=2\sin(x)\cos(x)\sin(x)+\cos(x)\sin^2(x)=3\sin^2(x)\cos(x)$$

   (c) $\frac{d}{dx}(\sin^4 x)=$

$$\frac{d}{dx}(\sin^4 x)=\frac{d}{dx}(\sin^3(x)\cdot\sin(x))=3\sin^2(x)\cos(x)\sin(x)+\cos(x)\sin^3(x)=$$

$$=4\sin^3(x)\cos(x)$$

Questions related to SECTION 3.6

1. Use chain rule to calculate $\frac{dy}{dx}$

   (a) $y = (5x^2 + 11x)^{20}$

   Let $u = 5x^2 + 11x$ then $y = u^{20}$

   $$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} \quad \Rightarrow \quad \frac{dy}{dx} = 20(5x^2 + 11x)^{19}(10x + 11)$$

   (b) $y = \sec(e^x)$

   Let $u = e^x$ then $y = \sec(u)$

   $$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} \quad \Rightarrow \quad \frac{dy}{dx} = \sec(e^x)\tan(e^x)e^x$$

   (c) $y = \cos^4(7x^3)$

   Let $u = 7x^3$ then $y = \cos^4(u)$

   $$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} \quad \Rightarrow \frac{dy}{dx} = 4\cos^3(7x^3)(-\sin(7x^3))21x^2 = -84x^2\cos^3(7x^3)\sin(7x^3)$$

   (d) $y = \tan(e^{\sqrt{3x}})$

   Let $u = e^{\sqrt{3x}}$ then $y = \tan(u)$

   $$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} \quad \Rightarrow \quad \frac{dy}{dx} = \sec^2(e^{\sqrt{3x}})\frac{3e^{\sqrt{3x}}}{2\sqrt{3x}}$$

   (e) $y = e^{-x^2}$

   Let $u = -x^2$ then $y = e^u$

   $$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} \quad \Rightarrow \quad \frac{dy}{dx} = -2xe^{-x^2}$$

(f) $y = e^{\tan x}$

Let $u = \tan x$ then $y = e^u$

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} \quad \Rightarrow \quad \frac{dy}{dx} = e^{\tan x}\sec^2 x$$

(g) $y = \sin(4x^3 + 3x + 1)$

Let $u = 4x^3 + 3x + 1$ then $y = \sin(u)$

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} \quad \Rightarrow \quad \frac{dy}{dx} = (12x^2 + 3)\cos(4x^3 + 3x + 1)$$

(h) $y = x^2 \sec(5x)$

$$\frac{dy}{dx} = (x^2)' \sec(5x) + (\sec(5x))' x^2$$

In order to find $(\sec(5x))'$, let $v = \sec(5x)$.

Then let $u = 5x$ thus $v = \sec(u)$

which gives us

$$\frac{dv}{dx} = \frac{dv}{du}\frac{du}{dx} \quad \Rightarrow \quad \frac{dv}{dx} = 5\sec(5x)\tan(5x)$$

Finally we get that

$$\frac{dy}{dx} = 2x\sec(5x) + 5x^2\sec(5x)\tan(5x)$$

Questions related to SECTION 3.7

1. Use implicit differentiation to find $\dfrac{dy}{dx}$.

   (a) $e^{xy} = 2y$

   $(xy)'e^{xy} = 2y' \quad \Rightarrow \quad (y + xy')e^{xy} = 2y' \quad \Rightarrow \quad ye^{xy} + xe^{xy}y' = 2y'$

   $\Rightarrow ye^{xy} = (2 - xe^{xy})y' \quad \Rightarrow \quad y' = \dfrac{ye^{xy}}{2 - xe^{xy}}$

   (b) $x^3 = \dfrac{x+y}{x-y}$

   $3x^2 = \dfrac{(1+y')(x-y) - (1-y')(x+y)}{(x-y)^2}$

   $\Rightarrow \quad 3x^2(x-y)^2 = x - y + xy' - yy' - x - y + xy' + yy'$

   $\Rightarrow \quad 3x^2(x-y)^2 = 2xy' - 2y \quad \Rightarrow \quad y' = \dfrac{3x^2(x-y)^2 + 2y}{2x}$

   (c) $(xy+1)^3 = x - y^2 + 8$

   $3(xy+1)^2(y + xy') = 1 - 2yy' \quad \Rightarrow \quad 3xy'(xy+1)^2 + 2yy' = 1 - 3y(xy+1)^2$

   $\Rightarrow y'(3x(xy+1)^2 + 2y) = 1 - 3y(xy+1)^2 \quad \Rightarrow \quad y' = \dfrac{1 - 3y(xy+1)^2}{3x(xy+1)^2 + 2y}$

2. (a) Verify that the given point lies on the curve.
   (b) Determine an equation of the line tangent to the curve at the given point.

$$x^3 + y^3 = 2xy \qquad (1,1)$$

   To check if the point lies on the curve or not let's substitude the given point to the function.

   $1^3 + 1^3 = 2(1)(1) \quad \Rightarrow \quad 2 = 2$ Thus the point lies on the curve.

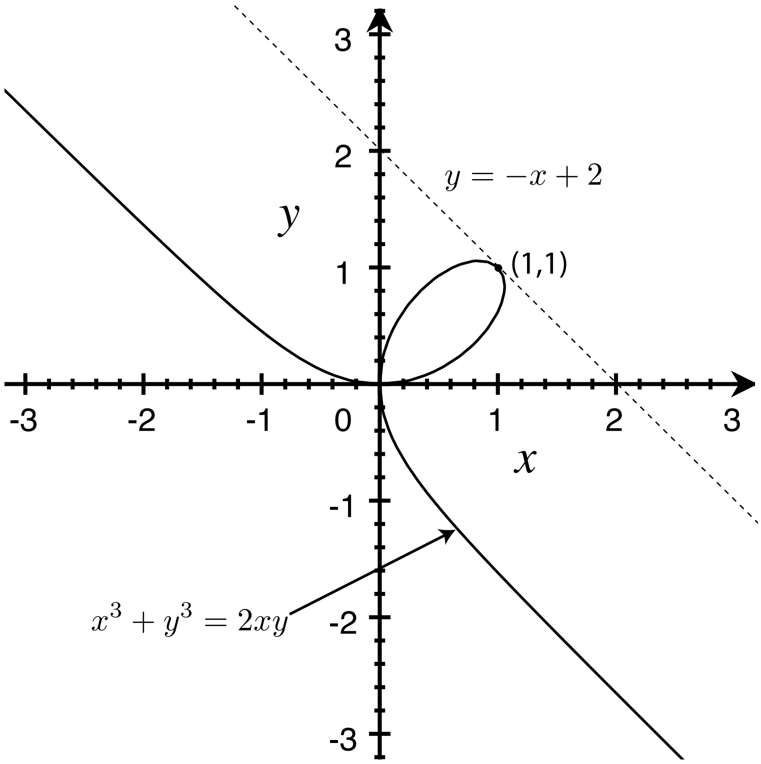
$$3x^2 + 3y^2y' = 2y + 2xy' \quad \Rightarrow \quad 3y^2y' - 2xy' = 2y - 3x^2$$

$$y'(3y^2 - 2x) = 2y - 3x^2 \quad \Rightarrow \quad y' = \frac{2y - 3x^2}{3y^2 - 2x}$$

$$m_{tan} = y'\Big|_{(1,1)} = \frac{2(1) - 3(1)^2}{3(1)^2 - 2(1)} = -1$$

Thus from the equation of the line we get :

$$y - y_0 = m_{tan}(x - x_0) \quad \Rightarrow \quad y - 1 = -1(x - 1) \quad \Rightarrow \quad y = -x + 2$$



3. Find $\dfrac{d^2y}{dx^2}$

   (a) $x^4 + y^4 = 64$

$$4x^3 + 4y^3y' = 0 \quad \Rightarrow \quad y' = -\frac{x^3}{y^3}$$

$$y'' = \frac{-3x^2y^3 + 3y^2y'x^3}{y^6} \quad \Rightarrow \quad y'' = \frac{3y'x^3 - 3x^2y}{y^4} \quad \Rightarrow \quad y'' = \frac{3x^3\left(-\frac{x^3}{y^3}\right) - 3x^2y}{y^4}$$

$$\Rightarrow \quad y'' = \frac{-3x^6 - 3x^2y^4}{y^7}$$

(b) $e^{2y} + x = y$

$$2y'e^{2y} + 1 = y' \quad \Rightarrow \quad 1 = (1 - 2e^{2y})y' \quad \Rightarrow \quad y' = \frac{1}{1 - 2e^{2y}}$$

$$y^{''} = \frac{4y'e^{2y}}{(1 - 2e^{2y})^2} = \frac{4\left(\frac{1}{1-2e^{2y}}\right)e^{2y}}{(1 - 2e^{2y})^2} = \frac{4e^{2y}}{(1 - 2e^{2y})^3}$$

4. (a) Use implicit differentiation to find the derivative $\dfrac{dy}{dx}$.

   (b) Find the slope of the curve at the given point.

   i. $y^2 + 3x = 2 \qquad (-1, \sqrt{5})$

   $2yy' + 3 = 0$

   $y' = \dfrac{-3}{2y}$

   slope $= y'\Big|_{(-1,\sqrt{5})} = \dfrac{-3}{2\sqrt{5}} = \dfrac{-3\sqrt{5}}{10}$

   ii. $5\sqrt{x} - 10\sqrt{y} = \sin(x) \qquad (4\pi, \pi)$

   $\dfrac{5}{2\sqrt{x}} - \dfrac{5y'}{\sqrt{y}} = \cos(x)$

   $\dfrac{5y'}{\sqrt{y}} = \dfrac{5}{2\sqrt{x}} - \cos(x) \quad \Rightarrow \quad y' = \dfrac{\sqrt{y}}{5}\left(\dfrac{5}{2\sqrt{x}} - \cos(x)\right)$

   slope $= y'\Big|_{(4\pi,\pi)} = \dfrac{\sqrt{\pi}}{5}\left(\dfrac{5}{2\sqrt{4\pi}} - \cos(4\pi)\right) = \dfrac{\sqrt{\pi}}{5}\dfrac{5}{2\cdot 2\sqrt{\pi}} - \dfrac{\sqrt{\pi}}{5} = \dfrac{1}{4} - \dfrac{\sqrt{\pi}}{5}$

Questions related to SECTION 3.8

1. Find the following derivatives.

   (a) $\dfrac{d}{dx}[\ln(\cos^2 x)]$

$$\frac{d}{dx}[\ln(\cos^2 x)] = \frac{1}{\cos^2 x}(-2\cos(x)\sin(x)) = -2\tan(x)$$

   (b) $\dfrac{d}{dx}[\ln(\ln(x))]$

$$\frac{d}{dx}[\ln(\ln(x))] = \frac{1}{\ln(x)}\frac{1}{x} = \frac{1}{x\ln(x)}$$

(c) $\frac{d}{dx}[\ln(x^2)]$

$$\frac{d}{dx}[\ln(x^2)] = \frac{1}{x^2}2x = \frac{2}{x}$$

(d) $\frac{d}{dx}[e^x \ln(x)]$

$$\frac{d}{dx}[e^x \ln(x)] = e^x \ln(x) + \frac{1}{x}e^x$$

2. Find the derivatives of the following functions.

   (a) $y = 5 \cdot 4^x$

   Note that if:

$$v = 4^x \Rightarrow \ln(v) = \ln(4^x) \Rightarrow \ln(v) = x\ln(4) \Rightarrow \frac{1}{v}v' = \ln(4) \Rightarrow v' = 4^x \ln(4)$$

   Then $\quad y' = 5\ln(4)4^x$

   (b) $y = 4^{-x}\sin(x)$

   By using the same arguments as in part (a) we get:

$$y' = -\ln(4)4^{-x}\sin(x) + \cos(x)4^{-x}$$

   (c) $y = x^3 3^x$

   By using the same arguments as in part (a) we get:

$$y' = 3x^2 3^x + x^3 \ln(3)3^x$$

   (d) $y = \ln(10^x)$

   By using the same arguments as in part (a) we get:

$$y' = \frac{1}{10^x}10^x \ln(10) \quad \Rightarrow \quad y' = \ln(10)$$

   (e) $y = x\ln(10)$

   $y' = \ln(10)$

3. Use logarithmic differentiation to evaluate $f'(x)$.

(a) $f(x) = \dfrac{(x+1)^{\frac{3}{2}}(x-4)^{\frac{5}{2}}}{(5x+3)^{\frac{2}{3}}}$

Applying $\ln$ on both sides of the equation we get :

$$\begin{aligned} \ln(f(x)) &= \ln\left(\frac{(x+1)^{\frac{3}{2}}(x-4)^{\frac{5}{2}}}{(5x+3)^{\frac{2}{3}}}\right) \\ \ln(f(x)) &= \frac{3}{2}\ln(x+1) + \frac{5}{2}\ln(x-4) - \frac{2}{3}\ln(5x+3) \end{aligned}$$

Differentiating both sides with respect to x gives:

$$\begin{aligned} \frac{1}{f(x)}f'(x) &= \frac{3}{2(x+1)} + \frac{5}{2(x-4)} - \frac{10}{3(5x+3)} \\ f'(x) &= f(x) \cdot \left(\frac{3}{2(x+1)} + \frac{5}{2(x-4)} - \frac{10}{3(5x+3)}\right) \\ f'(x) &= \frac{(x+1)^{\frac{3}{2}}(x-4)^{\frac{5}{2}}}{(5x+3)^{\frac{2}{3}}}\left[\frac{3}{2(x+1)} + \frac{5}{2(x-4)} - \frac{10}{3(5x+3)}\right] \end{aligned}$$

(b) $f(x) = \left(1 + \dfrac{1}{x}\right)^{2x}$

Applying $\ln$ on both sides of the equation we get :

$$\begin{aligned} \ln(f(x)) &= \ln\left(1+\frac{1}{x}\right)^{2x} \\ \ln(f(x)) &= 2x\ln\left(1+\frac{1}{x}\right) \end{aligned}$$

Differentiating both sides of the equation with respect to x:

$$\begin{aligned} \frac{1}{f(x)}f'(x) &= 2\ln\left(1+\frac{1}{x}\right) + 2x\frac{1}{1+\frac{1}{x}}\left(-\frac{1}{x^2}\right) \\ f'(x) &= f(x)\cdot\left(2\ln\left(1+\frac{1}{x}\right) - \frac{2}{x+1}\right) \\ f'(x) &= \left(1+\frac{1}{x}\right)^{2x}\left(2\ln\left(1+\frac{1}{x}\right) - \frac{2}{x+1}\right) \end{aligned}$$

(c) $f(x) = \ln(\sec^4 x\ \tan^2 x)$

$$\begin{aligned} f(x) &= \ln(\sec^4 x) + \ln(\tan^2 x) \\ f(x) &= 4\ln(\sec\ x) + 2\ln(\tan\ x) \end{aligned}$$

Differentiating both sides of the equation with respect to x:

$$\begin{aligned} f'(x) &= 4\frac{1}{\sec\, x}\sec\, x\tan\, x + 2\frac{1}{\tan\, x}\sec^2 x \\ f'(x) &= 4\tan\, x + 2\frac{\cos\, x}{\sin\, x}\frac{1}{\cos^2 x} \\ f'(x) &= 4\tan\, x + 2\frac{1}{\sin\, x}\frac{1}{\cos\, x} \\ f'(x) &= 4\tan\, x + 2\csc\, x\sec\, x \end{aligned}$$

(d) $f(x) = x^{\ln(x)}$

Applying $\ln$ on both sides of the equation we get :

$$\begin{aligned} \ln(f(x)) &= \ln\left(x^{\ln(x)}\right) \\ \ln(f(x)) &= \ln(x)\cdot\ln(x) \\ \ln(f(x)) &= (\ln(x))^2 \end{aligned}$$

Differentiating both sides with respect to x gives:

$$\begin{aligned} \frac{1}{f(x)}f'(x) &= 2\ln(x)\frac{1}{x} \\ f'(x) &= f(x)\cdot 2\ln(x)\frac{1}{x} \\ f'(x) &= x^{\ln(x)}\left(\frac{2\ln(x)}{x}\right) \end{aligned}$$

(e) $f(x) = \dfrac{x^8\cos^3 x}{\sqrt{x-1}}$

Applying $\ln$ on both sides of the equation we get :

$$\begin{aligned} \ln(f(x)) &= \ln\left(\frac{x^8\cos^3 x}{\sqrt{x-1}}\right) \\ \ln(f(x)) &= 8\ln(x) + 3\ln(\cos(x)) - \frac{1}{2}\ln(x-1) \end{aligned}$$

Differentiating both sides of the equation with respect to x:

$$\begin{aligned} \frac{1}{f(x)}f'(x) &= \frac{8}{x} - \frac{3\sin(x)}{\cos(x)} - \frac{1}{2(x-1)} \\ f'(x) &= f(x)\cdot\left(\frac{8}{x} - \frac{3\sin(x)}{\cos(x)} - \frac{1}{2(x-1)}\right) \\ f'(x) &= \frac{x^8\cos^3 x}{\sqrt{x-1}}\left[\frac{8}{x} - \frac{3\sin(x)}{\cos(x)} - \frac{1}{2(x-1)}\right] \end{aligned}$$

Questions related to SECTION 3.9

1. Evaluate the derivatives of the following functions.

   (a) $f(x) = \sin^{-1}(2x)$

   $$f'(x) = \frac{1}{\sqrt{1-(2x)^2}} \cdot 2 = \frac{2}{\sqrt{1-4x^2}}$$

   (b) $f(x) = \sin^{-1}(e^{\sin(x)})$

   $$f'(x) = \frac{1}{\sqrt{1-(e^{\sin(x)})^2}} \cdot (\cos(x)e^{\sin(x)}) = \frac{\cos(x)e^{\sin(x)}}{\sqrt{1-e^{2\sin(x)}}}$$

   (c) $f(x) = \sec^{-1}(\ln(x))$

   $$f'(x) = \frac{1}{|\ln x|\sqrt{(\ln x)^2-1}} \cdot \frac{1}{x} = \frac{1}{x|\ln x|\sqrt{(\ln x)^2-1}}$$

   (d) $f(y) = \tan^{-1}(2y^2-4)$

   $$f'(y) = \frac{1}{1+(2y^2-4)^2} \cdot 4y = \frac{4y}{1+(2y^2-4)^2}$$

   (e) $f(z) = \cot^{-1}(\sqrt{z})$

   $$f'(z) = -\frac{1}{1+(\sqrt{z})^2} \cdot \frac{1}{2\sqrt{z}} = \frac{-1}{2\sqrt{z}(1+z)}$$

   (f) $f(t) = \ln(\tan^{-1} t)$

   $$f'(t) = \frac{1}{\tan^{-1} t} \cdot \frac{d}{dt}(\tan^{-1} t) = \frac{1}{\tan^{-1} t} \cdot \frac{1}{1+t^2}$$

   (g) $f(x) = \csc^{-1}(\tan e^x)$

   $$f'(x) = -\frac{1}{|\tan e^x|\sqrt{(\tan e^x)^2-1}} \cdot e^x \sec^2 e^x = -\frac{e^x \sec^2 e^x}{|\tan e^x|\sqrt{(\tan e^x)^2-1}}$$