# Examples for Graph Sketching

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## 1 Graph of a Polynomial

Sketch the graph of $y = x^4 - 12x^3 + 48x^2 - 64x = x(x-4)^3$.

1. Domain: $x \in \mathbb{R}$
2. $x$- and $y$-intercepts:

   **$x$-intercept:**
   $$y = 0 \quad \Longleftrightarrow x(x-4)^3 = 0 \Longleftrightarrow x = 0 \text{ or } x = 4.$$
   So the $x$-intercepts are: $(0,0)$ and $(4,0)$.

   **$y$-intercept:**
   $$x = 0 \Longrightarrow y = 0$$
   So the $y$-intercept is: $(0,0)$

3. Critical Points and Local Extreme Values:

   (a) Find $f'(x)$
   $$f'(x) = (x-4)^3 + x \cdot 3(x-4)^2 = (x-4+3x)\,(x-4)^2 = 4(x-1)(x-4)^2$$

   (b) Find all critical points:
   $f'(x) = 0$**:**
   $$f'(x) = 4(x-1)(x-4)^2 = 0 \Longleftrightarrow x = 1 \text{ or } x = 4.$$
   $f'(x)$ **is not defined:** $f'(x)$ is defined for all real $x$.

   (c) Increasing, Decreasing intervals:

   | | $-\infty < x <$ | $1$ | $< x <$ | $4$ | $< x < \infty$ |
   |---|---|---|---|---|---|
   | $f'(x)$ | $-$ | $0$ | $+$ | $0$ | $+$ |
   | $f(x)$ | $\searrow$ | loc. min. | $\nearrow$ | | $\nearrow$ |

   (d) Evaluate $f(x)$ at each critical point:
   local min at $(1, -27)$.
   We have neither a local maximum or Minimum at $(4,0)$.

4. Determine Concavity and Points of Inflection:

   (a) Find $f''(x)$

$$f''(x) = 4(x-4)^2 + 4(x-1)\cdot 2(x-4) = 4(x-4)(x-4+2x-2) =$$
$$= 4(x-4)(3x-6) = 12(x-2)(x-4)$$

   (b) Find all potential points of inflection:

   $f''(x) = 0$**:**

$$f''(x) = 0 \iff 12(x-2)(x-4) = 0 \iff x = 2 \text{ or } x = 4$$

   $f''(x)$ **is not defined:** As $f(x)$ is a polynomial $f''(x)$ is defined for all real x.

   (c) Intervals of concavity

| | $-\infty < x <$ | 2 | $< x <$ | 4 | $< x < \infty$ |
|---|---|---|---|---|---|
| $f''(x)$ | $+$ | 0 | $-$ | 0 | $+$ |
| $f(x)$ | $\cup$ | infl. point | $\cap$ | infl. point | $\cup$ |

   (d) The points of inflection are: $(2, -16)$ and $(4, 0)$

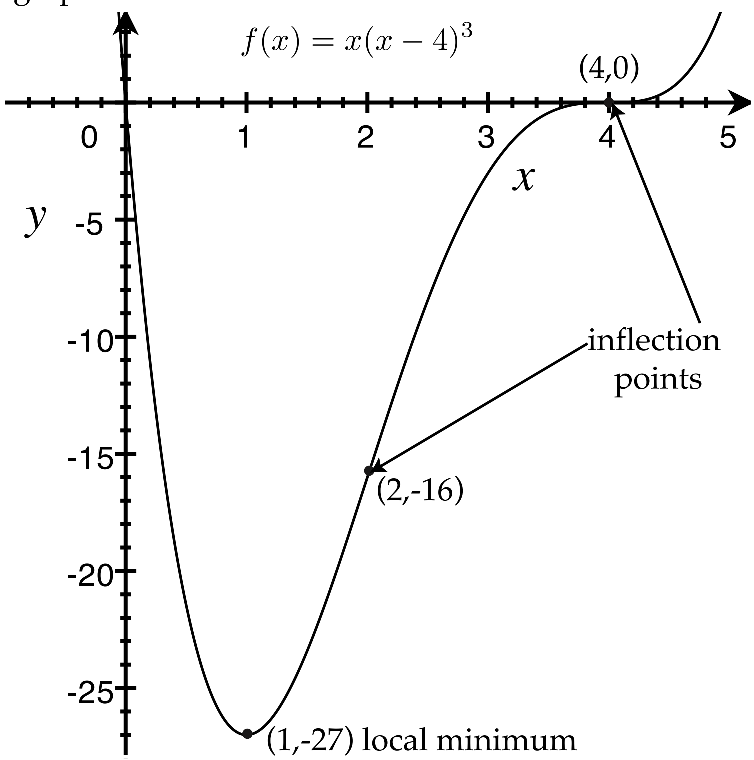
5. Asymptotes

   **Horizontal Asymptote:** None

   **Vertical Asymptote:** None

   **Oblique Asymptote:** None

6. Sketch the graph:




7. The range of $f(x)$ is $[-27, \infty)$.

## 2 Graph of a function with vertical and horizontal asymptotes

Sketch the graph of $y = \dfrac{x^2-1}{x^2-4}$

1. Domain: $x \in \mathbb{R}\backslash\{-2, 2\}$

2. $x$- and $y$-intercepts:

   **$x$-intercept:**

   $$y = 0 \quad \Longleftrightarrow x^2 - 1 = 0 \Longleftrightarrow x = -1 \text{ or } x = 1.$$

   So the $x$-intercepts are: $(-1, 0)$ and $(1, 0)$.

   **$y$-intercept:**

   $$x = 0 \Longrightarrow y = \frac{1}{4}$$

   So the $y$-intercept is: $\left(0, \frac{1}{4}\right)$

3. Critical Points and Local Extreme Values:

   (a) Find $f'(x)$

   $$f'(x) = \frac{2x(x^2-4) - (x^2-1)2x}{(x^2-4)^2} = -\frac{6x}{(x^2-4)^2}$$

   (b) Find all critical points:

   $f'(x) = 0$**:**

   $$f'(x) = -\frac{6x}{(x^2-4)^2} = 0 \Longleftrightarrow 6x = 0 \Longleftrightarrow x = 0.$$

   $f'(x)$ **is not defined:** $f'(x)$ is defined at $x = \pm 2$, which are outside the domain and therefore not critical points.

   (c) Increasing, Decreasing intervals:

| | $-\infty < x <$ | $-2$ | $< x <$ | $0$ | $< x <$ | $2$ | $< x < \infty$ |
|---|---|---|---|---|---|---|---|
| $f'(x)$ | $+$ | n.d. | $+$ | $0$ | - | n.d. | $-$ |
| $f(x)$ | $\nearrow$ | | $\nearrow$ | loc. max. | $\searrow$ | | $\searrow$ |

   (d) Evaluate $f(x)$ at each critical point:
   local max at $\left(0, \frac{1}{4}\right)$.
   We have neither a local maximum or minimum at $x = \pm 2$.

4. Determine Concavity and Points of Inflection:

   (a) Find $f''(x)$

$$f''(x) = -\frac{6\left(x^2-4\right)^2 - 6x \cdot 2\left(x^2-4\right)2x}{\left(x^2-4\right)^4} = -\frac{\left(x^2-4\right)\left(6\left(x^2-4\right)-24x^2\right)}{\left(x^2-4\right)^4} =$$
$$= \frac{6\left(3x^2+4\right)}{\left(x^2-4\right)^3}$$

   (b) Find all potential points of inflection:

   $f''(x) = 0$**:** There is no $x$ in the domain, s.t. $f''(x) = 0$

   $f''(x)$ **is not defined:** $f''(x)$ is not defined at $x = \pm 2$, which are outside the domain and therefore no possible inflection points.

   (c) Intervals of concavity

| | $-\infty < x <$ | $-2$ | $< x <$ | $2$ | $< x < \infty$ |
|---|---|---|---|---|---|
| $f''(x)$ | $+$ | n.d. | $-$ | n.d. | $+$ |
| $f(x)$ | $\cup$ | | $\cap$ | | $\cup$ |

   (d) There are no inflection points.

5. Asymptotes

   **Horizontal Asymptote:**

$$\lim_{x\to\pm\infty} \frac{x^2-1}{x^2-4} = \lim_{x\to\pm\infty} \frac{x^2\left(1-\frac{1}{x^2}\right)}{x^2\left(1-\frac{4}{x^2}\right)} = \lim_{x\to\pm\infty} \frac{1-\frac{1}{x^2}}{1-\frac{4}{x^2}} = 1$$

   So we have a horizontal asymptote $y = 1$.

   **Vertical Asymptote:** Vertical asymptotes appear where the function is not defined. The function is not defined at $x = \pm 2$. Therefore we have to examine the following limits.

$$\lim_{x\to -2^-} \frac{(x+1)(x-1)}{(x+2)(x-2)} = \infty, \quad \lim_{x\to -2^+} \frac{(x+1)(x-1)}{(x+2)(x-2)} = -\infty$$

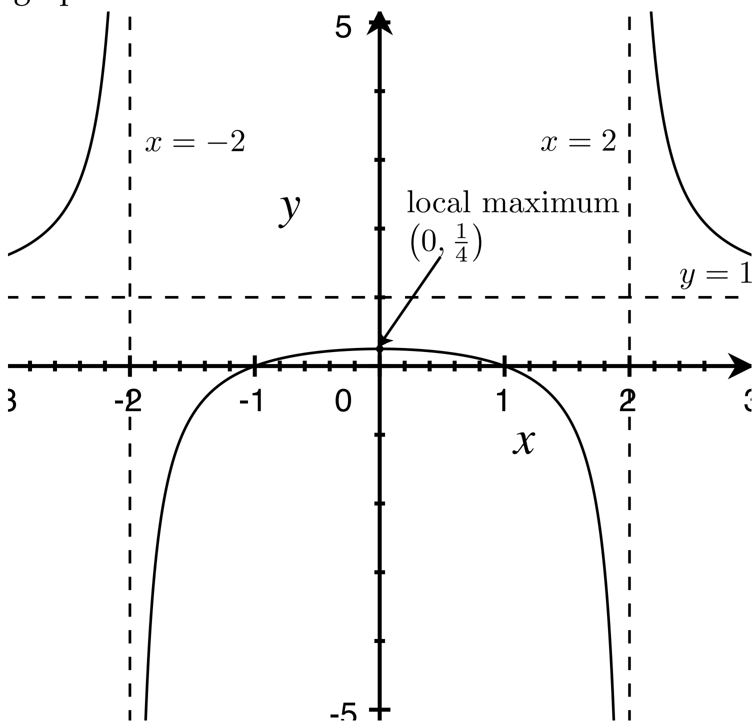
   so we have a vertical asymptote $x = -2$.

$$\lim_{x\to 2^-} \frac{(x+1)(x-1)}{(x+2)(x-2)} = -\infty, \quad \lim_{x\to 2^+} \frac{(x+1)(x-1)}{(x+2)(x-2)} = +\infty$$

   so we have a vertical asymptote $x = 2$.

   **Oblique Asymptote:** None

6. Sketch the graph:




7. The range of $f(x)$ is $\left(-\infty, \frac{1}{4}\right] \cup (1, \infty)$.

# 3 Graph of a function with vertical and oblique asymptotes

Sketch the graph of $y = \dfrac{x^2+2x+4}{2x} = \dfrac{x}{2} + 1 + \dfrac{2}{x}$

1. Domain: $x \in \mathbb{R}\backslash\{0\}$

2. $x$- and $y$-intercepts:

   **$x$-intercept:** no $x$-intercept

   **$y$-intercept:** no $y$-intercept

   So there are no $x$- and $y$- intercepts.

3. Critical Points and Local Extreme Values:

   (a) Find $f'(x)$

   $$f'(x) = \frac{1}{2} - \frac{2}{x^2} = \frac{x^2-4}{2x^2} = \frac{(x+2)(x-2)}{2x^2}$$

   (b) Find all critical points:

   **$f'(x) = 0$:**

   $$f'(x) = \frac{x^2-4}{2x^2} = 0 \Longleftrightarrow x^2 - 4 = 0 \Longleftrightarrow x = \pm 2.$$

   **$f'(x)$ is not defined:** $f'(x)$ is defined at $x = 0$, which are outside the domain and therefore not a critical point.

   (c) Increasing, Decreasing intervals:

| | $-\infty < x <$ | $-2$ | $< x <$ | $0$ | $< x <$ | $2$ | $< x < \infty$ |
|---|---|---|---|---|---|---|---|
| $f'(x)$ | $+$ | $0$ | $-$ | n.d. | $-$ | $0$ | $+$ |
| $f(x)$ | $\nearrow$ | loc. max. | $\searrow$ | | $\searrow$ | loc. min. | $\nearrow$ |

   (d) Evaluate $f(x)$ at each critical point:
   local maximum at $(-2, -1)$.
   local minimum at $(2, 3)$ We have neither a local maximum or minimum at $x = 0$.

4. Determine Concavity and Points of Inflection:

   (a) Find $f''(x)$

   $$f''(x) = \frac{d}{dx}\left(\frac{1}{2} - \frac{2}{x^2}\right) = \frac{4}{x^3}$$

   (b) Find all potential points of inflection:

   $f''(x) = 0$**:** There is no $x$ in the domain, s.t. $f''(x) = 0$

   $f''(x)$ **is not defined:** $f''(x)$ is not defined at $x = 0$, which is outside the domain and therefore no possible inflection point.

   (c) Intervals of concavity

   | | $-\infty < x <$ | $0$ | $< x < \infty$ |
   |---|---|---|---|
   | $f''(x)$ | $-$ | n.d. | $+$ |
   | $f(x)$ | $\cap$ | | $\cup$ |

   (d) There are no inflection points.

5. Asymptotes

   **Horizontal Asymptote:**

   $$\lim_{x\to\pm\infty}\left(\frac{x}{2} + 1 + \frac{2}{x}\right) = \pm\infty$$

   So we have no horizontal asymptote.

   **Vertical Asymptote:** Vertical asymptotes appear where the function is not defined. The function is not defined at $x = 0$. Therefore we have to examine the following limits.

   $$\lim_{x\to 0^-}\left(\frac{x}{2} + 1 + \frac{2}{x}\right) = -\infty, \lim_{x\to 0^+}\left(\frac{x}{2} + 1 + \frac{2}{x}\right) = \infty$$

   so we have a vertical asymptote $x = 0$.

   **Oblique Asymptote:** The oblique asymptote can be found easily by considering the function:

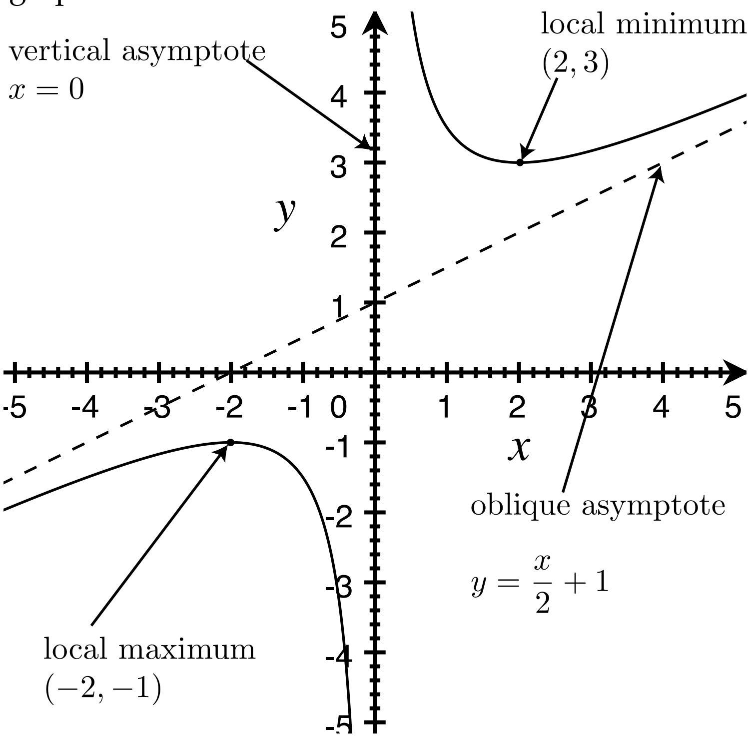
   $$f(x) = \underbrace{\frac{x}{2} + 1}_{\text{oblique asymptote}} + \frac{2}{x}$$

   Let us test this claim. Therefore we have to check the limit
   $\lim_{x\to\pm\infty}(f(x) - (ax + b)) = 0$.

   $$\lim_{x\to\pm\infty}\left[\left(\frac{x}{2} + 1 + \frac{2}{x}\right) - \left(\frac{x}{2} + 1\right)\right] = \lim_{x\to\pm\infty}\left[\frac{x}{2} + 1 + \frac{2}{x} - \frac{x}{2} - 1\right] = \lim_{x\to\pm\infty}\frac{2}{x} = 0$$

   Therefore $y = \frac{x}{2} + 1$ is an oblique asymptote to the graph of $f(x)$.

6. Sketch the graph:




7. The range of $f(x)$ is $(-\infty, -1] \cup [3, \infty)$.

# 4 Graph of a root function

Sketch the graph of $y = (x^2 - 1)^{2/3}$

1. Domain: $x \in \mathbb{R}$

2. $x$- and $y$-intercepts:

   **$x$-intercept:**

   $$f(x) = 0 \iff (x^2 - 1)^{2/3} = 0 \iff x^2 - 1 = 0 \iff x = \pm 1$$

   So we have two $x$-intercepts at $(-1, 0)$ and $(1, 0)$

   **$y$-intercept:**

   $$f(0) = (0 - 1)^{2/3} = 1$$

   So we have a $y$-intercept at $(0, 1)$.

3. Critical Points and Local Extreme Values:

   (a) Find $f'(x)$

   $$f'(x) = \frac{2}{3}(x^2 - 1)^{-1/3} 2x = \frac{4x}{3(x^2 - 1)^{1/3}}$$

   (b) Find all critical points:

   $f'(x) = 0$**:**

   $$f'(x) = \frac{4x}{3(x^2 - 1)^{1/3}} = 0 \iff 4x = 0 \iff x = 0.$$

   $f'(x)$ **is not defined:** $f'(x)$ is defined at $x = \pm 1$, which are inside the domain and therefore critical points.

   (c) Increasing, Decreasing intervals:

   | | $-\infty < x <$ | $-1$ | $< x <$ | $0$ | $< x <$ | $1$ | $< x < \infty$ |
   |---|---|---|---|---|---|---|---|
   | $f'(x)$ | $-$ | n.d. | $+$ | $0$ | $-$ | n.d. | $+$ |
   | $f(x)$ | $\searrow$ | loc. min. | $\nearrow$ | loc. max. | $\searrow$ | loc. min. | $\nearrow$ |

   (d) Evaluate $f(x)$ at each critical point:
   local maximum at $(0, 1)$.
   local minima at $(-1, 0)$ and $(1, 0)$.

4. Determine Concavity and Points of Inflection:

   (a) Find $f''(x)$

   $$f''(x) = \frac{4}{3} \frac{(x^2 - 1)^{1/3} - x \cdot \frac{1}{3}(x^2 - 1)^{-2/3} 2x}{(x^2 - 1)^{2/3}} = \frac{4}{3} \frac{\frac{x^2}{3} - 1}{(x^2 - 1)^{4/3}} = \frac{4}{9} \frac{x^2 - 3}{(x^2 - 1)^{4/3}}$$

   (b) Find all potential points of inflection:

   $f''(x) = 0$**:**

   $$f''(x) = 0 \iff \frac{4}{9} \frac{x^2 - 3}{(x^2 - 1)^{4/3}} = 0 \iff x^2 - 3 = 0 \iff x = \pm\sqrt{3}$$

$f''(x)$ **is not defined:** $f''(x)$ is not defined at $x = \pm 1$, which is inside the domain and therefore possible inflection points.

(c) Intervals of concavity

| | $-\infty < x <$ | $-\sqrt{3}$ | $< x <$ | $-1$ | $< x <$ | $1$ | $< x <$ | $\sqrt{3}$ | $< x < \infty$ |
|---|---|---|---|---|---|---|---|---|---|
| $f''(x)$ | $+$ | 0 | $-$ | n.d. | $-$ | n.d. | $-$ | 0 | $+$ |
| $f(x)$ | $\cup$ | infl. point | $\cap$ | | $\cap$ | | $\cap$ | infl. point | $\cup$ |

(d) There are two points of inflection at $\left(-\sqrt{3}, 2^{2/3}\right)$ and $\left(\sqrt{3}, 2^{2/3}\right)$.

5. Asymptotes

**Horizontal Asymptote:**

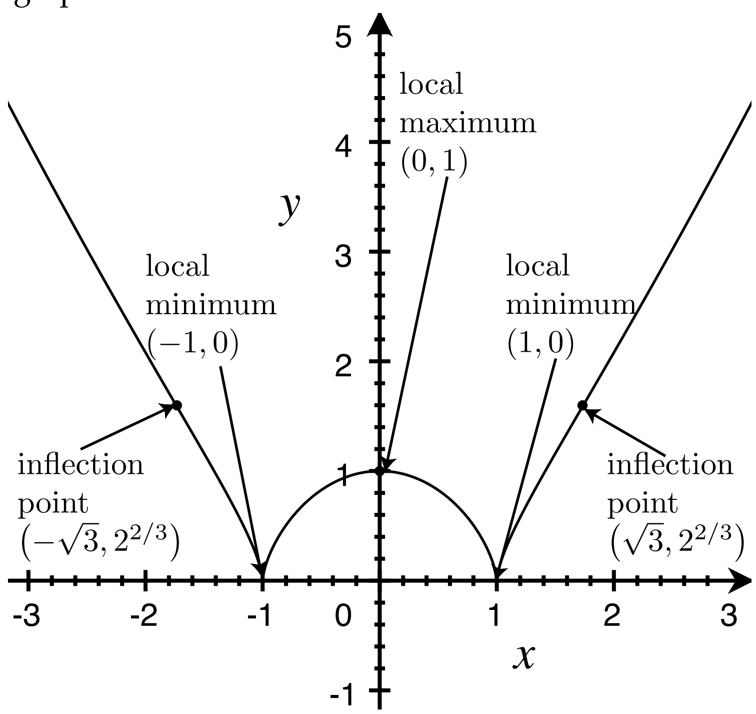
$$\lim_{x \to \pm\infty} \left(x^2 - 1\right)^{2/3} = \infty$$

So we have no horizontal asymptote.

**Vertical Asymptote:** Vertical asymptotes appear where the function is not defined. As the function is defined for all real numbers, there is no vertical asymptote.

**Oblique Asymptote:** As the function is not a rational function, there is also no oblique asymptote.

6. Sketch the graph:




7. The range of $f(x)$ is $[0, \infty)$.