Questions related to SECTION 4.3

1. Sketch the graph of $f(x) = x^4 - 6x^2$

   (a) Domain: $(-\infty, \infty)$

   (b) $x-$ and $y-$ intercepts:

   $x-$intercept: $y = 0 \quad \Leftrightarrow \quad x^4 - 6x^2 = 0 \quad \Leftrightarrow \quad x = 0, \quad x = -\sqrt{6}, \quad x = \sqrt{6}$

   So the $x-$intercepts are $(0,0), \quad (-\sqrt{6}, 0), \quad (\sqrt{6}, 0)$

   $y-$intercept: $x = 0 \quad \Leftrightarrow y = 0$

   So the $y-$intercept is $(0,0)$.

   (c) Critical Points and Local Extreme Values:

   i. Find $f'(x)$

   $f'(x) = 4x^3 - 12x = 4x(x^2 - 3)$

   ii. Find all critical points:

   $f'(x) = 0 : f'(x) = 4x(x^2 - 3) = 0 \quad \Leftrightarrow \quad x = 0, \quad x = -\sqrt{3}, \quad x = \sqrt{3}$

   $f'(x)$ is not defined: $f'(x)$ is defined for all real x.

   iii. Increasing and Decreasing Intervals:

x	$x <$	$-\sqrt{3}$	$< x <$	0	$< x <$	$\sqrt{3}$	$< x$
$f'(x)$	$-$	0	$+$	0	$-$	0	$+$
$f(x)$	$\searrow$	loc. min	$\nearrow$	loc. max	$\searrow$	loc. min	$\nearrow$

   iv. Evaluate $f(x)$ at each critical point:

   local minimums at $(-\sqrt{3}, -9)$, $(\sqrt{3}, -9)$ and local maximum at $(0,0)$

   (d) Determine Concavity and Points of Inflection:

   i. Find $f''(x)$

   $f''(x) = 12x^2 - 12 = 12(x^2 - 1) = 12(x-1)(x+1)$

ii. Find all potential points of inflection:

$$f''(x) = 0: \quad f''(x) = 12(x-1)(x+1) = 0 \quad \Leftrightarrow \quad x = 1, \quad x = -1$$

$f''(x)$ is not defined: $f''(x)$ is defined for all real x.

iii. Intervals of Concavity:

x	$x <$	-1	$< x <$	1	$< x$
$f''(x)$	$+$	0	$-$	0	$-$
$f(x)$	$\cup$	infl. point	$\cap$	infl. point	$\cup$

iv. The points of inflection are : $(-1, -5)$ and $(1, -5)$

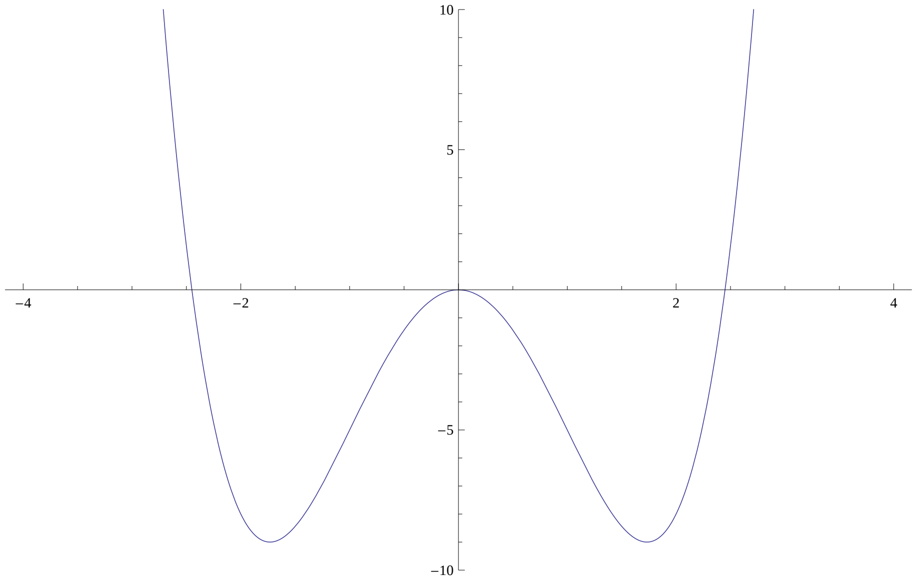
(e) Asymptotes:

Horizontal Asymptote: None

Vertical Asymptote: None

Oblique Asymptote: None

(f) Sketch the graph:



(g) The range of $f(x)$ is $[-9, \infty)$.

2. Sketch the graph of $y = \dfrac{(x+1)^2}{1+x^2}$

   (a) Domain: $x \in \mathbb{R}$

   (b) $x-$ and $y-$ intercepts:

   $x-$intercept: $y = 0 \quad \Leftrightarrow \dfrac{(x+1)^2}{1+x^2} = 0 \quad \Leftrightarrow (x+1)^2 = 0 \quad \Leftrightarrow x = -1$

   So the $x-$intercept is $(-1, 0)$.

   $y-$intercept: $x = 0 \quad \Leftrightarrow y = \dfrac{(0+1)^2}{1+0^2} \quad \Leftrightarrow y = 1$

   So the $y-$intercept is $(0, 1)$.

   (c) Critical Points and Local Extreme Values:

   i. Find $f'(x)$

   $$f'(x) = \frac{(1+x^2)\cdot 2(x+1) - (x+1)^2 \cdot 2x}{(1+x^2)^2} = \frac{2(1-x^2)}{(1+x^2)^2}$$

   ii. Find all critical points:

   $$f'(x) = 0: \quad f'(x) = \frac{2(1-x^2)}{(1+x^2)^2} = 0 \quad \Leftrightarrow \quad x = -1 \text{ and } x = 1$$

   $f'(x)$ is not defined: $f'(x)$ is defined for all real x.

   iii. Increasing and Decreasing Intervals:

x	$x <$	-1	$< x <$	1	$< x$
$f'(x)$	$-$	0	$+$	0	$-$
$f(x)$	$\searrow$	loc. min.	$\nearrow$	loc. max.	$\searrow$

   iv. Evaluate $f(x)$ at each critical point:

   local minimum at $(-1, 0)$ and local maximum at $(1, 2)$

   (d) Determine Concavity and Points of Inflection:

   i. Find $f''(x)$

   $$f''(x) = \frac{(1+x^2)^2 \cdot 2(-2x) - 2(1-x^2)[2(1+x^2)\cdot 2x]}{(1+x^2)^4} = \frac{4x(x^2-3)}{(1+x^2)^3}$$

ii. Find all potential points of inflection:

$f''(x) = 0: \quad f''(x) = \dfrac{4x(x^2-3)}{(1+x^2)^3} = 0 \quad \Leftrightarrow x = 0, x = \pm\sqrt{3}$

$f''(x)$ is not defined: $f''(x)$ is defined for all real x.

iii. Intervals of Concavity:

x	$x <$	$-\sqrt{3}$	$< x <$	0	$< x <$	$\sqrt{3}$	$< x$
$f''(x)$	$-$	0	$+$	0	$-$	0	$+$
$f(x)$	$\cap$	infl. point	$\cup$	infl. point	$\cap$	infl. point	$\cup$

iv. The points of inflection are : $(-\sqrt{3}, 0.1340)$, $(0, 1)$ and $(\sqrt{3}, 1.8660)$

(e) Asymptotes:

Horizontal Asymptote:

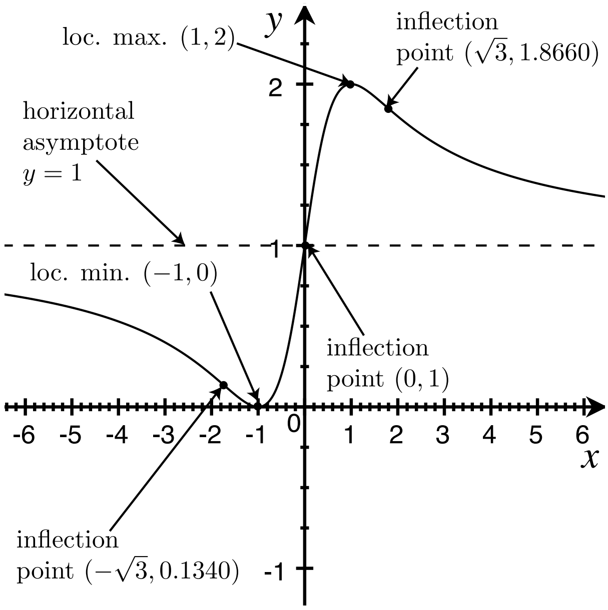
$$\lim_{x\to\pm\infty} \frac{(x+1)^2}{1+x^2} = \lim_{x\to\pm\infty} \frac{x^2+2x+1}{1+x^2} = \lim_{x\to\pm\infty} \frac{x^2\left(1+\frac{2}{x}+\frac{1}{x^2}\right)}{x^2\left(1+\frac{1}{x^2}\right)} = 1$$

So we have a horizontal asymptote at $y = 1$.

Vertical Asymptote: None

Oblique Asymptote: None

(f) Sketch the graph:



(g) The range of $f(x)$ is $[0, 2]$.

Questions related to SECTION 4.5

1. (a) Write the equation of the line that represents the linear approximation to the following functions at the given point a.

   (b) Use the linear approximation to estimate the given function value.

   (c) Compute the percent error in your approximation by the formula:

$$100 \cdot \frac{|approx - exact|}{|exact|}$$

i. $f(x) = 12 - x^2; \quad a = 2; \quad f(2.1)$

(a) $f(a) = f(2) = 8$ and $f'(a) = -2a = -4$

so the linear approximation has the equation:

$$y = L(x) = f(a) + f'(a)(x - a) = 8 - 4(x - 2) = -4x + 16$$

(b) $f(2.1) \approx L(2.1) = 7.6$

(c) The percent error is : $\quad 100 \cdot \frac{|7.6 - 7.59|}{|7.59|} \approx 0.13\%$

ii. $f(x) = \sin x; \quad a = \pi/4; \quad f(0.75)$

(a) $f(a) = f(\pi/4) = \sin(\pi/4) = \frac{\sqrt{2}}{2}$ and $f'(a) = \cos a = \cos(\pi/4) = \frac{\sqrt{2}}{2}$

so the linear approximation has the equation:

$$y = L(x) = f(a) + f'(a)(x-a) = \sin\frac{\pi}{4} + \cos\frac{\pi}{4}\left(x - \frac{\pi}{4}\right) =$$
$$= \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\left(x - \frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}\left(x + 1 - \frac{\pi}{4}\right)$$

(b) $f(0.75) \approx L(0.75) \approx 0.68$

(c) The percent error is : $\quad 100 \cdot \dfrac{|0.68 - \sin 0.75|}{|\sin 0.75|} \approx 0.064\%$

iii. $f(x) = \ln(1+x); \quad a = 0; \quad f(0.9)$

(a) $f(a) = f(0) = \ln 1 = 0$ and $f'(a) = \dfrac{1}{1+a} = \dfrac{1}{1+0} = 1$

so the linear approximation has the equation:

$$y = L(x) = f(a) + f'(a)(x-a) = 0 + 1(x-0) = x$$

(b) $f(0.9) \approx L(0.9) = 0.9$

(c) The percent error is : $\quad 100 \cdot \dfrac{|0.9 - \ln 1.9|}{|\ln 1.9|} \approx 40\%$

iv. $f(x) = e^x; \quad a = 0; \quad f(0.05)$

(a) $f(a) = f(0) = e^0 = 1$ and $f'(a) = e^a = e^0 = 1$

so the linear approximation has the equation:

$$y = L(x) = f(a) + f'(a)(x-a) = 1 + 1(x-0) = 1 + x$$

(b) $f(0.05) \approx L(0.05) = 1.05$

(c) The percent error is : $\quad 100 \cdot \dfrac{|1.05 - e^{0.05}|}{|e^{0.05}|} \approx 0.12\%$

2. Use linear approximations to estimate the following quantities. Choose a value of 'a' to produce a small error.

(a) $\sqrt{146}$

Let $f(x) = \sqrt{x}$ and $a = 144$. Then $f(a) = \sqrt{144} = 12$

$f'(x) = \frac{1}{2\sqrt{x}} \quad \Rightarrow \quad f'(a) = \frac{1}{2\sqrt{144}} = \frac{1}{24}$

So the linear approximation near $a = 144$ is

$L(x) = f(a) + f'(a)(x - a) = 12 + \frac{1}{24}(x - 144)$

Therefore

$$\sqrt{146} = f(146) \approx L(146) = 12 + \frac{1}{24}(146 - 144) = 12\frac{1}{12}$$

(b) $\sqrt[3]{65}$

Let $f(x) = x^{\frac{1}{3}}$ and $a = 64$. Then $f(a) = 64^{\frac{1}{3}} = 4$

$f'(x) = \frac{1}{3}x^{\frac{-2}{3}} \quad \Rightarrow \quad f'(a) = \frac{1}{3}64^{\frac{-2}{3}} = \frac{1}{48}$

So the linear approximation near $a = 64$ is

$L(x) = f(a) + f'(a)(x - a) = 4 + \frac{1}{48}(x - 64)$

Therefore

$$\sqrt[3]{65} = f(65) \approx L(65) = 4 + \frac{1}{48}(65 - 64) = 4\frac{1}{48}$$

(c) $e^{0.06}$

Let $f(x) = e^x$ and $a = 0$. Then $f(a) = e^0 = 1$

$f'(x) = e^x \quad \Rightarrow \quad f'(a) = e^0 = 1$

So the linear approximation near $a = 0$ is

$L(x) = f(a) + f'(a)(x - a) = 1 + 1(x - 0)$

Therefore

$$e^{0.06} = f(0.06) \approx L(0.06) = 1 + 1(0.06 - 0) = 1.06$$

(d) $\cos 31°$

Let $f(x) = \cos x$ and $a = \frac{\pi}{6}(= 30)$. Then $f(a) = \frac{\sqrt{3}}{2}$

$f'(x) = -\sin x \quad \Rightarrow \quad f'(a) = -\sin 30 = \frac{-1}{2}$

So the linear approximation near $a = 30°$ is :

$L(x) = f(a) + f'(a)(x - a) = \frac{\sqrt{3}}{2} - \frac{1}{2}(x - \frac{\pi}{6})$

Therefore

$$\cos 31° = \cos\left(\frac{31\pi}{180}\right) = f\left(\frac{31\pi}{180}\right) \approx L\left(\frac{31\pi}{180}\right) = \frac{\sqrt{3}}{2} - \frac{1}{2}\left(\frac{31\pi}{180} - \frac{\pi}{6}\right) \approx 0.857$$

Note that we must convert $31°$ to radians before applying the linear approximation formula.

