

**MATH151 Midterm Exam Solutions**  
**Department of Mathematics**  
**Spring 2014 - April, 14, 2014**

Question No.	1	2	3	4	5	6	Total
Weight	20	20	20	20	20	20	120
Received							

1. Calculate the following limits

(a)

$$\begin{aligned} \lim_{x \rightarrow 5} \frac{x^2 - 3x - 10}{x - 5} &= \lim_{x \rightarrow 5} \frac{(x - 5)(x + 2)}{x - 5} \\ &= \lim_{x \rightarrow 5} (x + 2) = 7 \end{aligned}$$

(b)

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{\sqrt{x+1} - 1}{x} &= \lim_{x \rightarrow 0} \frac{(\sqrt{x+1} - 1)(\sqrt{x+1} + 1)}{x(\sqrt{x+1} + 1)} = \\ &= \lim_{x \rightarrow 0} \frac{x + 1 - 1}{x(\sqrt{x+1} + 1)} = \lim_{x \rightarrow 0} \frac{x}{x(\sqrt{x+1} + 1)} = \\ &= \lim_{x \rightarrow 0} \frac{1}{(\sqrt{x+1} + 1)} = \frac{1}{2} \end{aligned}$$

(c)

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{\sin 2x}{3x} &= \lim_{x \rightarrow 0} \frac{2 \sin 2x}{3 \cdot 2x} = \\ \frac{2}{3} \lim_{x \rightarrow 0} \frac{\sin 2x}{2x} &= \frac{2}{3} \end{aligned}$$

(d)

$$\lim_{x \rightarrow \infty} \frac{4x^3 + 2x^2 + 3}{5x^3 + x - 1} = \frac{4}{5},$$

since the limit of the ratio of polynomials is equal to the ratio of the leading coefficients in case when the degrees of the polynomials are equal.

2. Determine whether the following functions are continuous at the indicated points:

(a)

$$f(x) = \begin{cases} \frac{x+2}{2}, & x \leq 3 \\ \frac{12-2x}{3}, & x > 3 \end{cases} \quad a = 3$$

$$\lim_{x \rightarrow 3^-} f(x) = \lim_{x \rightarrow 3^-} \frac{x+2}{2} = \frac{5}{2},$$

$$\lim_{x \rightarrow 3^+} f(x) = \lim_{x \rightarrow 3^+} \frac{12-2x}{3} = 2,$$

limits are not equal, the function is discontinuous at  $a = 3$ .

(b)

$$f(x) = \begin{cases} x^3 + 1 & x \leq 1 \\ x + 1, & x > 1 \end{cases}, \quad a = 1$$

$$\lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1^-} (x^3 + 1) = 2,$$

$$\lim_{x \rightarrow 1^+} f(x) = \lim_{x \rightarrow 1^+} (x + 1) = 2.$$

$$f(1) = 1^3 + 1 = 2$$

the function is continuous at  $a = 1$ .

3. Find the derivatives of the following functions

(a)

$$f(x) = (3x^3 + 4x^2 + 5)^9$$

$$f'(x) = 9(3x^3 + 4x^2 + 5)^8(9x^2 + 8x)$$

(b)

$$g(x) = e^x \sin x$$

$$g'(x) = e^x \sin x + e^x \cos x$$

(c)

$$h(x) = \frac{\ln x}{x^2 + 1}$$

$$h'(x) = \frac{1/x(x^2 + 1) - 2x \ln x}{(x^2 + 1)^2}$$

(d)

$$u(x) = \cos(e^x)$$
$$u'(x) = -e^x \sin(e^x)$$

4. Write an equation of the tangent line to the graph of the function given implicitly by the equation

$$x^3 + y^3 = 6xy - 1$$

at the point  $(2, 3)$ .

Solution.

$$3x^2 + 3y^2y' = 6y + 6xy',$$
$$3x^2 - 6y = (-3y^2 + 6x)y',$$
$$y' = \frac{3x^2 - 6y}{-3y^2 + 6x},$$
$$y'((2, 3)) = \frac{2}{5},$$
$$y - 3 = \frac{2}{5}(x - 2).$$

5. Use logarithmic differentiation to find the derivatives of the functions:

(a)

$$f(x) = x^x$$

$$y = x^x, \quad \ln y = x \ln x, \quad \frac{y'}{y} = \ln x + 1$$
$$y' = x^x (\ln x + 1).$$

(b)

$$g(x) = \frac{x^5 (x - 1)^3}{(x + 1)^2}$$

$$\begin{aligned}
y &= \frac{x^5(x-1)^3}{(x+1)^2}, & \ln y &= \ln \frac{x^5(x-1)^3}{(x+1)^2}, \\
\ln y &= 5 \ln x + 3 \ln(x-1) - 2 \ln(x+1), \\
\frac{y'}{y} &= \frac{5}{x} + \frac{3}{x-1} - \frac{2}{x+1}, \\
y' &= \frac{x^5(x-1)^3}{(x+1)^2} \left( \frac{5}{x} + \frac{3}{x-1} - \frac{2}{x+1} \right).
\end{aligned}$$

6. Consider the function

$$f(x) = \frac{x}{x-1}$$

- (a) Find the domain of the function (the interval on which it is defined)

$$\text{Domain : } (-\infty, 1) \cup (1, \infty)$$

- (b) Find  $x$  and  $y$ - intercepts

$$x = 0, y = 0$$

is the only intercept.

- (c) Find the derivative of  $f$ ,

$$f' = \frac{x-1-x}{(x-1)^2} = \frac{-1}{(x-1)^2} = -(x-1)^{-2}$$

- (d) Determine the intervals of increase and decrease of  $f$

Interval	$(-\infty, 1)$	$(1, \infty)$
$f'(x)$	$< 0$	$< 0$
$f(x)$	decreasing	decreasing

- (e) Find the second derivative

$$f'' = 2(x-1)^{-3} = \frac{2}{(x-1)^3}.$$

- (f) Determine the intervals of concavity

Interval	$(-\infty, 1)$	$(1, \infty)$
$f''(x)$	$< 0$	$> 0$
$f(x)$	concave down	concave up

- (g) Find the vertical asymptote of the graph of  $f$ , finding the necessary limits

$$\lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1^-} \frac{x}{x-1} = -\infty, \quad \lim_{x \rightarrow 1^+} f(x) = \lim_{x \rightarrow 1^+} \frac{x}{x-1} = \infty$$

$x = 1$  is a vertical asymptote.

- (h) Find the horizontal asymptote of the graph of  $f$ , finding the necessary limits

$$\begin{aligned} \lim_{x \rightarrow \infty} f(x) &= \lim_{x \rightarrow \infty} \frac{x}{x-1} = 1, \\ \lim_{x \rightarrow -\infty} f(x) &= \lim_{x \rightarrow -\infty} \frac{x}{x-1} = 1. \end{aligned}$$

$y = 1$  is a horizontal asymptote.

- (i) Sketch the graph of the function

