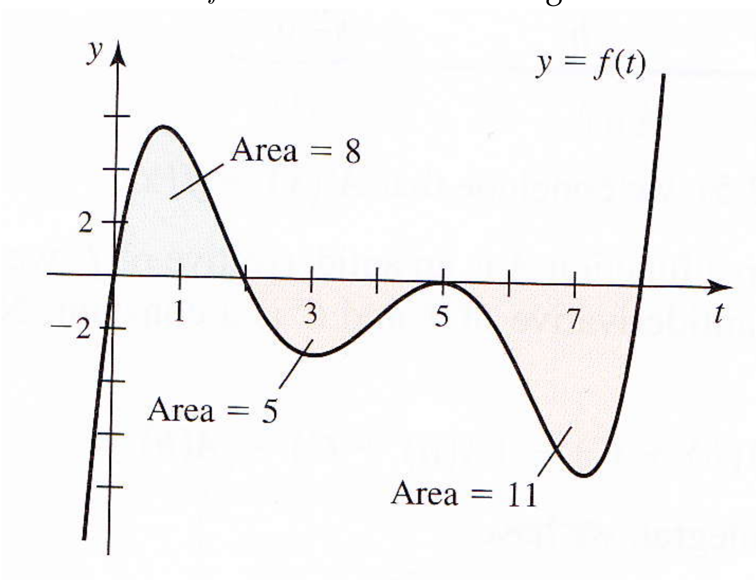
Questions related to SECTION 5.3

1. The graph of f is shown in the figure. Let

$$A(x) = \int_0^x f(t)\,dt \qquad \text{and} \qquad F(x) = \int_2^x f(t)\,dt$$

be two area functions of f. Evaluate the following area functions.



(a) $A(2) = \int_0^2 f(t)\,dt = 8$

(b) $F(5) = \int_2^5 f(t)\,dt = -5$

(c) $A(0) = \int_0^0 f(t)\,dt = 0$

(d) $A(8) = \int_0^8 f(t)\,dt = 8 - 5 - 11 = -8$

(e) $F(8) = \int_2^8 f(t)\,dt = -5 - 11 = -16$

(f) $A(8) = \int_0^5 f(t)\,dt = 8 - 5 = 3$

2. Evaluate the following integrals using the Fundamental Theorem of Calculus.

(a) $\int_{-2}^{2} (x^2 - 4)\,dx$

$$\int_{-2}^{2} (x^2 - 4)\,dx = \left(\frac{x^3}{3} - 4x\right)\Bigg|_{-2}^{2} = \frac{8}{3} - 8 + \frac{8}{3} - 8 = \frac{16}{3} - 16 = -\frac{32}{3}$$

(b) $\int_{\frac{1}{2}}^{1} (x^{-3} - 8)\, dx$

$$\int_{\frac{1}{2}}^{1} (x^{-3} - 8)\, dx = \left(\frac{x^{-2}}{-2} - 8x \right) \Bigg|_{\frac{1}{2}}^{1} = -\frac{1}{2} - 8 + 2 + 4 = -\frac{5}{2}$$

(c) $\int_{0}^{\frac{\pi}{4}} \sec^2(\theta)\, d\theta$

$$\int_{0}^{\frac{\pi}{4}} \sec^2(\theta)\, d\theta = (\tan\theta)\Big|_{0}^{\frac{\pi}{4}} = \tan\left(\frac{\pi}{4}\right) - \tan(0) = 1 - 0 = 1$$

(d) $\int_{0}^{\frac{1}{2}} \frac{dx}{\sqrt{1-x^2}}$

$$\int_{0}^{\frac{1}{2}} \frac{dx}{\sqrt{1-x^2}} = \left(\sin^{-1} x\right)\Big|_{0}^{\frac{1}{2}} = \sin^{-1}\left(\frac{1}{2}\right) - \sin^{-1}(0) = \frac{\pi}{6} - 0 = \frac{\pi}{6}$$

(e) $\int_{1}^{2} \frac{3}{t}\, dt$

$$\int_{1}^{2} \frac{3}{t}\, dt = (3\ln \mid t \mid)\Big|_{1}^{2} = 3\ln(2) - 3\ln(1) = 3\ln(2) = \ln(8)$$

(f) $\int_{4}^{9} \frac{x - \sqrt{x}}{x^3}\, dx$

$$\int_{4}^{9} \frac{x - \sqrt{x}}{x^3}\, dx = \int_{4}^{9} \left(\frac{1}{x^2} - \frac{1}{x^{\frac{4}{5}}} \right) dx = \int_{4}^{9} \left(x^{-2} - x^{-\frac{5}{2}} \right) dx =$$

$$= \left(\frac{x^{-1}}{-1} - \frac{x^{-\frac{3}{2}}}{-\frac{3}{2}} \right) \Bigg|_{4}^{9} = \left(-\frac{1}{x} + \frac{2}{3x^{\frac{3}{2}}} \right) \Bigg|_{4}^{9} = -\frac{1}{9} + \frac{2}{81} + \frac{1}{4} - \frac{2}{24} = \frac{13}{162}$$

3. Simplify the following expressions.

(a) $\frac{d}{dx} \int_{3}^{x} (t^2 + t + 1)\, dt$

$$\frac{d}{dx} \int_{3}^{x} (t^2 + t + 1)\, dt = x^2 + x + 1$$

(b) $\frac{d}{dx} \int_{x^2}^{10} \frac{dz}{z^2 + 1}$

$$\frac{d}{dx} \int_{x^2}^{10} \frac{dz}{z^2+1} = -\frac{d}{dx} \int_{10}^{x^2} \frac{dz}{z^2+1} = -\frac{1}{x^4+1} 2x = -\frac{2x}{x^4+1}$$

(c) $\dfrac{d}{dx}\displaystyle\int_a^{\tan^{-1}x} \cos(\ln t)^{\tan t}\,dt$

$$\frac{d}{dx}\int_a^{\tan^{-1}x} \cos(\ln\ t)^{\tan\ t}\,dt = \cos(\ln(\tan^{-1}x))^{\tan(\tan^{-1}x)}\frac{1}{1+x^2} = \frac{\cos(\ln(\tan^{-1}x))^x}{1+x^2}$$

Questions related to SECTION 5.4

1. Use symmetry to evaluate the following integrals.

(a) $\displaystyle\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \cos\ x\,dx$

Because $\cos(-x) = \cos(x)$, $\cos(x)$ function is even. Therefore the integral can be calculated as:

$$\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \cos\ x\,dx = 2\int_0^{\frac{\pi}{4}} \cos\ x\,dx = 2(\sin(x))\Big|_0^{\frac{\pi}{4}} = 2\left(\frac{\sqrt{2}}{2} - 0\right) = \sqrt{2}$$

(b) $\displaystyle\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \sin^5(x)\,dx$

Because $\sin(-x) = -\sin(x)$, the $\sin x$ function is odd. Therefore the integral can be calculated as:

$$\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \sin^5(x)\,dx = 0$$

Questions related to SECTION 5.5

1. Find the following integrals using substitution rule.

(a) $\displaystyle\int 2x(x^2+1)^4\,dx$

Let

$$u = x^2+1 \quad\Rightarrow\quad \frac{du}{dx} = 2x \quad\Rightarrow\quad dx = \frac{du}{2x}$$

$$\int 2x(x^2+1)^4\,dx = \int 2x\,u^4\frac{du}{2x} = \int u^4\,du = \frac{u^5}{5} + C = \frac{(x^2+1)^5}{5} + C$$

(b) $\int \sin^3(x)\cos x\,dx$

Let

$$u=\sin(x) \quad \Rightarrow \quad \frac{du}{dx}=\cos x \quad \Rightarrow \quad dx=\frac{du}{\cos x}$$
$$\int \sin^3(x)\cos x\,dx=\int u^3\cos x\frac{du}{\cos x}=\int u^3\,du=\frac{u^4}{4}+C=\frac{\sin^4(x)}{4}+C$$

(c) $\int (x^2+x)^{10}(2x+1)\,dx$

Let

$$u=x^2+x \quad \Rightarrow \quad \frac{du}{dx}=2x+1 \quad \Rightarrow \quad dx=\frac{du}{2x+1}$$
$$\int (x^2+x)^{10}(2x+1)\,dx=\int u^{10}(2x+1)\frac{du}{2x+1}=\int u^{10}\,du=\frac{u^{11}}{11}+C=\frac{(x^2+x)^{11}}{11}+C$$

(d) $\int \frac{1}{\sqrt{1-9x^2}}\,dx$

$$\int \frac{1}{\sqrt{1-9x^2}}\,dx=\int \frac{1}{\sqrt{1-(3x)^2}}\,dx$$

Let

$$u=3x \quad \Rightarrow \quad \frac{du}{dx}=3 \quad \Rightarrow \quad dx=\frac{du}{3}$$
$$\frac{1}{3}\int \frac{1}{\sqrt{1-(u)^2}}\,du=\frac{1}{3}\sin^{-1}(u)+C=\frac{\sin^{-1}(3x)}{3}+C$$

(e) $\int x^9\sin x^{10}\,dx$

Let

$$u=x^{10} \quad \Rightarrow \quad \frac{du}{dx}=10x^9 \quad \Rightarrow \quad dx=\frac{du}{10x^9}$$
$$\int x^9\sin x^{10}\,dx=\int x^9\sin u\frac{du}{10x^9}=\frac{1}{10}\int \sin u\,du=-\frac{1}{10}\cos u+C=-\frac{1}{10}\cos x^{10}+C$$

(f) $\int \frac{e^x-e^{-x}}{e^x+e^{-x}}\,dx$

Let

$$u=e^x+e^{-x} \quad \Rightarrow \quad \frac{du}{dx}=e^x-e^{-x} \quad \Rightarrow \quad dx=\frac{du}{e^x-e^{-x}}$$
$$\int \frac{e^x-e^{-x}}{e^x+e^{-x}}\,dx=\int \frac{e^x-e^{-x}}{u}\frac{du}{e^x-e^{-x}}dx=\int \frac{1}{u}\,du=\ln\mid u\mid+C=\ln\mid e^x+e^{-x}\mid+C$$

(g) $\displaystyle\int_0^2 \frac{2x}{(x^2+1)^2}\,dx$

Let

$$u = x^2+1 \quad \Rightarrow \quad \frac{du}{dx} = 2x \quad \Rightarrow \quad dx = \frac{du}{2x}$$

$$\int_0^2 \frac{2x}{(x^2+1)^2}\,dx = \int_{x=0}^{x=2} \frac{2x}{u^2}\frac{du}{2x} = \int_{u=1}^{u=5} u^{-2}\,du = \left(\frac{u^{-1}}{-1}\right)\Bigg|_1^5 = -\frac{1}{5}+1 = \frac{4}{5}$$

(h) $\displaystyle\int_0^{\frac{\pi}{2}} \sin^2(\theta)\cos\theta\,d\theta$

Let

$$u = \sin\theta \quad \Rightarrow \quad \frac{du}{d\theta} = \cos\theta \quad \Rightarrow \quad d\theta = \frac{du}{\cos\theta}$$

$$\int_0^{\frac{\pi}{2}} \sin^2(\theta)\cos\theta\,d\theta = \int_{\theta=0}^{\theta=\frac{\pi}{2}} u^2\cos\theta\,\frac{du}{\cos\theta} = \int_{u=0}^{u=1} u^2\,du = \left(\frac{u^3}{3}\right)\Bigg|_0^1 = \frac{1}{3}$$

(i) $\displaystyle\int_0^{\frac{\pi}{4}} \frac{\sin x}{\cos^2(x)}\,dx$

Let

$$u = \cos x \quad \Rightarrow \quad \frac{du}{dx} = -\sin x \quad \Rightarrow \quad dx = \frac{du}{-\sin x}$$

$$\int_0^{\frac{\pi}{4}} \frac{\sin x}{\cos^2(x)}\,dx = \int_{x=0}^{x=\frac{\pi}{4}} \frac{\sin x}{u^2}\frac{du}{-\sin x} = -\int_{u=1}^{u=\frac{\sqrt{2}}{2}} u^{-2}\,du = -\left(\frac{u^{-1}}{-1}\right)\Bigg|_1^{\frac{\sqrt{2}}{2}} =$$

$$= \frac{2}{\sqrt{2}} - 1 = \frac{2-\sqrt{2}}{\sqrt{2}} = \sqrt{2}-1$$

(j) $\displaystyle\int_{-1}^2 x^2 e^{x^3+1}\,dx$

Let

$$u = x^3+1 \quad \Rightarrow \quad \frac{du}{dx} = 3x^2 \quad \Rightarrow \quad dx = \frac{du}{3x^2}$$

$$\int_{-1}^2 x^2 e^{x^3+1}\,dx = \int_{x=-1}^{x=2} x^2 e^u \frac{du}{3x^2} = \frac{1}{3}\int_{u=0}^{u=9} e^u\,du = \frac{e^u}{3}\Bigg|_0^9 = \frac{e^9-1}{3}$$

(k) $\displaystyle\int_0^4 \frac{p}{\sqrt{9+p^2}}\,dp$

Let

$$u = 9 + p^2 \quad \Rightarrow \quad \frac{du}{dp} = 2p \quad \Rightarrow \quad dp = \frac{du}{2p}$$

$$\int_0^4 \frac{p}{\sqrt{9+p^2}}\,dp = \int_{p=0}^{p=4} \frac{p}{\sqrt{u}}\,\frac{du}{2p} = \frac{1}{2}\int_{u=9}^{u=25} \frac{1}{\sqrt{u}}\,du = \frac{1}{2}\left(2\sqrt{u}\right)\Big|_9^{25} = \sqrt{25} - \sqrt{9} = 2$$