Questions related to SECTION 7.7

1. Evaluate the following integrals or state that they diverge.

(a) $\displaystyle\int_1^\infty x^{-2}\,dx$

$$\int_1^\infty x^{-2}\,dx \;=\; \lim_{b\to\infty}\int_1^b x^{-2}\,dx \;=\; \lim_{b\to\infty}\left(-\frac{1}{x}\right)\Bigg|_1^b \;=\; \lim_{b\to\infty}\left(-\frac{1}{b}+1\right) \;=\; 1$$

(b) $\displaystyle\int_0^\infty \frac{dx}{(x+1)^3}$

$$\int_0^\infty \frac{dx}{(x+1)^3} = \lim_{b\to\infty}\int_0^b \frac{dx}{(x+1)^3} = \lim_{b\to\infty}\left(-\frac{1}{2(x+1)^2}\right)\Bigg|_0^b =$$

$$= \lim_{b\to\infty}\left(-\frac{1}{2(b+1)^2}+\frac{1}{2}\right) = \frac{1}{2}$$

(c) $\displaystyle\int_2^\infty \frac{dx}{x\ln x}$

$$\int_2^\infty \frac{dx}{x\ln x} = \lim_{b\to\infty}\int_2^b \frac{dx}{x\ln x}$$

We need to apply substitution method to the integral. Thus, let

$$u=\ln x \quad\Rightarrow\quad \frac{du}{dx}=\frac{1}{x} \quad\Rightarrow\quad dx = x\,du$$

$$\int_2^\infty \frac{dx}{x\ln x} = \lim_{b\to\infty}\int_2^b \frac{dx}{x\ln x} = \lim_{b\to\infty}\int_{x=2}^{x=b}\frac{x\,du}{x\,u} = \lim_{b\to\infty}\int_{x=2}^{x=b}\frac{du}{u} = \lim_{b\to\infty}(\ln|u|)\Bigg|_{x=2}^{x=b} =$$

$$= \lim_{b\to\infty}(\ln|\ln x|)\Bigg|_2^b = \lim_{b\to\infty}(\ln|\ln b| - \ln|\ln 2|) = \infty$$

Thus we say that the integral diverges.

(d) $\displaystyle\int_{e^2}^\infty \frac{dx}{x\ln^p x}, \quad p>1$

$$\int_{e^2}^\infty \frac{dx}{x\ln^p x} = \lim_{b\to\infty}\int_{e^2}^b \frac{dx}{x\ln^p x}$$

We need to apply substitution method to the integral. Thus, let

$$u = \ln x \quad \Rightarrow \quad \frac{du}{dx} = \frac{1}{x} \quad \Rightarrow \quad dx = x\,du$$

$$\int_{e^2}^{\infty} \frac{dx}{x \ln x} = \lim_{b\to\infty} \int_{e^2}^{b} \frac{dx}{x \ln^p x} = \lim_{b\to\infty} \int_{x=e^2}^{x=b} \frac{x\,du}{x\,u} = \lim_{b\to\infty} \int_{x=e^2}^{x=b} \frac{du}{u^p} =$$

$$= \lim_{b\to\infty} \left(\frac{1}{1-p} u^{1-p} \right) \Bigg|_{x=e^2}^{x=b} = \lim_{b\to\infty} \left(\frac{1}{1-p} \ln^{1-p} x \right) \Bigg|_{e^2}^{b} =$$

$$= \lim_{b\to\infty} \left(\frac{1}{1-p} \ln^{1-p} b - \frac{1}{1-p} (\ln e^2)^{1-p} \right) = \frac{1}{p-1} 2^{1-p}$$

(e) $\displaystyle\int_0^{\infty} \cos x\,dx$

$$\int_0^{\infty} \cos x\,dx = \lim_{b\to\infty} \int_0^{b} \cos x\,dx = \lim_{b\to\infty} (\sin x) \Bigg|_0^{b} =$$

$$= \lim_{b\to\infty} (\sin b - \sin 0) = \lim_{b\to\infty} (\sin b)$$

the limit does not exist, so the integral diverges.

(f) $\displaystyle\int_2^{\infty} \frac{\cos\left(\frac{\pi}{x}\right)}{x^2}\,dx$

$$\int_2^{\infty} \frac{\cos\left(\frac{\pi}{x}\right)}{x^2}\,dx = \lim_{b\to\infty} \int_2^{b} \frac{\cos\left(\frac{\pi}{x}\right)}{x^2}\,dx$$

We need to apply substitution method to the integral. Thus, let

$$u = \frac{\pi}{x} \quad \Rightarrow \quad \frac{du}{dx} = \frac{-\pi}{x^2} \quad \Rightarrow \quad dx = -\frac{x^2}{\pi} du$$

$$\int_2^{\infty} \frac{\cos\left(\frac{\pi}{x}\right)}{x^2}\,dx = \lim_{b\to\infty} \int_2^{b} \frac{\cos\left(\frac{\pi}{x}\right)}{x^2}\,dx = \lim_{b\to\infty} \int_{x=2}^{x=b} \frac{\cos\,u}{x^2} \left(-\frac{x^2}{\pi} \right) du =$$

$$= \lim_{b\to\infty} \int_{x=2}^{x=b} -\frac{\cos\,u}{\pi}\,du = \lim_{b\to\infty} \left(-\frac{1}{\pi} \sin\,u \right) \Bigg|_{x=2}^{x=b} = \lim_{b\to\infty} \left(-\frac{1}{\pi} \sin\left(\frac{\pi}{x}\right) \right) \Bigg|_2^{b} =$$

$$= \lim_{b\to\infty} \left(-\frac{1}{\pi} \sin\left(\frac{\pi}{b}\right) + \frac{1}{\pi} \right) = \frac{1}{\pi}$$

(g) $\displaystyle\int_0^\infty \frac{dx}{1+x^2}$

$$\int_0^\infty \frac{dx}{1+x^2} = \lim_{b\to\infty}\int_0^b \frac{dx}{1+x^2} = \lim_{b\to\infty}\left(\tan^{-1} x\right)\Big|_0^b =$$
$$= \lim_{b\to\infty}\left(\tan^{-1} b - \tan^{-1} 0\right) = \lim_{b\to\infty}\left(\tan^{-1} b\right) = \frac{\pi}{2}$$

(h) $\displaystyle\int_1^\infty \frac{\tan^{-1} x}{1+x^2}\,dx$

$$\int_1^\infty \frac{\tan^{-1} x}{1+x^2}\,dx = \lim_{b\to\infty}\int_1^b \frac{\tan^{-1} x}{1+x^2}\,dx$$

We need to apply substitution method to the integral. Thus, let

$$u = \tan^{-1} x \quad\Rightarrow\quad \frac{du}{dx} = \frac{1}{1+x^2} \quad\Rightarrow\quad dx = (1+x^2)du$$

$$\lim_{b\to\infty}\int_1^b \frac{\tan^{-1} x}{1+x^2}\,dx = \lim_{b\to\infty}\int_{x=1}^{x=b} \frac{u}{1+x^2}\,(1+x^2)\,du = \lim_{b\to\infty}\int_{x=1}^{x=b} u\,du =$$
$$= \lim_{b\to\infty}\left(\frac{u^2}{2}\right)\Bigg|_{x=1}^{x=b} = \lim_{b\to\infty}\left(\frac{(\tan^{-1} x)^2}{2}\right)\Bigg|_1^b =$$
$$= \lim_{b\to\infty}\left(\frac{(\tan^{-1} b)^2}{2} - \frac{(\tan^{-1} 1)^2}{2}\right) = \frac{\pi^2}{8} - \frac{\pi^2}{32} = \frac{3\pi^2}{32}$$

(i) $\displaystyle\int_0^1 \frac{x^3}{x^4-1}\,dx$

$$\int_0^1 \frac{x^3}{x^4-1}\,dx = \lim_{c\to 1^-}\int_0^c \frac{x^3}{x^4-1}\,dx$$

We need to apply substitution method to the integral. Thus, let

$$u = x^4 - 1 \quad\Rightarrow\quad \frac{du}{dx} = 4x^3 \quad\Rightarrow\quad dx = \frac{du}{4x^3}$$

$$\lim_{c\to 1^-}\int_0^c \frac{x^3}{x^4-1}\,dx = \lim_{c\to 1^-}\int_{x=0}^{x=c} \frac{x^3}{u}\frac{du}{4x^3} = \lim_{c\to 1^-}\int_{x=0}^{x=c} \frac{1}{4u}\,du =$$
$$= \lim_{c\to 1^-}\left(\frac{1}{4}\ln|u|\right)\Bigg|_{x=0}^{x=c} = \lim_{c\to 1^-}\left(\frac{1}{4}\ln|x^4-1|\right)\Bigg|_0^c =$$
$$= \lim_{c\to 1^-}\left(\frac{1}{4}\ln|c^4-1| - \frac{1}{4}\ln|0^4-1|\right) = \lim_{c\to 1^-}\left(\frac{1}{4}\ln|c^4-1|\right) = -\infty$$

so the integral diverges.

(j) $\displaystyle\int_0^1 \ln x^2\,dx$

$$\int_0^1 \ln x^2\,dx = 2\lim_{c\to 0^+}\int_c^1 \ln x\,dx = 2\lim_{c\to 0^+}(x\ln x - x)\Big|_c^1 = 2\lim_{c\to 0^+}(-1 - c\ln c + c) = -2$$

where

$$\lim_{c\to 0^+}(c\ln c) = 0$$

can be evaluated by using L'Hopital's rule.

(k) $\displaystyle\int_1^{11} \frac{dx}{(x-3)^{2/3}}$

It is clear that the integral is improper at $x = 3$, so we can write

$$\begin{aligned}
\int_1^{11} \frac{dx}{(x-3)^{2/3}} &= \int_1^3 \frac{dx}{(x-3)^{2/3}} + \int_3^{11} \frac{dx}{(x-3)^{2/3}} \\
&= \lim_{c\to 3^-}\int_1^c (x-3)^{-2/3}\,dx + \lim_{c\to 3^+}\int_c^{11} (x-3)^{-2/3}\,dx \\
&= \lim_{c\to 3^-}\left(3(x-3)^{1/3}\right)\Big|_1^c + \lim_{c\to 3^+}\left(3(x-3)^{1/3}\right)\Big|_c^{11} \\
&= 3\lim_{c\to 3^-}\left((c-3)^{1/3} + 2^{1/3}\right) + 3\lim_{c\to 3^+}\left(8^{1/3} - (c-3)^{1/3}\right) \\
&= 3\cdot 2^{1/3} + 3\cdot 8^{1/3}
\end{aligned}$$

Thus we have

$$\int_1^{11} \frac{dx}{(x-3)^{2/3}} = 6 + 3\cdot 2^{1/3}$$

(l) $\displaystyle\int_1^{11} \frac{dx}{(x-3)^{3/2}}$

Again the integral is improper at $x = 3$, so we can write

$$\begin{aligned}
\int_1^{11} \frac{dx}{(x-3)^{3/2}} &= \int_1^3 \frac{dx}{(x-3)^{3/2}} + \int_3^{11} \frac{dx}{(x-3)^{3/2}} \\
&= \lim_{c\to 3^-}\int_1^c (x-3)^{-3/2}\,dx + \lim_{c\to 3^+}\int_c^{11} (x-3)^{-3/2}\,dx \\
&= \lim_{c\to 3^-}\left(-2(x-3)^{-1/2}\right)\Big|_1^c + \lim_{c\to 3^+}\left(-2(x-3)^{-1/2}\right)\Big|_c^{11} \\
&= -2\lim_{c\to 3^-}\left((c-3)^{-1/2} - (1-3)^{-1/2}\right) - 2\lim_{c\to 3^+}\left(8^{-1/2} - (c-3)^{-1/2}\right) \\
&= \infty
\end{aligned}$$

Thus we have

$$\int_1^{11} \frac{dx}{(x-3)^{3/2}} = \infty$$

We say that the integral diverges.

