Questions related to SECTION 8.1

1. (a) Find the next two terms of the sequence.
(b) Find a recurrence relation that generates the relation.
(c) Find an explicit formula for the general nth term of the sequence.

i. \( \left\{ \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots \right\} \)
   
   (a) \( \frac{1}{32} \) and \( \frac{1}{64} \)
   
   (b) \( a_1 = 1; \quad a_{n+1} = \frac{a_n}{2} \)
   
   (c) \( a_n = \frac{1}{2^{n-1}} \)

ii. \( \left\{ 1, -2, 3, -4, 5, \ldots \right\} \)

   (a) \(-6\) and \(7\)

   (b) \( a_1 = 1; \quad a_{n+1} = (-1)^n(|a_n| + 1) \)

   (c) \( a_n = (-1)^{n+1}n \)

2. Write the terms \( a_1, a_2, a_3 \) and \( a_4 \) of the following sequences. If the sequence appears to converge, make a conjecture about its limit. If the sequence diverges, explain why.

   (a) \( a_n = \frac{(-1)^n}{n}; \quad n = 1, 2, 3, \ldots \)

   \[ a_1 = -1, \quad a_2 = \frac{1}{2}, \quad a_3 = -\frac{1}{3} \quad and \quad a_4 = \frac{1}{4} \]

   This sequence converges to 0 since each term is smaller in absolute value than the preceding term and they get arbitrarily close to 0.

   (b) \( a_n = 1 - 10^{-n}; \quad n = 1, 2, 3, \ldots \)

   \[ a_1 = 0.9, \quad a_2 = 0.99, \quad a_3 = 0.999 \quad and \quad a_4 = 0.9999 \]

   This sequence converges to 1.
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(c) \(a_{n+1} = \frac{a_n^2}{10} \), \(a_0 = 1\)

Rewrite the recurrence as \(a_{n+1} = 10^{-1}a_n^2\). Then we have

\[
\begin{align*}
a_0 &= 1 , \quad a_1 = 10^{-1} = \frac{1}{10} , \quad a_2 = 10^{-1}(10^{-1})^2 = 10^{-3} = \frac{1}{1000} , \quad a_3 = 10^{-1}(10^{-3})^2 = 10^{-7} = \frac{1}{10000000} , \quad a_4 = 10^{-1}(10^{-7})^2 = 10^{-15} = \frac{1}{1000000000000000} \\
\end{align*}
\]

This sequence converges to 0.

(d) \(a_{n+1} = 0.5a_n(1 - a_n); \quad a_0 = 0.8\)

\[
\begin{align*}
a_0 &= 0.8 , \quad a_1 = 0.5 \cdot 0.8(1 - 0.8) = 0.5 \cdot 0.8 \cdot 0.2 = 0.08 , \quad a_2 = 0.5 \cdot 0.08(1 - 0.08) = 0.5 \cdot 0.08 \cdot 0.92 = 0.0368 , \quad a_3 = 0.5 \cdot 0.0368(1 - 0.0368) = 0.01772288 , \quad a_4 = 0.5 \cdot 0.01772288(1 - 0.01772288) \approx 0.0087 \\
\end{align*}
\]

3. Consider the following recurrence relations.

(a) Find the terms \(a_0, a_1, a_2\) and \(a_3\) of the sequence.

(b) If possible, find an explicit formula for the \(n^{th}\) term of the sequence.

i. \(a_{n+1} = a_n + 2; \quad a_0 = 3\)

ii. \(a_{n+1} = 2a_n + 1; \quad a_0 = 0\)

i.

\[
\begin{align*}
a_0 &= 3 , \quad a_1 = 5 , \quad a_2 = 7 \quad \text{and} \quad a_3 = 9 \\
& \quad a_n = 2n + 3 \\
\end{align*}
\]

ii.

\[
\begin{align*}
a_0 &= 0 , \quad a_1 = 1 , \quad a_2 = 3 , \quad a_3 = 7 \quad \text{and} \quad a_4 = 15 \\
& \quad a_n = 2^n - 1 \\
\end{align*}
\]
Questions related to SECTION 8.2

1. Find the limit of the following sequences or determine that the limit does not exist.

   (a) \( \left\{ \frac{3n^3 - 1}{2n^3 + 1} \right\} \)

   Dividing the numerator and the denominator by \( n^3 \) we get:
   \[
   \lim_{n \to \infty} \frac{3 - n^{-3}}{2 + n^{-3}} = \frac{3}{2}
   \]

   (b) \( \left\{ \left( 1 + \frac{2}{n} \right)^n \right\} \)

   Find the limit of the logarithm of the expression, which is \( n \ln \left( 1 + \frac{2}{n} \right) \), using L’Hopital’s rule.
   \[
   \lim_{n \to \infty} n \ln \left( 1 + \frac{2}{n} \right) = \lim_{n \to \infty} \ln \left( 1 + \frac{2}{n} \right) = \lim_{n \to \infty} \frac{1}{1 + \frac{2}{n}} \left( -\frac{2}{n^2} \right) = \lim_{n \to \infty} \frac{2}{1 - \frac{2}{n}} = 2
   \]

   Thus the limit of the original expression is \( e^2 \).

   (c) \( \left\{ \sqrt{\left( 1 + \frac{1}{2n} \right)^n} \right\} \)

   Take the logarithm of the expression and use L’Hopital’s rule.
   \[
   \lim_{n \to \infty} \frac{n}{2} \ln \left( 1 + \frac{1}{2n} \right) = \lim_{n \to \infty} \ln \left( 1 + \frac{1}{2n} \right) = \lim_{n \to \infty} \frac{1}{1 + \frac{1}{2n}} \left( -\frac{1}{2n^2} \right) = \frac{1}{4}
   \]

   Thus the limit of the original expression is \( e^{\frac{1}{4}} \).

   (d) \( \left\{ \frac{\ln \left( \frac{1}{n} \right)}{n} \right\} \)

   Since \( \ln \left( \frac{1}{n} \right) = -\ln n \), we get
   \[
   \lim_{n \to \infty} \frac{\ln \left( \frac{1}{n} \right)}{n} = \lim_{n \to \infty} \frac{-\ln n}{n}
   \]

   Then by applying L’Hopital’s rule we get:
   \[
   \lim_{n \to \infty} \frac{\ln \left( \frac{1}{n} \right)}{n} = \lim_{n \to \infty} \frac{-\ln n}{n} = \lim_{n \to \infty} \frac{-\frac{1}{n}}{1} = -\lim_{n \to \infty} \frac{1}{n} = 0
   \]
(e) \( \left\{ \left( \frac{1}{n} \right)^{1/n} \right\} \)

Find the limit of the logarithm of the expression, which is \( \frac{1}{n} \ln \left( \frac{1}{n} \right) \), using L’Hospital’s rule.

\[
\lim_{n \to \infty} \frac{1}{n} \ln \left( \frac{1}{n} \right) = \lim_{n \to \infty} -\ln n = \lim_{n \to \infty} \frac{-1}{1} = \lim_{n \to \infty} \frac{-1}{n} = 0
\]

Thus the limit of the original expression is \( e^0 \).

(f) \( \left\{ \left( 1 - \frac{4}{n} \right)^n \right\} \)

Find the limit of the logarithm of the expression, which is \( n \ln \left( 1 - \frac{4}{n} \right) \), using L’Hospital’s rule.

\[
\lim_{n \to \infty} n \ln \left( 1 - \frac{4}{n} \right) = \lim_{n \to \infty} \frac{\ln \left( 1 - \frac{4}{n} \right)}{\frac{1}{n}} = \lim_{n \to \infty} \frac{\frac{4}{n^2}}{1 - \frac{4}{n}} = \lim_{n \to \infty} \frac{-4}{1 - \frac{4}{n}} = -4
\]

Thus the limit of the original expression is \( e^{-4} \).

(g) \( a_n = e^{-n} \cos n \)

The sequence is

\[
a_n = e^{-n} \cos n = \frac{\cos n}{e^n}
\]

The numerator of the sequence is bounded by 1 and the denominator increases without any bound, so:

\[
\lim_{n \to \infty} e^{-n} \cos n = \lim_{n \to \infty} \frac{\cos n}{e^n} = 0
\]

(h) \( a_n = \frac{\ln n}{n^{1.1}} \)

Using L’Hospital’s rule, we have

\[
\lim_{n \to \infty} \frac{\ln n}{n^{1.1}} = \lim_{n \to \infty} \frac{\frac{1}{n}}{(1.1)n^{0.1}} = \lim_{n \to \infty} \frac{1}{(1.1)n^{1.1}} = 0
\]
Questions related to SECTION 8.3

1. Evaluate the following geometric sums.

(a) \[ \sum_{k=0}^{20} \left( \frac{2}{5} \right)^k \]
   
   We have \( a_1 = 1 \), \( r = \frac{4}{25} \) and \( n = 21 \)
   
   \[ S = \frac{a_1 (1 - r^n)}{1 - r} = \frac{1 - \left( \frac{4}{25} \right)^{21}}{1 - \frac{4}{25}} = \frac{25^{21} - 4^{21}}{25^{21} - 4 \cdot 25^{20}} \approx 1.1905 \]

(b) \[ \sum_{k=4}^{12} 2^k \]
   
   We have \( a_1 = 16 \), \( r = 2 \) and \( n = 9 \)
   
   \[ S = \frac{a_1 (1 - r^n)}{1 - r} = 16 \cdot \frac{1 - 2^9}{1 - 2} = 512 \cdot 16 = 8176 \]

(c) \[ \sum_{k=0}^{9} \left( -\frac{3}{4} \right)^k \]
   
   We have \( a_1 = 1 \), \( r = -\frac{3}{4} \) and \( n = 10 \)
   
   \[ S = \frac{a_1 (1 - r^n)}{1 - r} = 1 \cdot \frac{1 - \left( -\frac{3}{4} \right)^{10}}{1 + \frac{3}{4}} = \frac{4^{10} - 3^{10}}{4^{10} + 3 \cdot 4^9} = \frac{141361}{262144} \approx 0.5392 \]

(d) \[ \sum_{k=0}^{20} (-1)^k \]
   
   We have \( a_1 = 1 \), \( r = -1 \) and \( n = 21 \)
   
   \[ S = \frac{a_1 (1 - r^n)}{1 - r} = 1 \cdot \frac{1 - (-1)^{21}}{1 + 1} = 1 \]

2. For the following telescoping series, find a formula for the \( n^{th} \) term of the sequence of the partial sums \( \{S_n\} \). Then evaluate \( \lim_{n \to \infty} S_n \) to obtain the value of the series or state that the series diverges.

(a) \[ \sum_{k=1}^{\infty} \left( \frac{1}{k+2} - \frac{1}{k+3} \right) \]
   
   When we write the terms of the sum we get:
   
   \[ \sum_{k=1}^{\infty} \left( \frac{1}{k+2} - \frac{1}{k+3} \right) = \left( \frac{1}{3} - \frac{1}{4} \right) + \left( \frac{1}{4} - \frac{1}{5} \right) + \left( \frac{1}{5} - \frac{1}{6} \right) + \left( \frac{1}{6} - \frac{1}{7} \right) + ... \]
It is clear that second term of each summand cancels with the first term of
the succeeding summand, so
\[ S_n = \frac{1}{3} - \frac{1}{n + 3} = \frac{n}{3n + 9} \Rightarrow \lim_{n \to \infty} \frac{n}{3n + 9} = \frac{1}{3} \]

Questions related to SECTION 8.4

1. Use the Divergence Test to determine whether the following series diverge or
   state that the test is inconclusive.

   (a) \( \sum_{k=1}^{\infty} \frac{k}{k^2 + 1} \)

   We have \( a_k = \frac{k}{k^2 + 1} \). It is clear that \( \lim_{k \to \infty} a_k = 0 \)
   Thus we say that the Divergence Test is inconclusive.

   (b) \( \sum_{k=2}^{\infty} \frac{k}{\ln k} \)

   We have \( a_k = \frac{k}{\ln k} \). It is clear that \( \lim_{k \to \infty} a_k = \infty \)
   Thus we say that the series diverges.

   (c) \( \sum_{k=1}^{\infty} \frac{k^2}{2^k} \)

   We have \( a_k = \frac{k^2}{2^k} \). It is clear that \( \lim_{k \to \infty} a_k = 0 \)
   Thus we say that the Divergence Test is inconclusive.

2. Use the Integral Test to determine the convergence or the divergence of the fol-
   lowing series. Check that the conditions of the test are satisfied.

   (a) \( \sum_{k=2}^{\infty} \frac{1}{k \ln k} \)

   Let \( f(x) = \frac{1}{x \ln x} \). Then \( f(x) \) is continuous and decreasing on \((1, \infty)\),
   since \( x \ln x \) is increasing there. We have:
   \[ \int_{1}^{\infty} f(x) \, dx = \infty \]
   thus, we say that the series diverges.
(b) \[ \sum_{k=1}^{\infty} \frac{k}{\sqrt{k^2 + 4}} \]

Let \( f(x) = \frac{x}{\sqrt{x^2 + 4}} \). \( f(x) \) is continuous for \( x \geq 1 \).

Note that \( f'(x) = \frac{4}{(\sqrt{x^2 + 4})^3} > 0 \). Thus \( f \) is increasing and the conditions of the Integral Test aren’t satisfied. Thus the given series diverges by the Divergence Test.

(c) \[ \sum_{k=1}^{\infty} ke^{-2k^2} \]

Let \( f(x) = xe^{-2x^2} \). This function is continuous for \( x \geq 1 \).

Note that \( f'(x) = e^{-2x^2}(1 - 4x^2) < 0 \), for \( x \geq 1 \), so \( f(x) \) is decreasing. Thus we have:

\[ \int_{1}^{\infty} xe^{-2x^2} \, dx = \frac{1}{4e^2} \]

So, we say that the series converges.

**Questions related to SECTION 8.5**

1. Use the Ratio Test to determine whether the following series converge.

(a) \[ \sum_{k=1}^{\infty} \frac{k^2}{4^k} \]

The ratio between successive terms is:

\[ \frac{a_{k+1}}{a_k} = \frac{(k+1)^2}{4^{k+1}} \cdot \frac{4^k}{k^2} = \frac{1}{4} \cdot \left( \frac{k+1}{k} \right)^2 \Rightarrow \lim_{k \to \infty} \frac{1}{4} \cdot \left( \frac{k+1}{k} \right)^2 = \frac{1}{4} \]

So the given series converges by the Ratio Test.

(b) \[ \sum_{k=1}^{\infty} ke^{-k} \]

The ratio between successive terms is:

\[ \frac{a_{k+1}}{a_k} = \frac{(k+1)e^{-(k+1)}}{ke^{-k}} = \frac{k+1}{k}e \Rightarrow \lim_{k \to \infty} \frac{k+1}{k}e = \frac{1}{e} < 1 \]

So the given series converges by the Ratio Test.
(c) $\sum_{k=1}^{\infty} \frac{k!}{k^k}$ The ratio between successive terms is:

$$\frac{a_{k+1}}{a_k} = \frac{(k+1)!}{k!} \cdot \frac{k^k}{(k+1)k!} = \left(1 - \frac{1}{k+1}\right)^k \Rightarrow \lim_{k \to \infty} \left(1 - \frac{1}{k+1}\right)^k = \frac{1}{e} < 1$$

So the given series converges by the Ratio Test.

(d) $\sum_{k=1}^{\infty} \frac{k^6}{k!}$

The ratio between successive terms is:

$$\frac{a_{k+1}}{a_k} = \frac{(k+1)^6}{(k+1)!} \cdot \frac{k!}{k^6} = \frac{1}{k+1} \left(1 + \frac{1}{k}\right)^6 \Rightarrow \lim_{k \to \infty} \frac{1}{k+1} \left(1 + \frac{1}{k}\right)^6 = 0$$

So the given series converges by the Ratio Test.

2. Use the Root Test to determine whether the following series converge.

(a) $\sum_{k=1}^{\infty} \frac{k^2}{2^k}$

$$\lim_{k \to \infty} \sqrt[k]{a_k} = \lim_{k \to \infty} \sqrt[k]{\frac{k^2}{2^k}} = \lim_{k \to \infty} \frac{k^2}{2^k} = \frac{1}{2} < 1$$

So the given series converges by the Root Test.

(b) $\sum_{k=1}^{\infty} \left(\frac{k+1}{2k}\right)^k$

$$\lim_{k \to \infty} \sqrt[k]{a_k} = \lim_{k \to \infty} \sqrt[k]{\left(\frac{k+1}{2k}\right)^k} = \lim_{k \to \infty} \frac{k+1}{2k} = \frac{1}{2} < 1$$

So the given series converges by the Root Test.

(c) $\sum_{k=1}^{\infty} \left(1 + \frac{3}{k}\right)^{k^2}$

$$\lim_{k \to \infty} \sqrt[k]{a_k} = \lim_{k \to \infty} \sqrt[k]{\left(1 + \frac{3}{k}\right)^{k^2}} = \lim_{k \to \infty} \left(1 + \frac{3}{k}\right)^k = e^3 > 1$$

So the given series diverges by the Root Test.
(d) \[ \sum_{k=1}^{\infty} \left( \frac{1}{\ln(k+1)} \right)^k \]

\[ \lim_{k \to \infty} \sqrt[k]{a_k} = \lim_{k \to \infty} \left( \frac{1}{\ln(k+1)} \right)^k = \lim_{k \to \infty} \frac{1}{\ln(k+1)} = 0 < 1 \]

So the given series converges by the Root Test.

3. Use the Comparison Test or Limit Comparison Test to determine whether the following series converge.

(a) \[ \sum_{k=1}^{\infty} \frac{k^2 - 1}{k^3 + 4} \]

Let us use the Limit Comparison Test with \( \left\{ \frac{1}{k} \right\} \). The ratio of the terms of the two series is:

\[ \frac{k^2 - 1}{k^3 + 4} = \frac{k^2 - 1}{k} \cdot \frac{1}{k^3 + 4} = \frac{k^2 - 1}{k} \cdot \frac{1}{k^3 + 4} = k \cdot \frac{k^2 - 1}{k^3 + 4} \Rightarrow \lim_{k \to \infty} \frac{k^2 - 1}{k^3 + 4} = 1 \]

Since the comparison series \( \left\{ \frac{1}{k} \right\} \) diverges, the given series diverges as well.

(b) \[ \sum_{k=1}^{\infty} \frac{k^2 + k - 1}{k^4 + 4k^2 - 3} \]

Let us use the Limit Comparison Test with \( \left\{ \frac{1}{k^2} \right\} \). The ratio of the terms of the two series is:

\[ \frac{k^2 + k - 1}{k^4 + 4k^2 - 3} = k^2 \cdot \frac{k^2 + k - 1}{k^4 + 4k^2 - 3} = \frac{k^4 + k^2 - k^2}{k^4 + 4k^2 - 3} = \lim_{k \to \infty} \frac{k^4 + k^2 - k^2}{k^4 + 4k^2 - 3} = 1 \]

Since the comparison series \( \left\{ \frac{1}{k^2} \right\} \) converges, the given series converges as well.
(c) \[ \sum_{k=1}^{\infty} \sqrt{\frac{k}{k^3 + 1}} \]

Let us use the Limit Comparison Test with \( \left\{ \frac{1}{k} \right\} \). The ratio of the terms of the two series is:

\[
\frac{\sqrt{\frac{k}{k^3 + 1}}}{\frac{1}{k}} = k \cdot \sqrt{\frac{k}{k^3 + 1}} = \sqrt{k^2 \sqrt{\frac{k}{k^3 + 1}}} = \sqrt{\frac{k^3}{k^3 + 1}} \Rightarrow \lim_{k \to \infty} \sqrt{\frac{k^3}{k^3 + 1}} = 1
\]

Since the comparison series \( \left\{ \frac{1}{k} \right\} \) diverges, the given series diverges as well.

(d) \[ \sum_{k=2}^{\infty} \frac{1}{(k \ln k)^2} \]

It is clear that, for all \( k \), \( \frac{1}{(k \ln k)^2} < \frac{1}{k^2} \).

Thus, since the series \( \left\{ \frac{1}{k^2} \right\} \) converges, the given series converges as well.

Questions related to SECTION 8.6

1. Determine whether the following series converge.

(a) \[ \sum_{k=1}^{\infty} (-1)^{k+1} \frac{k^2}{k^3 + 1} \]

It is clear that, the terms of the series decrease in magnitude, and

\[
\lim_{k \to \infty} \frac{k^2}{k^3 + 1} = \lim_{k \to \infty} \frac{1}{k + \frac{1}{k^2}} = 0
\]

So the given series converges.

(b) \[ \sum_{k=2}^{\infty} (-1)^k \frac{\ln k}{k^2} \]

The terms of the series decrease in magnitude, since if

\[
f(x) = \frac{\ln x}{x^2} \Rightarrow f'(x) = \frac{x(1 - 2 \ln x)}{x^4} = \frac{1 - 2 \ln x}{x^3}
\]
which is negative for large enough \( x \). Then

\[
\lim_{k \to \infty} \frac{\ln k}{k^2} = \lim_{k \to \infty} \frac{\frac{1}{2k}}{2k} = \lim_{k \to \infty} \frac{1}{2k^2} = 0
\]

Thus the given series converges.

(c) \[\sum_{k=2}^{\infty} (-1)^k \left(1 + \frac{1}{k}\right)\]

Since we have:

\[
\lim_{k \to \infty} \left(1 + \frac{1}{k}\right) = 1
\]

the given series diverges.

2. Determine whether the following series converge absolutely or conditionally.

(a) \[\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^{3/2}}\]

The series of absolute values is a \( p \)-series with \( p = \frac{3}{2} \). Thus the series converges absolutely.

(b) \[\sum_{k=2}^{\infty} \frac{(-1)^k}{\ln k}\]

The series of absolute values is \[\sum \frac{1}{\ln k}\], which diverges, so the series does not converge absolutely.

However, since \[\lim_{k \to \infty} \frac{1}{\ln k} \to 0\] and the terms are non-increasing, the series converge conditionally.

(c) \[\sum_{k=1}^{\infty} \frac{(-1)^k \tan^{-1} k}{k^3}\]

The series of absolute values is \[\sum \frac{\tan^{-1} k}{k^3}\], which converges by the Comparison Test since

\[
\frac{\tan^{-1} k}{k^3} < \frac{\pi}{2k^3}
\]

and \[\sum \frac{\pi}{2k^3}\] converges since it is a constant multiple of a convergent \( p \)-series. Thus the given series converges absolutely.