Questions related to SECTION 8.1

1. (a) Find the next two terms of the sequence.
   (b) Find a recurrence relation that generates the relation.
   (c) Find an explicit formula for the general nth term of the sequence.

   i. $\left\{1, \frac{1}{2}, , \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, ...\right\}$

      (a) $\frac{1}{32}$ and $\frac{1}{64}$

      (b) $a_1 = 1; \qquad a_{n+1} = \frac{a_n}{2}$

      (c) $a_n = \frac{1}{2^{n-1}}$

   ii. $\left\{1, -2, 3, -4, 5, ...\right\}$

      (a) -6 and 7

      (b) $a_1 = 1; \qquad a_{n+1} = (-1)^n(|a_n| + 1)$

      (c) $a_n = (-1)^{n+1}n$

2. Write the terms a_1, a_2, a_3 and a_4 of the following sequences. If the sequence appears to converge, make a conjecture about its limit. If the sequence diverges, explain why.

   (a) $a_n = \dfrac{(-1)^n}{n}; \quad n = 1, 2, 3, ...$

   $$a_1 = -1 \quad , \quad a_2 = \frac{1}{2} \quad , \quad a_3 = -\frac{1}{3} \quad and \quad a_4 = \frac{1}{4}$$

   This sequence converges to 0 since each term is smaller in absolute value than the preceding term and they get arbitrarily close to 0.

   (b) $a_n = 1 - 10^{-n}; \quad n = 1, 2, 3, ...$

   $$a_1 = 0.9 \quad , \quad a_2 = 0.99 \quad , \quad a_3 = 0.999 \quad and \quad a_4 = 0.9999$$

   This sequence converges to 1.

(c) $a_{n+1} = \frac{a_n^2}{10}; \quad a_0 = 1$

Rewrite the recurrence as $\quad a_{n+1} = 10^{-1}a_n^2$. Then we have

$$a_0 = 1 \quad , \quad a_1 = 10^{-1} = \frac{1}{10} \quad , \quad a_2 = 10^{-1}(10^{-1})^2 = 10^{-3} = \frac{1}{1000} \quad ,$$
$$a_3 = 10^{-1}(10^{-3})^2 = 10^{-7} = \frac{1}{10000000},$$
$$a_4 = 10^{-1}(10^{-7})^2 = 10^{-15} = \frac{1}{1000000000000000}$$

This sequence converges to 0.

(d) $a_{n+1} = 0.5a_n(1 - a_n); \quad a_0 = 0.8$

$$a_0 = 0.8 \quad , \quad a_1 = 0.5 \cdot 0.8(1 - 0.8) = 0.5 \cdot 0.8 \cdot 0.2 = 0.08 \quad ,$$
$$a_2 = 0.5 \cdot 0.08(1 - 0.08) = 0.5 \cdot 0.08 \cdot 0.92 = 0.0368 \quad ,$$
$$a_3 = 0.5 \cdot 0.0368(1 - 0.0368) = 0.01772288$$
$$a_4 = 0.5 \cdot 0.01772288(1 - 0.01772288) \approx 0.0087$$

3. Consider the following recurrence relations.

   (a) Find the terms a_0, a_1, a_2 and a_3 of the sequence.

   (b) If possible, find an explicit formula for the n^{th} term of the sequence.

   i. $a_{n+1} = a_n + 2; \quad a_0 = 3$

   ii. $a_{n+1} = 2a_n + 1; \quad a_0 = 0$

   i.

$$a_0 = 3 \quad , \quad a_1 = 5 \quad , \quad a_2 = 7 \quad and \quad a_3 = 9$$
$$a_n = 2n + 3$$

   ii.

$$a_0 = 0 \quad , \quad a_1 = 1 \quad , \quad a_2 = 3 \quad , \quad a_3 = 7 \quad and \quad a_4 = 15$$
$$a_n = 2^n - 1$$

Questions related to SECTION 8.2

1. Find the limit of the following sequences or determine that the limit does not exist.

   (a) $\left\{\dfrac{3n^3-1}{2n^3+1}\right\}$

   Dividing the numerator and the denominator by n^3 we get :

   $$\lim_{n\to\infty}\frac{3-n^{-3}}{2+n^{-3}}=\frac{3}{2}$$

   (b) $\left\{\left(1+\dfrac{2}{n}\right)^n\right\}$

   Find the limit of the logarithm of the expression, which is $n\ln\left(1+\dfrac{2}{n}\right)$, using L'Hopital's rule.

   $$\lim_{n\to\infty} n\ln\left(1+\frac{2}{n}\right)=\lim_{n\to\infty}\frac{\ln\left(1+\frac{2}{n}\right)}{\frac{1}{n}}=\lim_{n\to\infty}\frac{\frac{1}{1+\frac{2}{n}}\left(-\frac{2}{n^2}\right)}{-\frac{1}{n^2}}=\lim_{n\to\infty}\frac{2}{1-\frac{2}{n}}=2$$

   Thus the limit of the original expression is e^2.

   (c) $\left\{\sqrt{\left(1+\dfrac{1}{2n}\right)^n}\right\}$

   Take the logarithm of the expression and use L'Hopital's rule.

   $$\lim_{n\to\infty}\frac{n}{2}\ln\left(1+\frac{1}{2n}\right)=\lim_{n\to\infty}\frac{\ln\left(1+\frac{1}{2n}\right)}{\frac{2}{n}}=\lim_{n\to\infty}\frac{\frac{1}{1+\frac{1}{2n}}\left(-\frac{1}{2n^2}\right)}{-\frac{2}{n^2}}=\frac{1}{4}$$

   Thus the limit of the original expression is $e^{\frac{1}{4}}$.

   (d) $\left\{\dfrac{\ln\left(\frac{1}{n}\right)}{n}\right\}$

   Since $\ln\left(\dfrac{1}{n}\right)=-\ln n$, we get

   $$\lim_{n\to\infty}\frac{\ln\left(\frac{1}{n}\right)}{n}=\lim_{n\to\infty}\frac{-\ln n}{n}$$

   Then by applying L'Hopital's rule we get :

   $$\lim_{n\to\infty}\frac{\ln\left(\frac{1}{n}\right)}{n}=\lim_{n\to\infty}\frac{-\ln n}{n}=\lim_{n\to\infty}\frac{-\frac{1}{n}}{1}=-\lim_{n\to\infty}\frac{1}{n}=0$$

(e) $\left\{ \left(\frac{1}{n}\right)^{1/n} \right\}$

Find the limit of the logarithm of the expression, which is $\frac{1}{n}\ln\left(\frac{1}{n}\right)$, using L'Hopital's rule.

$$\lim_{n\to\infty} \frac{1}{n}\ln\left(\frac{1}{n}\right) = \lim_{n\to\infty} \frac{-\ln\, n}{n} = \lim_{n\to\infty} \frac{-\frac{1}{n}}{1} = \lim_{n\to\infty} \frac{-1}{n} = 0$$

Thus the limit of the original expression is e^0.

(f) $\left\{ \left(1-\frac{4}{n}\right)^{n} \right\}$

Find the limit of the logarithm of the expression, which is $n\ln\left(1-\frac{4}{n}\right)$, using L'Hopital's rule.

$$\lim_{n\to\infty} n\ln\left(1-\frac{4}{n}\right) = \lim_{n\to\infty} \frac{\ln\left(1-\frac{4}{n}\right)}{\frac{1}{n}} = \lim_{n\to\infty} \frac{\frac{1}{1-\frac{4}{n}}\left(\frac{4}{n^2}\right)}{-\frac{1}{n^2}} = \lim_{n\to\infty} \frac{-4}{1-\frac{4}{n}} = -4$$

Thus the limit of the original expression is e^{-4}.

(g) $a_n = e^{-n}\cos\, n$

The sequence is

$$a_n = e^{-n}\cos\, n = \frac{\cos\, n}{e^n}$$

The numerator of the sequence is bounded by 1 and the denominator increases without any bound, so:

$$\lim_{n\to\infty} e^{-n}\cos\, n = \lim_{n\to\infty} \frac{\cos\, n}{e^n} = 0$$

(h) $a_n = \frac{\ln\, n}{n^{1.1}}$

Using L'Hopital's rule, we have

$$\lim_{n\to\infty} \frac{\ln\, n}{n^{1.1}} = \lim_{n\to\infty} \frac{\frac{1}{n}}{(1.1)n^{0.1}} = \lim_{n\to\infty} \frac{1}{(1.1)n^{1.1}} = 0$$

Questions related to SECTION 8.3

1. Evaluate the following geometric sums.

   (a) $\sum_{k=0}^{20} \left(\frac{2}{5}\right)^{2k}$

   We have $a_1 = 1$, $r = \frac{4}{25}$ and $n = 21$

   $$S = \frac{a_1 (1 - r^n)}{1 - r} = 1 \cdot \frac{1 - \left(\frac{4}{25}\right)^{21}}{1 - \frac{4}{25}} = \frac{25^{21} - 4^{21}}{25^{21} - 4 \cdot 25^{20}} \approx 1.1905$$

   (b) $\sum_{k=4}^{12} 2^k$

   We have $a_1 = 16$, $r = 2$ and $n = 9$

   $$S = \frac{a_1 (1 - r^n)}{1 - r} = 16 \cdot \frac{1 - 2^9}{1 - 2} = 511 \cdot 16 = 8176$$

   (c) $\sum_{k=0}^{9} \left(-\frac{3}{4}\right)^{k}$

   We have $a_1 = 1$, $r = -\frac{3}{4}$ and $n = 10$

   $$S = \frac{a_1 (1 - r^n)}{1 - r} = 1 \cdot \frac{1 - \left(-\frac{3}{4}\right)^{10}}{1 + \frac{3}{4}} = \frac{4^{10} - 3^{10}}{4^{10} + 3 \cdot 4^9} = \frac{141361}{262144} \approx 0.5392$$

   (d) $\sum_{k=0}^{20} (-1)^k$

   We have $a_1 = 1$, $r = -1$ and $n = 21$

   $$S = \frac{a_1 (1 - r^n)}{1 - r} = 1 \cdot \frac{1 - (-1)^{21}}{1 + 1} = 1$$

2. For the following telescoping series, find a formula for the n^{th} term of the sequence of the partial sums $\{S_n\}$. Then evaluate $\lim_{n\to\infty} S_n$ to obtain the value of the series or state that the series diverges.

   (a) $\sum_{k=1}^{\infty} \left(\frac{1}{k+2} - \frac{1}{k+3}\right)$

   When we write the terms of the sum we get:

   $$\sum_{k=1}^{\infty} \left(\frac{1}{k+2} - \frac{1}{k+3}\right) = \left(\frac{1}{3} - \frac{1}{4}\right) + \left(\frac{1}{4} - \frac{1}{5}\right) + \left(\frac{1}{5} - \frac{1}{6}\right) + \left(\frac{1}{6} - \frac{1}{7}\right) + \ldots$$

It is clear that second term of each summand cancels with the first term of the succeeding summand,so

$$S_n = \frac{1}{3} - \frac{1}{n+3} = \frac{n}{3n+9} \quad \Rightarrow \quad \lim_{n\to\infty} \frac{n}{3n+9} = \frac{1}{3}$$

Questions related to SECTION 8.4

1. Use the Divergence Test to determine whether the following series diverge or state that the test is inconclusive.

   (a) $\displaystyle\sum_{k=1}^{\infty} \frac{k}{k^2+1}$

   We have $\quad a_k = \dfrac{k}{k^2+1}$. It is clear that $\lim_{k\to\infty} a_k = 0$

   Thus we say that the Divergence Test is inconclusive.

   (b) $\displaystyle\sum_{k=2}^{\infty} \frac{k}{\ln\, k}$

   We have $\quad a_k = \dfrac{k}{\ln\, k}$. It is clear that $\lim_{k\to\infty} a_k = \infty$

   Thus we say that the series diverges.

   (c) $\displaystyle\sum_{k=1}^{\infty} \frac{k^2}{2^k}$

   We have $\quad a_k = \dfrac{k^2}{2^k}$. It is clear that $\lim_{k\to\infty} a_k = 0$

   Thus we say that the Divergence Test is inconclusive.

2. Use the Integral Test to determine the convergence or the divergence of the following series. Check that the conditions of the test are satisfied.

   (a) $\displaystyle\sum_{k=2}^{\infty} \frac{1}{k\,\ln\, k}$

   Let $f(x) = \dfrac{1}{x\,\ln\, x}$. Then $f(x)$ is continuous and decreasing on $(1,\infty)$, since $x\,\ln\, x$ is increasing there. We have:

   $$\int_1^{\infty} f(x)\, dx = \infty$$

   thus, we say that the series diverges.

(b) $\displaystyle\sum_{k=1}^{\infty} \frac{k}{\sqrt{k^2+4}}$

Let $f(x) = \dfrac{x}{\sqrt{x^2+4}}$. $f(x)$ is continuous for $x \geq 1$.

Note that $f'(x) = \dfrac{4}{(\sqrt{x^2+4})^3} > 0$. Thus f is increasing and the conditions of the Integral Test aren't satisfied. Thus the given series diverges by the Divergence Test.

(c) $\displaystyle\sum_{k=1}^{\infty} ke^{-2k^2}$

Let $f(x) = x\,e^{-2x^2}$. This function is continuous for $x \geq 1$.

Note that $f'(x) = e^{-2x^2}(1-4x^2) < 0$, for $x \geq 1$, so $f(x)$ is decreasing. Thus we have:

$$\int_1^{\infty} x\,e^{-2x^2}\,dx = \frac{1}{4e^2}$$

So, we say that the series converges.

Questions related to SECTION 8.5

1. Use the Ratio Test to determine whether the following series converge.

(a) $\displaystyle\sum_{k=1}^{\infty} \frac{k^2}{4^k}$

The ratio between successive terms is :

$$\frac{a_{k+1}}{a_k} = \frac{(k+1)^2}{4^{k+1}} \cdot \frac{4^k}{k^2} = \frac{1}{4} \cdot \left(\frac{k+1}{k}\right)^2 \quad \Rightarrow \quad \lim_{k\to\infty} \frac{1}{4} \cdot \left(\frac{k+1}{k}\right)^2 = \frac{1}{4}$$

So the given series converges by the Ratio Test.

(b) $\displaystyle\sum_{k=1}^{\infty} ke^{-k}$

The ratio between successive terms is :

$$\frac{a_{k+1}}{a_k} = \frac{(k+1)e^{-(k+1)}}{ke^{-k}} = \frac{k+1}{k\,e} \quad \Rightarrow \quad \lim_{k\to\infty} \frac{k+1}{k\,e} = \frac{1}{e} < 1$$

So the given series converges by the Ratio Test.

(c) $\sum_{k=1}^{\infty} \frac{k!}{k^k}$ The ratio between successive terms is :

$$\frac{a_{k+1}}{a_k} = \frac{(k+1)!}{(k+1)^{(k+1)}} \cdot \frac{k^k}{k!} = \left(\frac{k}{k+1}\right)^k = \left(1 - \frac{1}{k+1}\right)^k \quad \Rightarrow \quad \lim_{k\to\infty} \left(1 - \frac{1}{k+1}\right)^k = \frac{1}{e} < 1$$

So the given series converges by the Ratio Test.

(d) $\sum_{k=1}^{\infty} \frac{k^6}{k!}$

The ratio between successive terms is :

$$\frac{a_{k+1}}{a_k} = \frac{(k+1)^6}{(k+1)!} \cdot \frac{k!}{k^6} = \frac{1}{k+1}\left(\frac{k+1}{k}\right)^6 \quad \Rightarrow \quad \lim_{k\to\infty} \frac{1}{k+1}\left(\frac{k+1}{k}\right)^6 = 0$$

So the given series converges by the Ratio Test.

2. Use the Root Test to determine whether the following series converge.

(a) $\sum_{k=1}^{\infty} \frac{k^2}{2^k}$

$$\lim_{k\to\infty} \sqrt[k]{a_k} = \lim_{k\to\infty} \sqrt[k]{\frac{k^2}{2^k}} = \lim_{k\to\infty} \frac{k^{\frac{2}{k}}}{2} = \frac{1}{2} < 1$$

So the given series converges by the Root Test.

(b) $\sum_{k=1}^{\infty} \left(\frac{k+1}{2k}\right)^k$

$$\lim_{k\to\infty} \sqrt[k]{a_k} = \lim_{k\to\infty} \sqrt[k]{\left(\frac{k+1}{2k}\right)^k} = \lim_{k\to\infty} \frac{k+1}{2k} = \frac{1}{2} < 1$$

So the given series converges by the Root Test.

(c) $\sum_{k=1}^{\infty} \left(1 + \frac{3}{k}\right)^{k^2}$

$$\lim_{k\to\infty} \sqrt[k]{a_k} = \lim_{k\to\infty} \sqrt[k]{\left(1 + \frac{3}{k}\right)^{k^2}} = \lim_{k\to\infty} \left(1 + \frac{3}{k}\right)^k = e^3 > 1$$

So the given series diverges by the Root Test.

(d) $\displaystyle\sum_{k=1}^{\infty}\left(\frac{1}{\ln(k+1)}\right)^k$

$$\lim_{k\to\infty}\sqrt[k]{a_k}=\lim_{k\to\infty}\sqrt[k]{\left(\frac{1}{\ln(k+1)}\right)^k}=\lim_{k\to\infty}\frac{1}{\ln(k+1)}=0<1$$

So the given series converges by the Root Test.

3. Use the Comparison Test or Limit Comparison Test to determine whether the following series converge.

(a) $\displaystyle\sum_{k=1}^{\infty}\frac{k^2-1}{k^3+4}$

Let us use the Limit Comparison Test with $\left\{\frac{1}{k}\right\}$.The ratio of the terms of the two series is:

$$\frac{\frac{k^2-1}{k^3+4}}{\frac{1}{k}}=k\cdot\frac{k^2-1}{k^3+4}=\frac{k^3-k}{k^3+4}\quad\Rightarrow\quad\lim_{k\to\infty}\frac{k^3-k}{k^3+4}=1$$

Since the comparison series $\left(\left\{\frac{1}{k}\right\}\right)$ diverges, the given series diverges as well.

(b) $\displaystyle\sum_{k=1}^{\infty}\frac{k^2+k-1}{k^4+4k^2-3}$

Let us use the Limit Comparison Test with $\left\{\frac{1}{k^2}\right\}$.The ratio of the terms of the two series is:

$$\frac{\frac{k^2+k-1}{k^4+4k^2-3}}{\frac{1}{k^2}}=k^2\cdot\frac{k^2+k-1}{k^4+4k^2-3}=\frac{k^4+k^3-k^2}{k^4+4k^2-3}\quad\Rightarrow\quad\lim_{k\to\infty}\frac{k^4+k^3-k^2}{k^4+4k^2-3}=1$$

Since the comparison series $\left\{\frac{1}{k^2}\right\}$ converges, the given series converges as well.

(c) $\sum_{k=1}^{\infty} \sqrt{\frac{k}{k^3+1}}$

Let us use the Limit Comparison Test with $\left\{\frac{1}{k}\right\}$.The ratio of the terms of the two series is:

$$\frac{\sqrt{\frac{k}{k^3+1}}}{\frac{1}{k}} = k \cdot \sqrt{\frac{k}{k^3+1}} = \sqrt{k^2}\sqrt{\frac{k}{k^3+1}} = \sqrt{\frac{k^3}{k^3+1}} \quad \Rightarrow \quad \lim_{k\to\infty} \sqrt{\frac{k^3}{k^3+1}} = 1$$

Since the comparison series $\left\{\frac{1}{k}\right\}$ diverges, the given series diverges as well.

(d) $\sum_{k=2}^{\infty} \frac{1}{(k \ln k)^2}$

It is clear that, for all k, $\frac{1}{(k \ln k)^2} < \frac{1}{k^2}$.

Thus, since the series $\left\{\frac{1}{k^2}\right\}$ converges, the given series converges as well.

Questions related to SECTION 8.6

1. Determine whether the following series converge.

(a) $\sum_{k=1}^{\infty} (-1)^{k+1} \frac{k^2}{k^3+1}$

It is clear that, the terms of the series decrease in magnitude, and

$$\lim_{k\to\infty} \frac{k^2}{k^3+1} = \lim_{k\to\infty} \frac{1}{k + \frac{1}{k^2}} = 0$$

So the given series converges.

(b) $\sum_{k=2}^{\infty} (-1)^k \frac{\ln k}{k^2}$

The terms of the series decrease in magnitude, since if

$$f(x) = \frac{\ln x}{x^2} \quad \Rightarrow \quad f'(x) = \frac{x(1 - 2\ln x)}{x^4} = \frac{1 - 2\ln, x}{x^3}$$

which is negative for large enough x. Then

$$\lim_{k\to\infty} \frac{\ln\ k}{k^2} = \lim_{k\to\infty} \frac{\frac{1}{k}}{2k} = \lim_{k\to\infty} \frac{1}{2k^2} = 0$$

Thus the given series converges.

(c) $\sum_{k=2}^{\infty} (-1)^k \left(1 + \frac{1}{k}\right)$

Since we have:

$$\lim_{k\to\infty} \left(1 + \frac{1}{k}\right) = 1$$

the given series diverges.

2. Determine whether the following series converge absolutely or conditionally.

(a) $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^{3/2}}$

The series of absolute values is a $p-series$ with $p = \frac{3}{2}$. Thus the series converges absolutely.

(b) $\sum_{k=2}^{\infty} \frac{(-1)^k}{\ln\ k}$

The series of absolute values is $\sum \frac{1}{\ln\ k}$, which diverges, so the series does not converge absolutely.

However, since $\lim_{k\to\infty} \frac{1}{\ln\ k} \to 0$ and the terms are non-increasing, the series converge conditionally.

(c) $\sum_{k=1}^{\infty} \frac{(-1)^k \tan^{-1}\ k}{k^3}$

The series of absolute values is $\sum \frac{\tan^{-1}\ k}{k^3}$, which converges by the Comparison Test since

$$\frac{\tan^{-1}\ k}{k^3} < \frac{\pi}{2}\frac{1}{k^3}$$

and $\sum \frac{\pi}{2}\frac{1}{k^3}$ converges since it is a constant multiple of a convergent $p-series$. Thus the given series converges absolutely.

