MATH151 Final Exam
Department of Mathematics
Spring 2014 - June, 9, 2014

1. Evaluate the area between the curves:

   (a)
   $$y = x^3 - x^2 - 2x, \qquad y = 0, \qquad -1 \le x \le 2$$

   Solution. The points of intersection of two graphs are found as

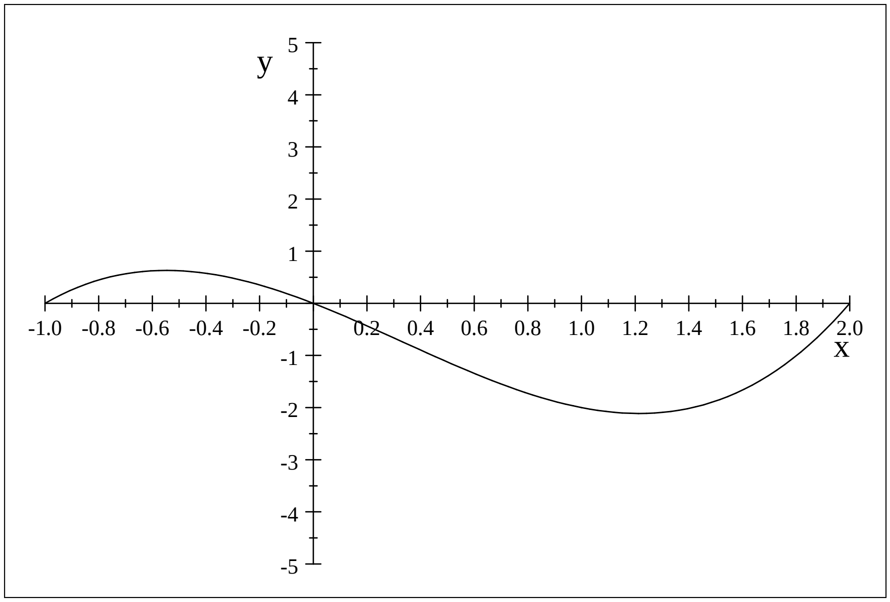
   $$\begin{aligned} x^3 - x^2 - 2x &= 0, \qquad x\left(x^2 - x - 2\right) = x(x-2)(x+1) = 0, \\ x &= 0, \qquad x = -1, \qquad x = 2. \end{aligned}$$

   We have the following inequalities:

   $$\begin{aligned} x(x-2)(x+1) &> 0, \qquad -1 < x < 0, \\ x(x-2)(x+1) &< 0, \qquad 0 < x < 2 \end{aligned}$$

   Therefore the area between the graphs is

   $$\begin{aligned} A &= \int_{-1}^{0} \left(x^3 - x^2 - 2x\right) dx - \int_{0}^{2} \left(x^3 - x^2 - 2x\right) dx \\ &= \left(\frac{x^4}{4} - \frac{x^3}{3} - x^2\right)\Bigg|_{-1}^{0} - \left(\frac{x^4}{4} - \frac{x^3}{3} - x^2\right)\Bigg|_{0}^{2} \\ &= -\left(\frac{1}{4} + \frac{1}{3} - 1\right) - \left(\frac{16}{4} - \frac{8}{3} - 4\right) \\ &= \frac{37}{12}. \end{aligned}$$



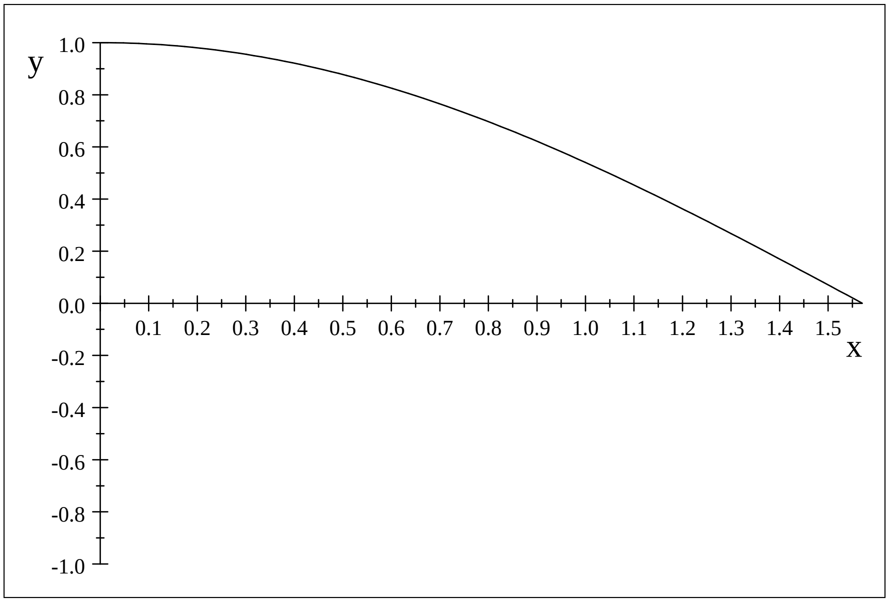
(b)

$$y = \cos x, \qquad y = 0, \qquad 0 \leq x \leq \frac{\pi}{2}$$

Solution.

$$A = \int_0^{\frac{\pi}{2}} \cos x dx = \sin x|_0^{\frac{\pi}{2}} = 1 - 0 = 1$$

$$\cos x$$



2. (a) Use the formula for linear approximation to find the value:

$$\sqrt[3]{1.009}$$

Solution. Taking $f(x) = \sqrt[3]{x}$ and $a = 1$ we use the formula for the linear approximation:

$$\begin{aligned} f(x) &\simeq f(a) + f'(a)(x-a), \\ f'(x) &= \frac{1}{3\sqrt[3]{x^2}}, \qquad f'(1) = \frac{1}{3\sqrt[3]{1^2}} = \frac{1}{3}. \\ f(1) &= 1, \qquad x - a = 1.009 - 1 = 0.009 \\ f(1.009) &\simeq 1 + \frac{1}{3}(0.009) = 1 + 0.003 = 1.003 \end{aligned}$$

(b) Use the L'Hopital Rule to calculate the limit

$$\lim_{x\to\infty}\left(1+\frac{5}{x}\right)^x$$

Solution. Let

$$L = \lim_{x\to\infty}\left(1+\frac{5}{x}\right)^x$$

Taking ln of both sides:

$$\begin{aligned}
\ln L &= \ln \lim_{x\to\infty}\left(1+\frac{5}{x}\right)^{x} = \lim_{x\to\infty} x\ln\left(1+\frac{5}{x}\right) \\
&= \lim_{x\to\infty}\frac{\ln\left(1+\frac{5}{x}\right)}{1/x} = \left[\frac{0}{0}\right].
\end{aligned}$$

Using L'Hopital's Rule

$$\begin{aligned}
\lim_{x\to\infty}\frac{\ln\left(1+\frac{5}{x}\right)}{1/x} &= \lim_{x\to\infty}\frac{\frac{x}{5+x}\left(-\frac{5}{x^2}\right)}{-1/x^2} \\
&= \lim_{x\to\infty}\frac{5x}{5+x} = 5, \\
L &= e^5.
\end{aligned}$$

3. Calculate the integrals:

   (a) using integration by parts twice:

$$\int\left(x^2+1\right)\cos x dx$$

Solution.

$$\begin{aligned}
\int\left(x^2+1\right)\cos x dx &= \left|\begin{array}{ll} u=x^2+1, & du=2xdx \\ dv=\cos x dx, & v=\sin x \end{array}\right| \\
&= \left(x^2+1\right)\sin x - \int 2x\sin x dx \\
&= \left|\begin{array}{ll} u=x, & du=dx \\ dv=\sin x dx, & v=-\cos x \end{array}\right| \\
&= \left(x^2+1\right)\sin x - 2\left(-x\cos x - \int -\cos x dx\right) \\
&= \left(x^2+1\right)\sin x + 2x\cos x - \int \cos x dx \\
&= \left(x^2+1\right)\sin x + 2x\cos x - \sin x + C
\end{aligned}$$

   (b) using partial fractions

$$\int\frac{2x+1}{(x+1)^2}dx$$

Solution.

$$\begin{aligned}\frac{2x+1}{(x+1)^2} &= \frac{A}{x+1}+\frac{B}{(x+1)^2}=\frac{A(x+1)+B}{(x+1)^2}\\ Ax+A+B &= 2x+1, \qquad A=2, \qquad B=-1\end{aligned}$$

$$\begin{aligned}\int \frac{2x+1}{(x+1)^2}dx &= \int \left[\frac{2}{x+1}-\frac{1}{(x+1)^2}\right]dx\\ &= 2\ln|x+1|+\frac{1}{x+1}+C\end{aligned}$$

(c) using the substitution method

$$\int_0^1 \frac{2\ln(2x+1)}{2x+1}dx$$

Solution.

$$\begin{aligned}\int_0^1 \frac{2\ln(2x+1)}{2x+1}dx &= \left|\begin{array}{c} u=\ln(2x+1)\\ du=\frac{2}{2x+1},\\ x=0, u=0\\ x=1, u=\ln 3\end{array}\right| = \int_0^{\ln 3} udu=\\ &= \left.\frac{u^2}{2}\right|_0^{\ln 3}=\frac{\ln^2 3}{2}\end{aligned}$$

4. Calculate the following improper integrals or state that they are divergent

(a)

$$\int_0^\infty \frac{1}{x^2+1}dx$$

Solution.

$$\begin{aligned}\int_0^\infty \frac{1}{x^2+1}dx &= \lim_{R\to\infty}\int_0^R \frac{1}{x^2+1}dx=\lim_{R\to\infty}\left[\tan^{-1}x\right]\Big|_0^R\\ &= \lim_{R\to\infty}\left[\tan^{-1}R-\tan^{-1}0\right]=\frac{\pi}{2}-0=\frac{\pi}{2}.\end{aligned}$$

(b)

$$\int_1^\infty \frac{1}{\sqrt[3]{x^4}}dx$$

Solution

$$\begin{aligned}\int_1^\infty \frac{1}{\sqrt[3]{x^4}}dx &= \lim_{b\to\infty}\int_1^b \frac{1}{\sqrt[3]{x^4}}dx\\ &= \lim_{b\to\infty}\left(-\frac{1}{3\sqrt[3]{x}}\right)\Bigg|_1^b = -\frac{1}{3}\end{aligned}$$

(c)

$$\int_0^1 \frac{dx}{x^{2/3}}$$

Solution

$$\int_0^1 \frac{dx}{x^{2/3}} = \lim_{a\to 0}\frac{1}{3}\left[\sqrt[3]{x}\right]\Big|_a^1 = \frac{1}{3}.$$

5. Determine whether the given series is convergent or divergent

(a) using the Integral Test

$$\sum_{k=2}^{\infty}\frac{\ln k}{k}$$

Solution.

$$f(x) = \frac{\ln x}{x}, \qquad f(k) = a_k,$$

$f(x)$ is nonincreasing positive function on $(3,\infty)$, since

$$\begin{aligned} f'(x) &= \frac{x\frac{1}{x} - \ln x}{x^2} = \frac{1-\ln x}{x^2} < 0,\\ if \qquad x &> 3.\end{aligned}$$

Therefore the series is convergent if the integral

$$\int_2^\infty \frac{\ln x}{x}dx$$

is convergent, and divergent if the integral is divergent. Calculating

$$\begin{aligned}
\int_2^{\infty} \frac{\ln x}{x} dx &= \lim_{R\to\infty} \int_2^R \frac{\ln x}{x} dx = \\
&= \left| \begin{array}{c} u = \ln x, du = \frac{1}{x} dx \\ x = 2, u = \ln 2 \\ x = R, u = \ln R \end{array} \right| \\
&= \lim_{R\to\infty} \int_{\ln 2}^{\ln R} u du = \lim_{R\to\infty} \left(\frac{1}{2} u^2 \right) \Big|_{\ln 2}^{\ln R} \\
&= \frac{1}{2} \lim_{R\to\infty} \left(\ln^2 R - \ln^2 2 \right) = \infty
\end{aligned}$$

The integral is divergent, the series is divergent.

(b) using Ratio Test

$$\sum_{k=1}^{\infty} e^{-k} k^3$$

Solution.

$$\begin{aligned}
a_k &= e^{-k} k^3, \qquad a_{k+1} = e^{-(k+1)} (k+1)^3, \\
L &= \lim_{k\to\infty} \frac{a_{k+1}}{a_k} = \lim_{k\to\infty} \frac{e^{-(k+1)} (k+1)^3}{e^{-k} k^3} \\
&= \frac{1}{e} \lim_{k\to\infty} \left(\frac{k+1}{k} \right)^3 = \frac{1}{e} < 1
\end{aligned}$$

The series is convergent.

6. Determine whether the given alternating series is convergent or divergent

(a)

$$\sum_{k=1}^{\infty} (-1)^{k+1} \frac{1}{3^k}$$

Solution.

$$\begin{aligned}
a_k &= \frac{1}{3^k}, \qquad a_{k+1} = \frac{1}{3^{k+1}} < \frac{1}{3^k} = a_k, \\
\lim_{k\to\infty} a_k &= \lim_{k\to\infty} \frac{1}{3^k} = 0.
\end{aligned}$$

By alternating series test the series is convergent.

(b)

$$\sum_{k=1}^{\infty} (-1)^{k+1} \frac{5k-1}{4k+1}$$

Solution.

$$\lim_{k\to\infty} \frac{5k-1}{4k+1} = \frac{5}{4}, \qquad \lim_{k\to\infty} a_k \neq 0$$

The seriews is divergent by $k-th$ term divergence test.