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Fundamentals of Vibration

Outline

- **Why vibration is important?**
- **Definition; mass, spring (or stiffness) dashpot**
- **Newton's laws of motion, 2nd order ODE**
- **Three types of vibration for SDOF sys.**
- **Alternative way to find eqn of motion: energy methods**
- **Examples**

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Why to study vibration

- Vibrations can lead to excessive deflections and failure on the machines and structures
- To reduce vibration through proper design of machines and their mountings
- To utilize profitably in several consumer and industrial applications
- To improve the efficiency of certain machining, casting, forging & welding processes
- To stimulate earthquakes for geological research and conduct studies in design of nuclear reactors

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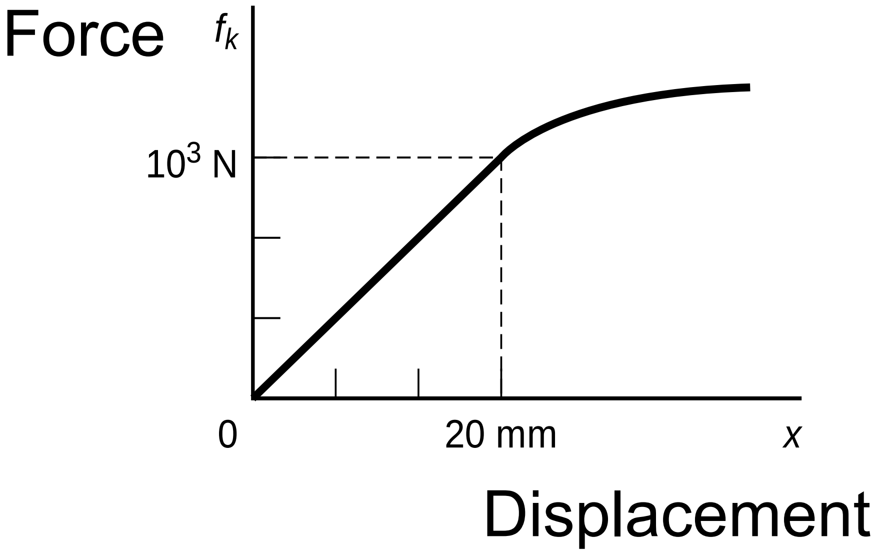
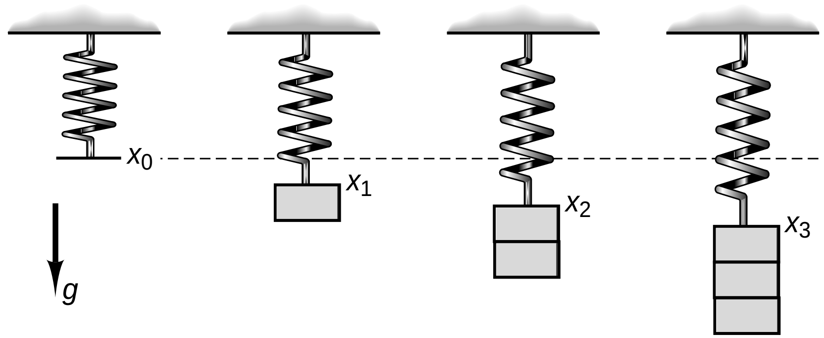
Why to study vibration

- Imbalance in the gas or diesel engines
- Blade and disk vibrations in turbines
- Noise and vibration of the hard-disks in your computers
- Cooling fan in the power supply
- Vibration testing for electronic packaging to conform Internatioal standard for quality control (QC)
- Safety eng.: machine vibration causes parts loose from the body

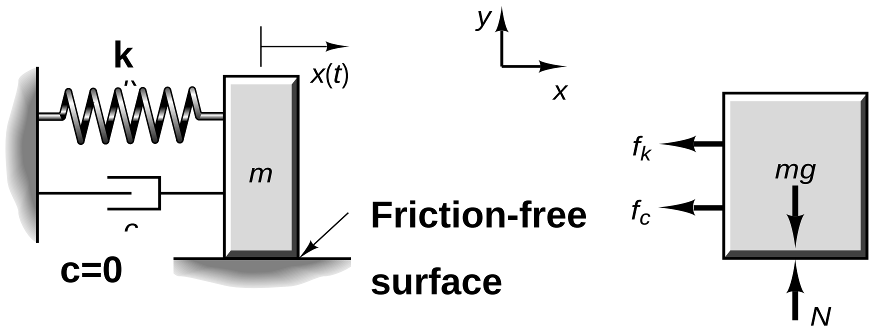
Stiffness

- From strength of materials (Solid Mech) recall:





Free-body diagram and equations of motion



- Newton's Law:

$$m\ddot{x}(t) = -kx(t)$$

$$m\ddot{x}(t) + kx(t) = 0$$

$$x(0) = x_0, \dot{x}(0) = v_0$$

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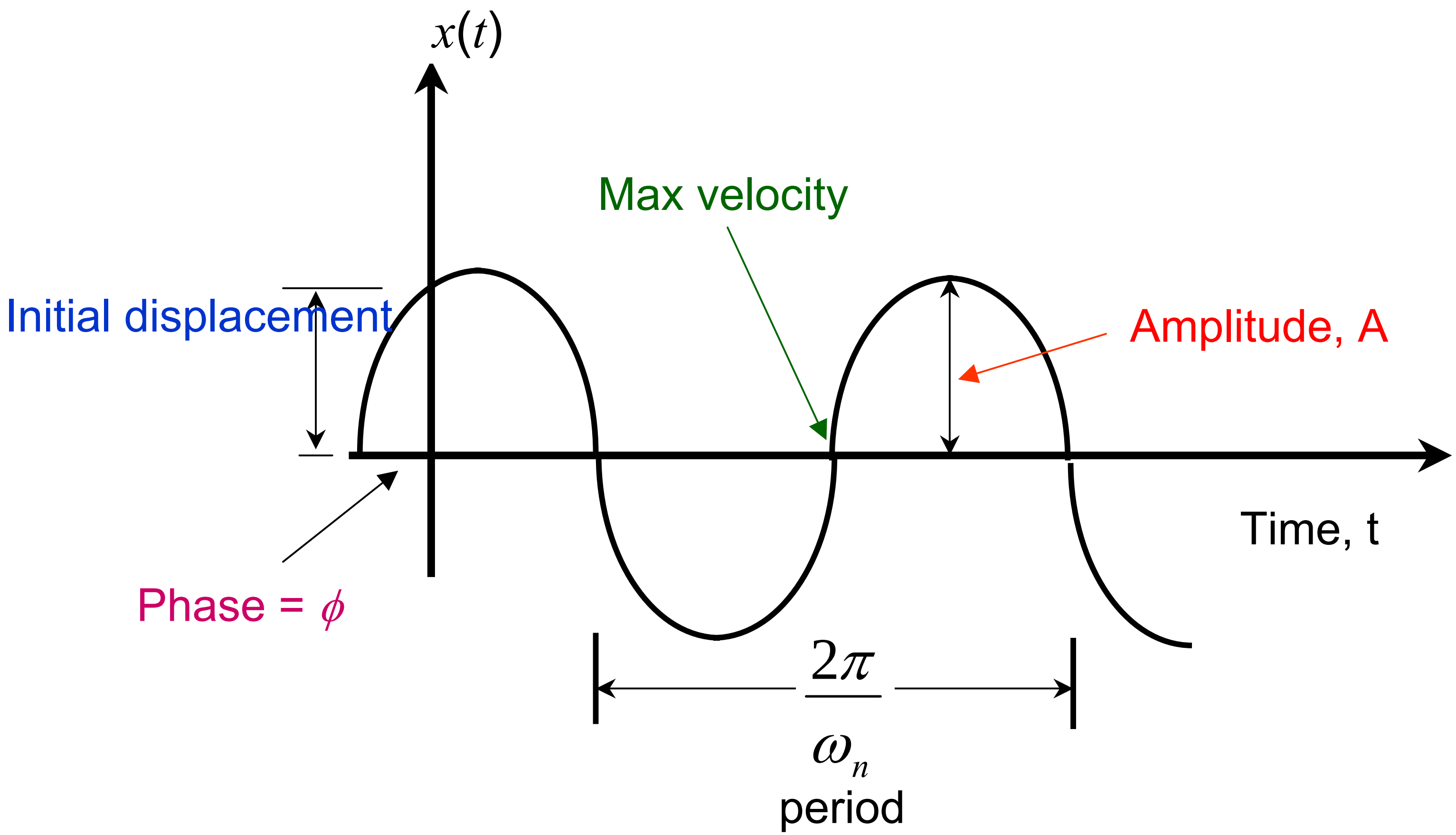
2nd Order Ordinary Differential Equation with Constant Coefficients

$$\text{Divide by } m: \ddot{x}(t) + \omega_n^2 x(t) = 0$$

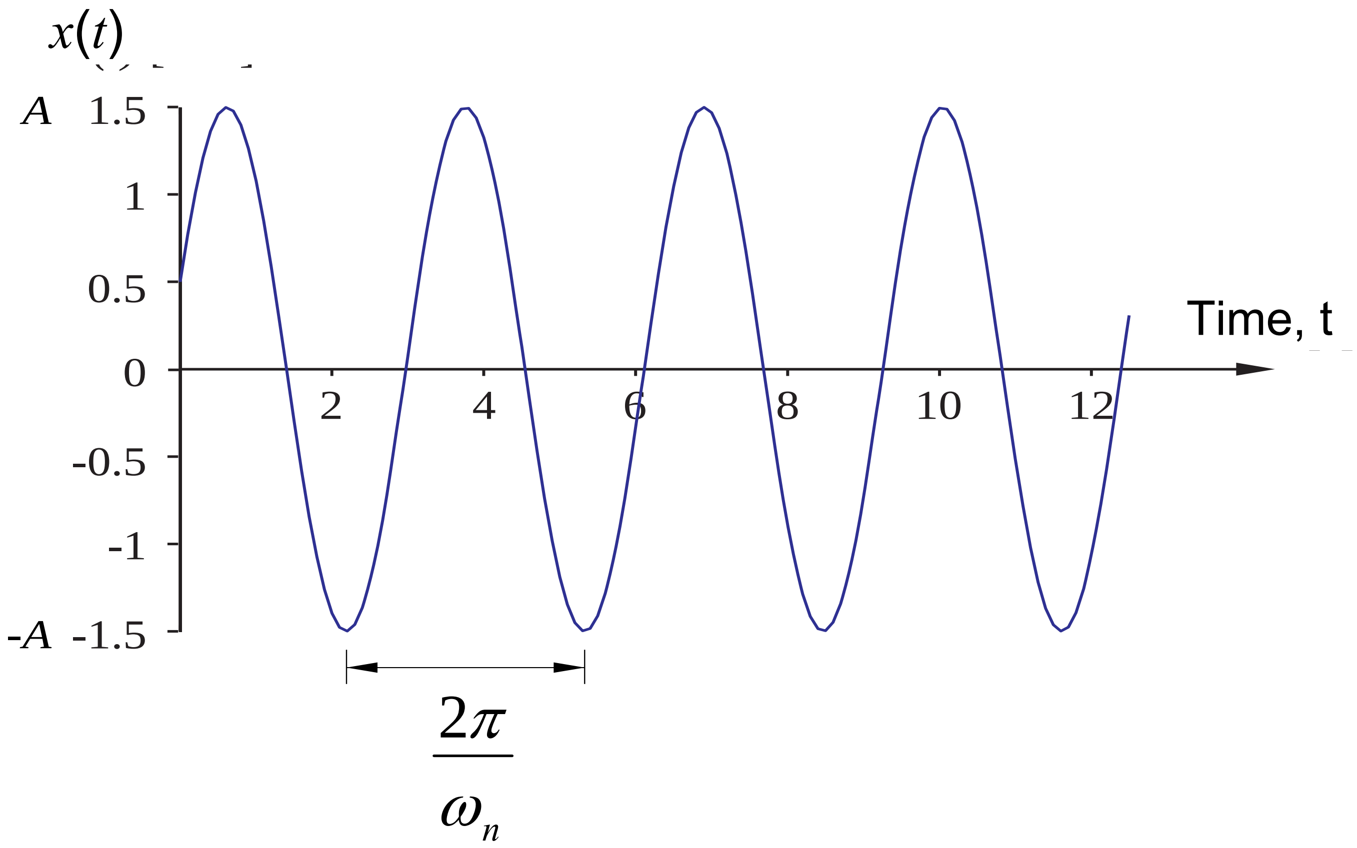
$$\omega_n = \sqrt{\frac{k}{m}} \ : \ \text{natural frequency in rad/s}$$

$$x(t) = A\sin(\omega_n t + \phi)$$

Periodic Motion



Periodic Motion



Frequency

ω_n is in rad/s is the natural frequency

$$f_n = \frac{\omega_n \text{ rad/s}}{2\pi \text{ rad/cycle}} = \frac{\omega_n \text{ cycles}}{2\pi \text{ s}} = \frac{\omega_n}{2\pi} \text{Hz}$$

$$T = \frac{2\pi}{\omega_n} \text{ s} \quad \text{is the period}$$

We often speak of frequency in Hertz, but we need rad/s in the arguments of the trigonometric functions (sin and cos function).

Amplitude & Phase from the initial conditions

$$x_0 = A\sin(\omega_n 0 + \phi) = A\sin\phi$$

$$v_0 = \omega_n A\cos(\omega_n 0 + \phi) = \omega_n A\cos\phi$$

Solving yields

$$\underbrace{A = \frac{1}{\omega_n}\sqrt{\omega_n^2 x_0^2 + v_0^2}}_{\text{Amplitude}}, \quad \underbrace{\phi = \tan^{-1}\left(\frac{\omega_n x_0}{v_0}\right)}_{\text{Phase}}$$

Phase Relationship between x, v, a

Displacement

$$x = A\cos(\omega_n t + \phi)$$

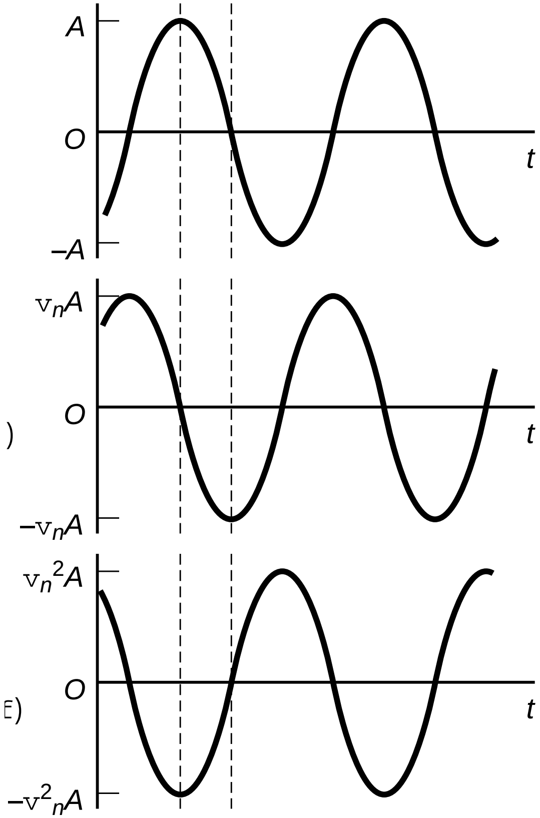
$x\ (t) = A$ sin(v$_n t$ + f)

Velocity

$$\dot{x} = -\omega_n A\sin(\omega_n t + \phi)$$

Acceleration

$$\ddot{x} = -\omega_n^2 A\cos(\omega_n t + \phi)$$



Example For m= 300 kg and ω_n =10 rad/s compute the stiffness:

$$\omega_n = \sqrt{\frac{k}{m}} \Rightarrow k = m\omega_n^2$$

$$= (300)10^2 \ \text{kg/s}^2$$

$$= 3 \times 10^4 \ \text{N/m}$$

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Other forms of the solution:

$$x(t) = A\sin(\omega_n t + \phi)$$

$$x(t) = A_1 \sin \omega_n t + A_2 \cos \omega_n t$$

$$x(t) = a_1 e^{j\omega_n t} + a_2 e^{-j\omega_n t}$$

Phasor: representation of a complex number in terms of a complex exponential

Ref: 1) Sec 1.10.2, 1.10.3

2) http://mathworld.wolfram.com/Phasor.html

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Some useful quantities

$$A = \text{ peak value}$$

$$\bar{x} = \lim_{T\to\infty} \frac{1}{T}\int_0^T x(t)dt \;=\; \text{average value}$$

$$\bar{x}^2 = \lim_{T\to\infty} \frac{1}{T}\int_0^T x^2(t)dt \;=\; \text{mean - square value}$$

$$x_{rms} = \sqrt{\bar{x}^2} \;=\; \text{root mean square value}$$

Peak Values

$$\text{max or peak value of :}$$

$$\text{displacement : } x_{\max} = A$$

$$\text{velocity : } \dot{x}_{\max} = \omega A$$

$$\text{acceleration : } \ddot{x}_{\max} = \omega^2 A$$

Example Hardware store spring, bolt: *m*= 49.2x10^{-3} kg,k=857.8 N/m and x_0 =10 mm. Compute ω_n and max amplitude of vibration.

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{857.8 \text{ N/m}}{49.2 \times 10^{-3} \text{ kg}}} = 132 \text{ rad/s}$$

$$f_n = \frac{\omega_n}{2\pi} = 21 \text{ Hz}$$

$$T = \frac{2\pi}{\omega_n} = \frac{1}{f_n} = \frac{1}{21\, cyles/\text{sec}} 0.0476 \text{ s}$$

$$x(t)_{\text{max}} = A = \frac{1}{\omega_n}\sqrt{\omega_n^2 x_0^2 + \cancel{v_0^2}} = x_0 = 10 \text{ mm}$$

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Compute the solution and max velocity and acceleration

$$v(t)_{\max} = \omega_n A = 1320 \text{ mm/s} = 1.32 \text{ m/s}$$

$$a(t)_{\max} = \omega_n^2 A = 174.24 \times 10^3 \text{ mm/s}^2$$

$$= 174.24 \text{ m/s}^2 \approx 17.8g!$$

$$\phi = \tan^{-1}\left(\frac{\omega_n x_0}{0}\right) = \frac{\pi}{2} \text{ rad}$$

$$x(t) = 10\sin(132t + \pi/2) = 10\cos(132t) \text{ mm}$$

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Derivation of the solution

Substitute $x(t) = ae^{\lambda t}$ into $m\ddot{x} + kx = 0 \Rightarrow$

$$m\lambda^2 ae^{\lambda t} + kae^{\lambda t} = 0 \Rightarrow$$

$$m\lambda^2 + k = 0 \Rightarrow$$

$$\lambda = \pm\sqrt{-\frac{k}{m}} = \pm\sqrt{\frac{k}{m}}j = \pm\omega_n j \Rightarrow$$

$x(t) = a_1 e^{\omega_n jt}$ and $x(t) = a_2 e^{-\omega_n jt} \Rightarrow$

$$x(t) = a_1 e^{\omega_n jt} + a_2 e^{-\omega_n jt}$$

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Damping Elements

- Viscous Damping:
  Damping force is proportional to the velocity of the vibrating body in a fluid medium such as air, water, gas, and oil.
- Coulomb or Dry Friction Damping:
  Damping force is constant in magnitude but opposite in direction to that of the motion of the vibrating body between dry surfaces
- Material or Solid or Hysteretic Damping:
  Energy is absorbed or dissipated by material during deformation due to friction between internal planes

Viscous Damping

- *Shear Stress (τ)* developed in the fluid layer at a distance y from the fixed plate is:

$$\tau = \mu \frac{du}{dy} \qquad (1.26)$$

where $du/dy = v/h$ is the velocity gradient.

- *Shear* or *Resisting Force (F)* developed at the bottom surface of the moving plate is:

$$F = \tau A = \mu \frac{Av}{h} = cv \qquad (1.27)$$

where A is the surface area of the moving plate.

$c = \frac{\mu A}{h}$ is the damping constant

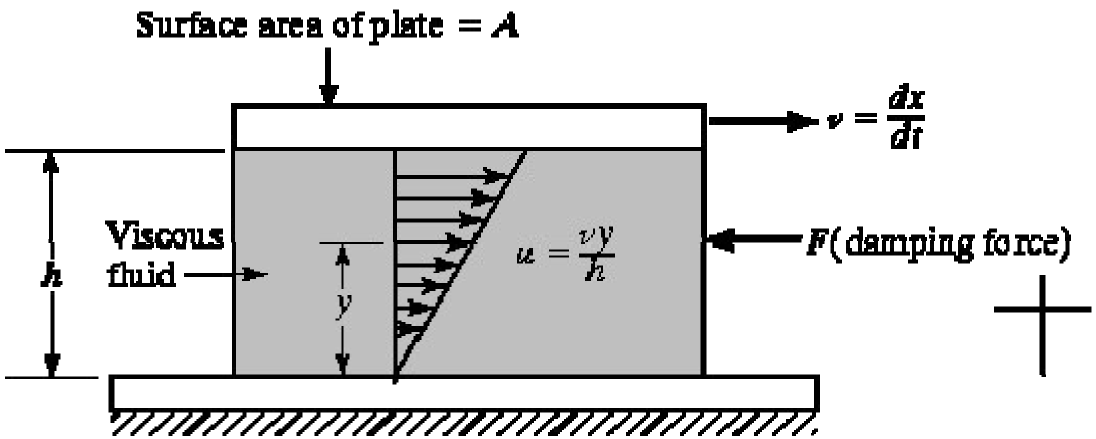
Viscous Damping

and $$c = \frac{\mu A}{h} \qquad (1.28)$$

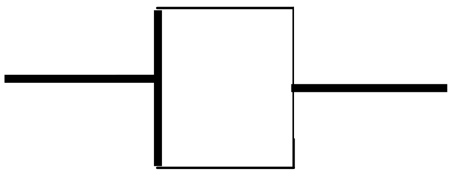
is called the damping constant.

❑ If a damper is nonlinear, a linearization process is used about the operating velocity (v^*) and the equivalent damping constant is:

$$c = \left.\frac{dF}{dv}\right|_{v^*} \qquad (1.29)$$



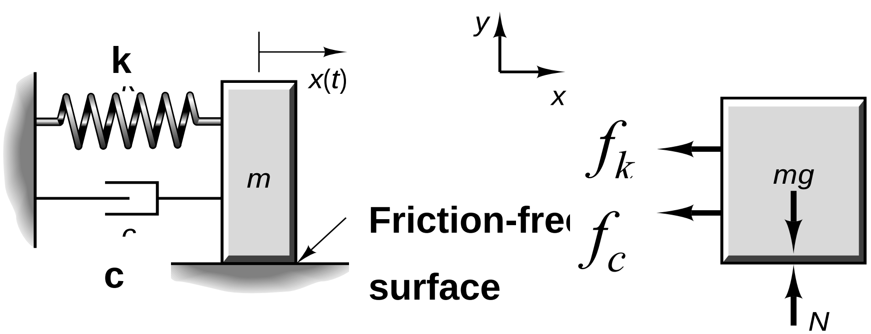
Linear Viscous Damping



$$f_c = c\dot{x}(t)$$

- A mathematical form
- Called a dashpot or viscous damper
- Somewhat like a shock absorber
- The constant c has units: Ns/m or kg/s

Spring-mass-damper systems



- From Newton's law:

$$m\ddot{x}(t) = -f_c - f_k$$
$$= -c\dot{x}(t) - kx(t)$$
$$m\ddot{x}(t) + c\dot{x}(t) + kx(t) = 0$$
$$x(0) = x_0, \;\; \dot{x}(0) = v_0$$

Derivation of the solution

$$\text{Substitute} \quad x(t) = ae^{\lambda t} \quad \text{into} \quad m\ddot{x} + c\dot{x} + kx = 0 \Rightarrow$$

$$m\lambda^2 ae^{\lambda t} + c\lambda ae^{\lambda t} + kae^{\lambda t} = 0 \Rightarrow$$

$$m\lambda^2 + c\lambda + k = 0 \Rightarrow$$

$$\lambda_{1,2} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1} \Rightarrow$$

$$x(t) = a_1 e^{\lambda_1 t} \quad \text{and} \quad x(t) = a_2 e^{\lambda_2 t} \Rightarrow$$

$$x(t) = a_1 e^{\lambda_1 t} + a_2 e^{\lambda_2 t}$$

Solution (dates to 1743 by Euler)

Divide equation of motion by m

$$\ddot{x}(t) + 2\zeta\omega_n\dot{x}(t) + \omega_n^2 x(t) = 0$$

$$\text{where } \omega_n = \sqrt{k/m} \text{ and}$$

$$\zeta = \frac{c}{2\sqrt{km}} = \text{damping ratio (dimensionless)}$$

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Let $x(t) = ae^{\lambda t}$ & subsitute into eq. of motion

$$\lambda^2 ae^{\lambda t} + 2\zeta\omega_n \lambda ae^{\lambda t} + \omega_n^2 ae^{\lambda t} = 0$$

which is now an algebraic equation in λ:

$$\lambda_{1,2} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$$

from the roots of a quadratic equation

Here the discriminant $\zeta^2 - 1$, determines

the nature of the roots λ

Three possibilities:

1) $\zeta = 1 \Rightarrow$ roots are equal & repeated

called critically damped

$$\zeta = 1 \Rightarrow c = c_{cr} = 2\sqrt{km} = 2m\omega_n$$

$$x(t) = a_1 e^{-\omega_n t} + a_2 t e^{-\omega_n t}$$

Using the initial conditions:

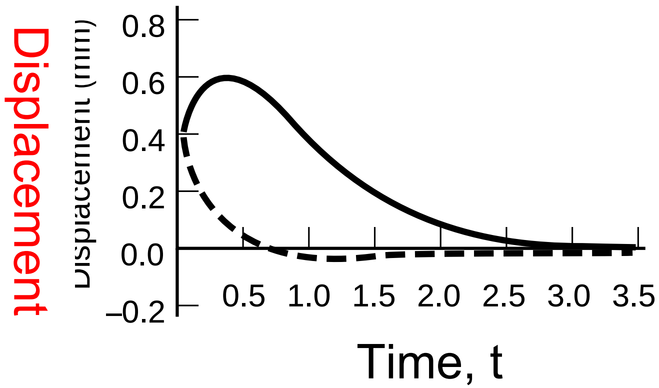
$$a_1 = x_0,\ \ a_2 = v_0 + \omega_n x_0$$

Sec. 2.6

Critical damping continued

- No oscillation occurs
- Useful in door mechanisms, analog gauges

$$x(t) = [x_0 + (v_0 + \omega_n x_0)t]e^{-\omega_n t}$$



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2) $\zeta > 1$, called overdamping - two distinct real roots :

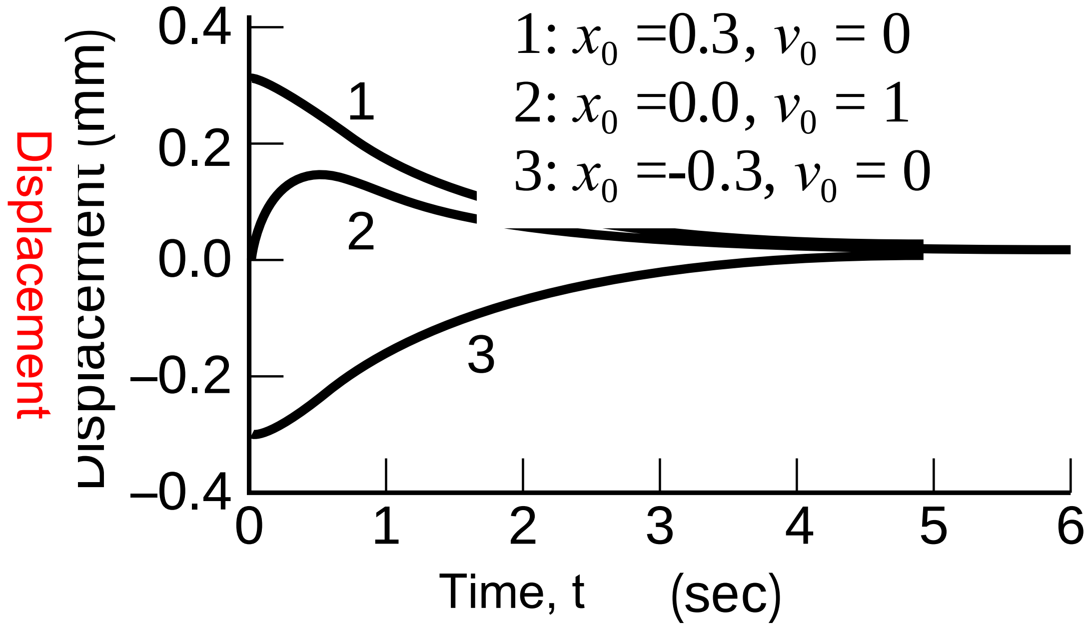
$$\lambda_{1,2} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$$

$$x(t) = e^{-\zeta\omega_n t}(a_1 e^{-\omega_n t\sqrt{\zeta^2-1}} + a_2 e^{\omega_n t\sqrt{\zeta^2-1}})$$

$$\text{where } a_1 = \frac{-v_0 + (-\zeta + \sqrt{\zeta^2 - 1})\omega_n x_0}{2\omega_n\sqrt{\zeta^2 - 1}}$$

$$a_2 = \frac{v_0 + (\zeta + \sqrt{\zeta^2 - 1})\omega_n x_0}{2\omega_n\sqrt{\zeta^2 - 1}}$$

The overdamped response



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3) $\zeta < 1$, called underdamped motion - most common

Two complex roots as conjugate pairs

write roots in complex form as:

$$\lambda_{1,2} = -\zeta\omega_n \pm \omega_n j\sqrt{1-\zeta^2}$$

where $j = \sqrt{-1}$

Underdamping

$$x(t) = e^{-\zeta\omega_n t}\left(a_1 e^{j\omega_n t\sqrt{1-\zeta^2}} + a_2 e^{-j\omega_n t\sqrt{1-\zeta^2}}\right)$$

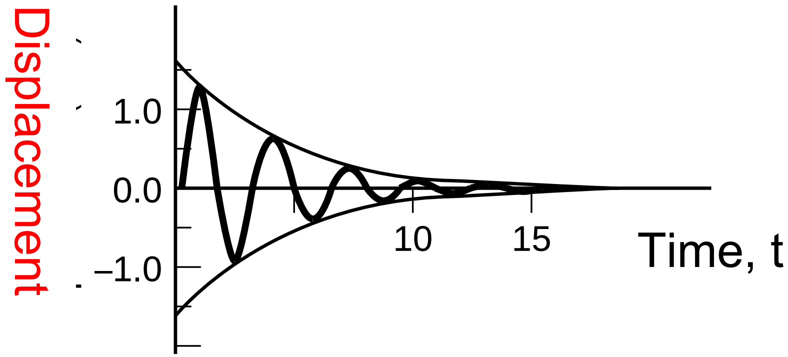
$$= A e^{-\zeta\omega_n t}\sin(\omega_d t + \phi)$$

$$\omega_d = \omega_n\sqrt{1-\zeta^2},\ \text{damped natural frequency}$$

$$A = \frac{1}{\omega_d}\sqrt{(v_0 + \zeta\omega_n x_0)^2 + (x_0\omega_d)^2}$$

$$\phi = \tan^{-1}\left(\frac{x_0\omega_d}{v_0 + \zeta\omega_n x_0}\right)$$

Underdamped-oscillation



- Gives an oscillating response with exponential decay
- Most natural systems vibrate with and underdamped response
- See textbook for details and other representations

Example consider the spring in Ex., if c = 0.11 kg/s,

determine the damping ratio of the spring-bolt system.

$$m = 49.2 \times 10^{-3} \text{ kg}, \quad k = 857.8 \text{ N/m}$$

$$c_{cr} = 2\sqrt{km} = 2\sqrt{49.2 \times 10^{-3} \times 857.8}$$

$$= 12.993 \text{ kg/s}$$

$$\zeta = \frac{c}{c_{cr}} = \frac{0.11 \text{ kg/s}}{12.993 \text{ kg/s}} = 0.0085 \Rightarrow$$

the motion is *underdamped*

and the bolt will oscillate

Example

The human leg has a measured natural frequency of around 20 Hz when in its rigid (knee locked) position, in the longitudinal direction (i.e., along the length of the bone) with a damping ratio of $\zeta = 0.224$.

Calculate the response of the tip if the leg bone to $v_0(t=0) = 0.6$ m/s and $x_0(t=0)=0$

This correspond to the vibration induced while landing on your feet, with your knees locked from a height of 18 mm) and plot the response. What is the maximum acceleration experienced by the leg assuming no damping?

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Solution:

V_0=0.6, X_0=0, ζ = 0.224

$$\omega_n = \frac{20}{1}\frac{\text{cycles}}{s}\frac{2\pi\,\text{rad}}{\text{cycles}} = 125.66\ \text{rad/s}$$

$$\omega_d = 125.66\sqrt{1-(.224)^2} = 122.467\ \ \text{rad/s}$$

$$A = \frac{\sqrt{(0.6+(0.224)(125.66)(0))^2+(0)(122.467)^2}}{122.467} = 0.005\ \text{m}$$

$$\boxed{A = \frac{1}{\omega_d}\sqrt{(v_0+\zeta\omega_n x_0)^2+(x_0\omega_d)^2}}$$

$$\phi = \tan^{-1}\left(\frac{(0)(\omega_d)}{v_0+\zeta\omega_n(0)}\right) = 0$$

$$\boxed{\phi = \tan^{-1}\left(\frac{x_0\omega_d}{v_0+\zeta\omega_n x_0}\right)}$$

$$\Rightarrow \underline{x(t) = 0.005e^{-28.148t}\sin(122.467t)}$$

Use <u>undamped formula</u> to get max acceleration:

$$A=\sqrt{x_0^2+\left(\frac{v_0}{\omega_n}\right)^2},\ \omega_n=125.66,\ v_0=0.6,\ x_0=0$$

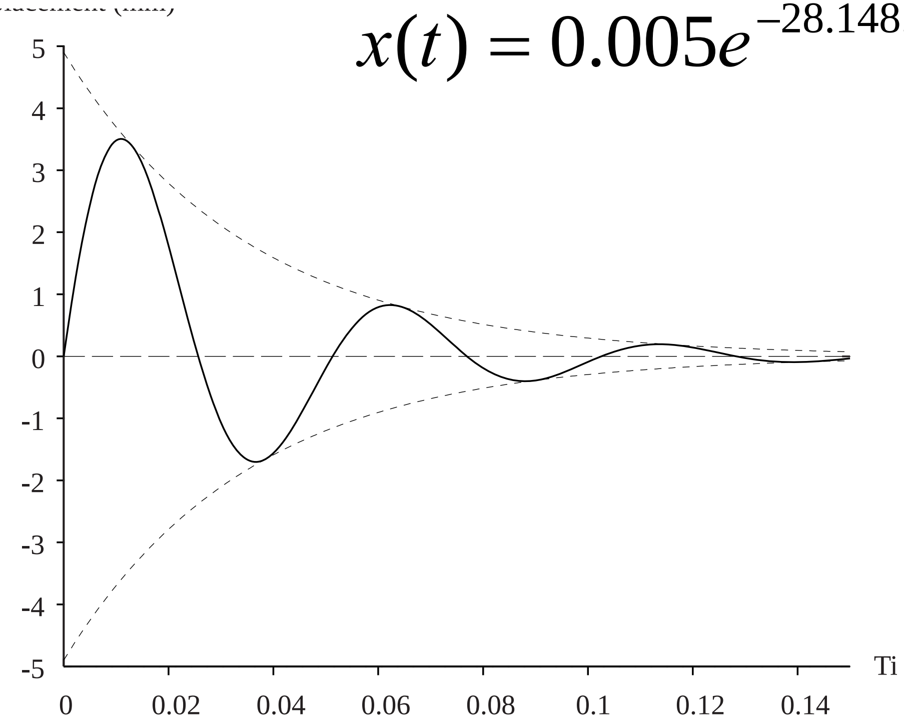
$$A=\frac{v_0}{\omega_n}\,\text{m}=\frac{0.6}{\omega_n}\,\text{m}$$

$$\max(\ddot{x})=\left|-\omega_n^2 A\right|=\left|-\omega_n^2\left(\frac{0.6}{\omega_n}\right)\right|=(0.6)\left(125.66\ \text{m/s}^2\right)=\underline{75.396\ \text{m/s}^2}$$

$$\text{maximum acceleration}=\frac{75.396\ \text{m/s}^2}{9.81\ \text{m/s}^2}\,g=7.68\,\text{g's}$$

Plot of the response:

$$x(t) = 0.005e^{-28.148t}\sin(122.467t)$$



Example Compute the form of the response of an underdamped system using the Cartesian form

$$\sin(x+y) = \sin x \cos y - \cos x \sin y \Rightarrow$$

$$x(t) = Ae^{-\zeta\omega_n t}\sin(\omega_d t + \phi) = e^{-\zeta\omega_n t}(A_1 \sin\omega_d t + A_2 \cos\omega_d t)$$

$$x(0) = x_0 = e^0(A_1 \sin(0) + A_2\cos(0)) \Rightarrow \underline{A_2 = x_0}$$

$$\dot{x} = -\zeta\omega_n e^{-\zeta\omega_n t}(A_1 \sin\omega_d t + A_2\cos\omega_d t)$$
$$+\omega_d e^{-\zeta\omega_n t}(A_1\cos\omega_d t - A_2 \sin\omega_d t)$$

$$v_0 = -\zeta\omega_n(A_1 \sin 0 + x_0 \cos 0) + \omega_d(A_1\cos 0 - x_0 \sin 0)$$

$$\Rightarrow \underline{A_1 = \frac{v_0 + \zeta\omega_n x_0}{\omega_d}} \Rightarrow$$

$$x(t) = e^{-\zeta\omega_n t}\left(\frac{v_0 + \zeta\omega_n x_0}{\omega_d}\sin\omega_d t + x_0\cos\omega_d t\right)$$

$\boxed{\text{Eq. 2.72}}$

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MODELING AND ENERGY METHODS

An alternative way to determine the equation of motion and an alternative way to calculate the natural frequency

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Modelling

- Newton’s Laws

$$\sum F_{xi} = m\ddot{x}$$

$$\sum M_{0i} = I_0\ddot{\theta}$$

Energy Methods

$$\int F dx = \int m\ddot{x} dx \Rightarrow$$

$$\text{work done} = \overbrace{U_1 - U_2}^{\text{Potential Energy}} = \frac{1}{2} m\dot{x}^2 \Big|_1^2 = \underbrace{T_2 - T_1}_{\text{Kinetic Energy}}$$

$$\Rightarrow T + U = \text{constant}$$

$$\text{or} \qquad \frac{d}{dt}(T+U) = 0$$

Alternate method of getting the eq. of motion

Rayleigh's Method

- $T_1 + U_1 = T_2 + U_2$
- Let t_1 be the time at which m moves through its static equilibrium position, then
- $U_1=0$, reference point
- Let t_2 be the time at which m undergoes its max displacement ($v=0$ so $T_2=0$), U_2 is max (T_1 must be max),
- Thus $U_{max}=T_{max}$

Ref: Section 2.5

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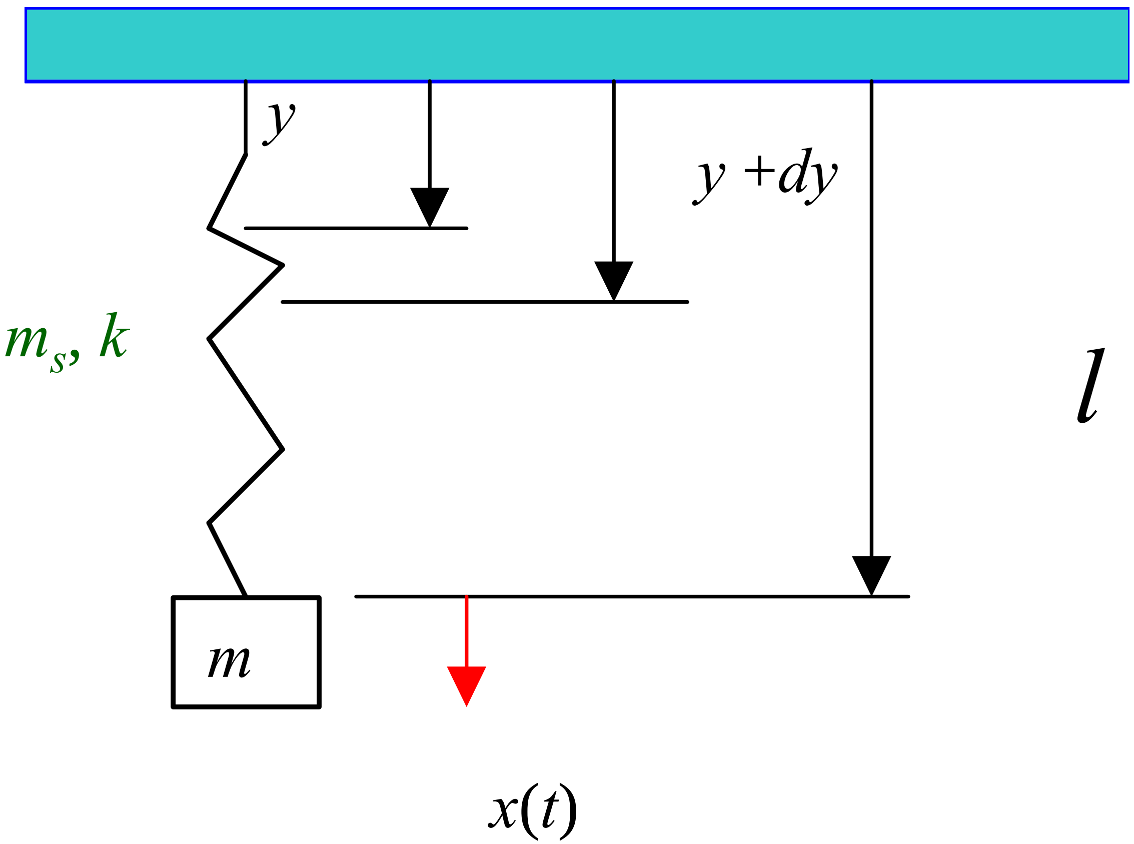
Example The effect of including the mass of the spring on the value of the frequency.



Ex. 2.8

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$$\left.\begin{array}{l}\text{mass of element } dy: \dfrac{m_s}{\ell}dy \\ \text{velocity of element } dy: v_{dy} = \dfrac{y}{\ell}\dot{x}(t),\end{array}\right\}\text{assumptions}$$

$$T_{spring} = \frac{1}{2}\int_0^{\ell}\frac{m_s}{\ell}\left[\frac{y}{\ell}\dot{x}\right]^2 dy \text{ (adds up the KE of each element)}$$

$$= \frac{1}{2}\left(\frac{m_s}{3}\right)\dot{x}^2$$

$$T_{mass} = \frac{1}{2}m\dot{x}^2 \Rightarrow T_{tot} = \left[\frac{1}{2}\left(\frac{m_s}{3}\right) + \frac{1}{2}m\right]\dot{x}^2 \Rightarrow T_{\max} = \frac{1}{2}\left(m + \frac{m_s}{3}\right)\omega_n^2 A_n^2$$

$$U_{\max} = \frac{1}{2}kA^2$$

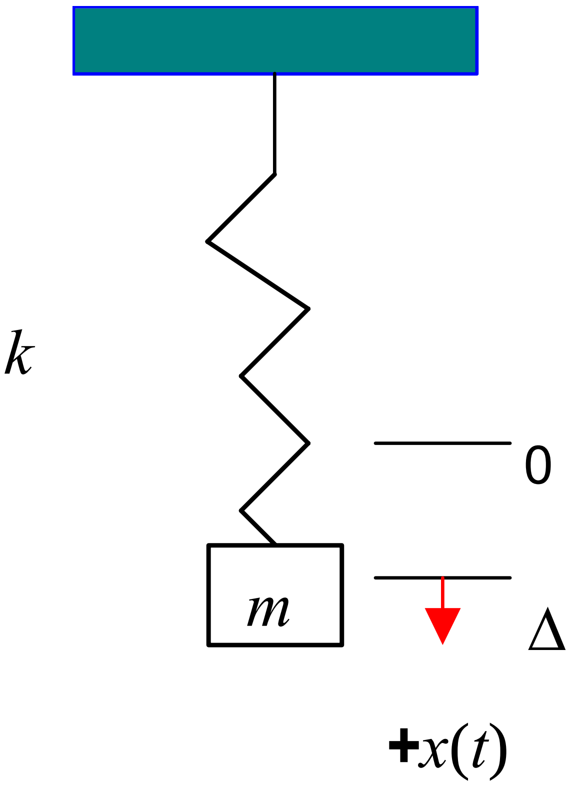
$$\Rightarrow \omega_n = \sqrt{\frac{k}{\boxed{m + \frac{m_s}{3}}}}$$

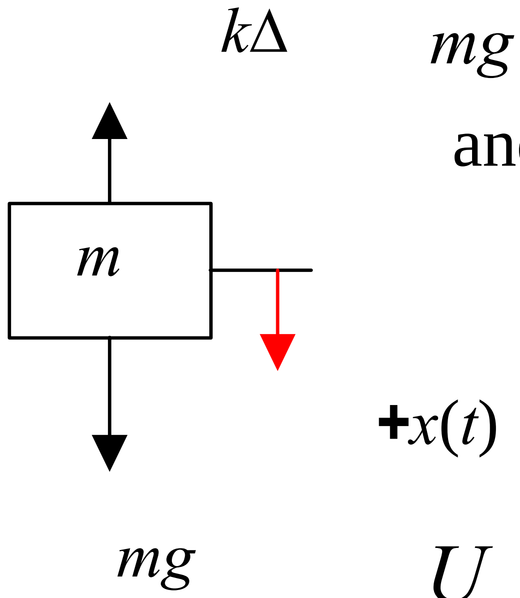
Provides some simple design and modeling guides

Ex. 2.8

Effect of the spring mass = add 1/3 of its mass to the main mass

What about gravity?





$mg - k\Delta = 0$, from FBD, and static equilibrium

$$U_{spring} = \frac{1}{2}k(\Delta + x)^2$$

$$U_{grav} = -mgx$$

$$T = \frac{1}{2}m\dot{x}^2$$

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Now use $\dfrac{d}{dt}(T+U)=0$

$$\Rightarrow \frac{d}{dt}\left[\frac{1}{2}m\dot{x}^2 - mgx + \frac{1}{2}k(\Delta + x)^2\right] = 0$$

$$\Rightarrow m\dot{x}\ddot{x} - mg\dot{x} + k(\Delta + x)\dot{x}$$

$$\Rightarrow \dot{x}(m\ddot{x} + kx) + \dot{x}(\underbrace{k\Delta - mg}_{\text{0 from static equ.}}) = 0$$

$$\Rightarrow m\ddot{x} + kx = 0$$

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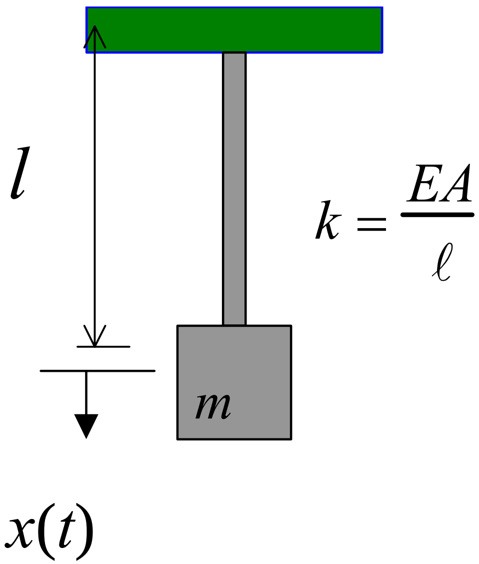
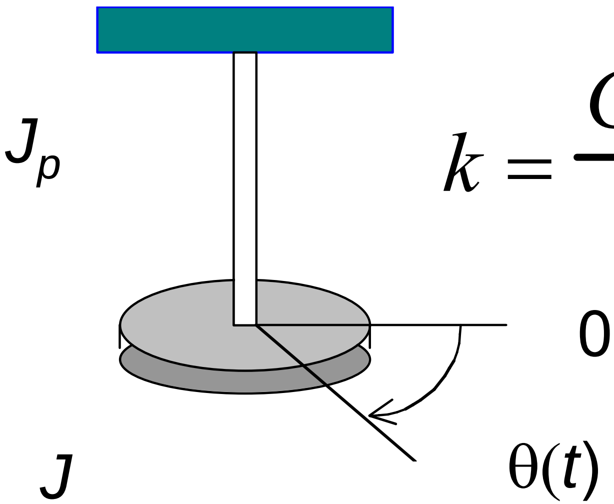
More on springs and stiffness



- Longitudinal motion
- A is the cross sectional area (m^2)
- E is the elastic modulus ($Pa=N/m^2$)
- l is the length (m)
- k is the stiffness (N/m)

Torsional Stiffness



$$k = \frac{GJ_p}{\ell}$$

- J_p is the polar moment of inertia of the rod
- J is the mass moment of inertia of the disk
- G is the shear modulus, l is the length

Sec. 2.3

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Example compute the frequency of a shaft/mass system {J = 0.5 kg m²}

$$\sum M = J\ddot{\theta} \Rightarrow J\ddot{\theta}(t) + k\theta(t) = 0$$

$$\Rightarrow \ddot{\theta}(t) + \frac{k}{J}\theta(t) = 0$$

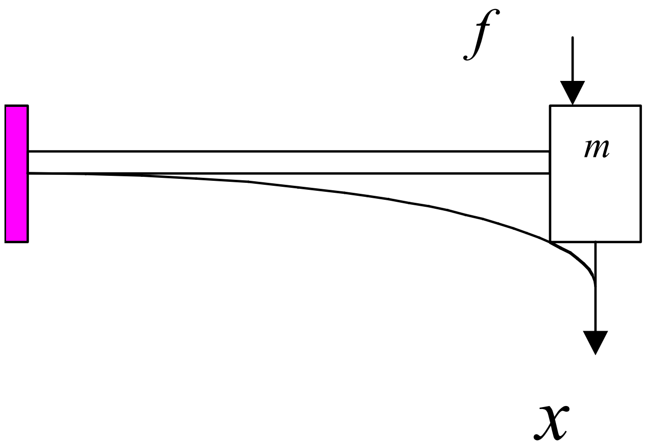
$$\Rightarrow \omega_n = \sqrt{\frac{k}{J}} = \sqrt{\frac{GJ_p}{\ell J}}, \quad J_p = \frac{\pi d^4}{32}$$

For a 2 m steel shaft, diameter of 0.5 cm $\Rightarrow$

$$\omega_n = \sqrt{\frac{GJ_p}{\ell J}} = \sqrt{\frac{(8\times 10^{10}\ \mathrm{N/m^2})[\pi(0.5\times 10^{-2}\ \mathrm{m})^4/32]}{(2\ \mathrm{m})(0.5\mathrm{kg}\cdot\mathrm{m^2})}}$$

$$= 2.2\ \mathrm{rad/s}$$

Transverse beam stiffness



- Strength of materials and experiments yield:

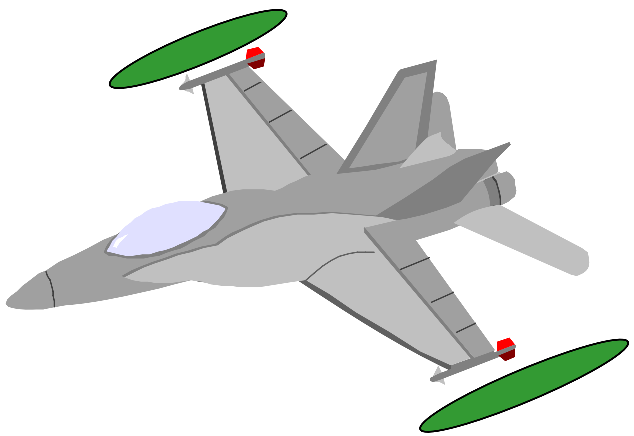
$$k = \frac{3EI}{\ell^3}$$

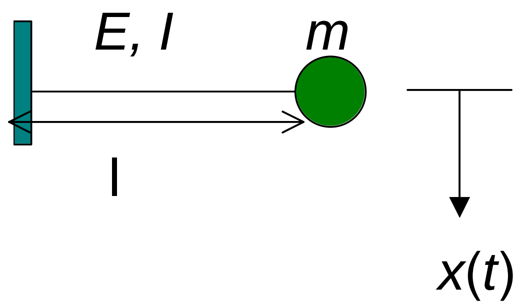
$$\omega_n = \sqrt{\frac{3EI}{m\ell^3}}$$

Samples of Vibrating Systems

- Deflection of continuum (beams, plates, bars, etc) such as airplane wings, truck chassis, disc drives, circuit boards…
- Shaft rotation
- Rolling ships
- See text for more examples.

Example : effect of fuel on frequency of an airplane wing





- Model wing as transverse beam
- Model fuel as tip mass
- Ignore the mass of the wing and see how the frequency of the system changes as the fuel is used up

-

Mass of pod 10 kg empty 1000 kg full
I = 5.2x10^{-5} m^4, E =6.9x10^9 N/m, l = 2 m

- Hence the natural frequency changes by an order of magnitude while it empties out fuel.

$$\omega_{\text{full}} = \sqrt{\frac{3EI}{m\ell^3}} = \sqrt{\frac{3(6.9\times10^9)(5.2\times10^{-5})}{1000\cdot2^3}}$$

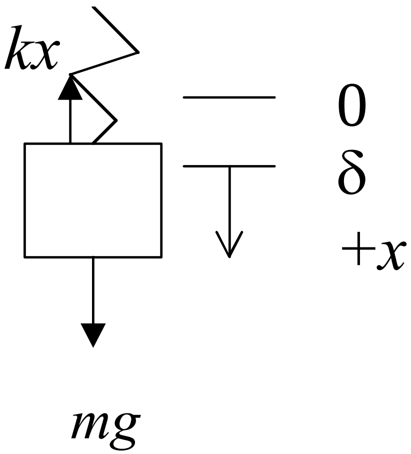
$$= 11.6 \text{ rad/s} = 1.8 \text{ Hz}$$

$$\omega_{\text{empty}} = \sqrt{\frac{3EI}{m\ell^3}} = \sqrt{\frac{3(6.9\times10^9)(5.2\times10^{-5})}{10\cdot2^3}}$$

$$= 115 \text{ rad/s} = 18.5 \text{ Hz}$$

Pod= a streamlined external housing that enclose engines or fuel

Does gravity effect frequency?



- Static equilibrium:

$$\sum F = 0 = -k\delta + mg$$

- Dynamic equation :

$$\sum F = m\ddot{x} = -k(x+\delta) + mg$$

$$m\ddot{x} + kx + k\delta - mg = 0$$

$$m\ddot{x} + kx = 0 \;\; !$$

-

Static Deflection

$\delta, \Delta =$ distance spring is stretched or compressed under the force of gravity by attaching a mass m to it.

$$\Delta = \delta = \delta_s = \frac{mg}{k}$$

Many symbols in use including x_s and x_0

Combining Springs

- Equivalent Spring

$$\text{series}: \; k_{AC} = \frac{1}{1/k_1 + 1/k_2}$$

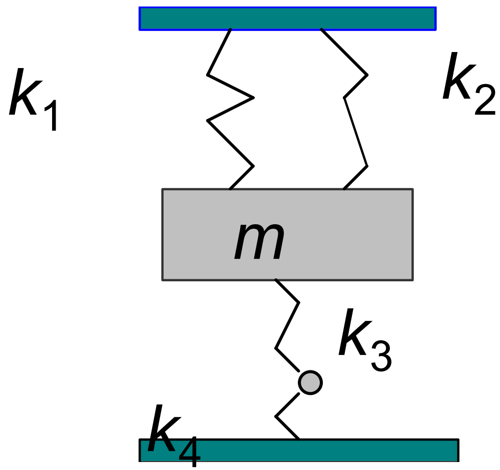
$$\text{parallel}: \; k_{ab} = k_1 + k_2$$

Use these to design from available parts

- Discrete springs available in standard values
- Dynamic requirements require specific frequencies
- Mass is often fixed or $\pm$ small amount
- Use spring combinations to adjust w_n
- Check static deflection

-

Example Design of a spring mass system using available springs: series vs parallel



- Let m = 10 kg
- Compare a series and parallel combination
- a) k_1 =1000 N/m, k_2 = 3000 N/m, $k_3 = k_4$ =0
- b) k_3 =1000 N/m, k_4 = 3000 N/m, $k_1 = k_2$ =0

-

Case a) parallel connection :

$$k_3 = k_4 = 0, k_{eq} = k_1 + k_2 = 1000 + 3000 = 4000 \text{ N/m}$$

$$\Rightarrow \omega_{parallel} = \sqrt{\frac{k_{eg}}{m}} = \sqrt{\frac{4000}{10}} = 20 \text{ rad/s}$$

Case b) series connection :

$$k_1 = k_2 = 0, k_{eq} = \frac{1}{(1/k_3) + (1/k_4)} = \frac{3000}{3+1} = 750 \text{ N/m}$$

$$\Rightarrow \omega_{series} = \sqrt{\frac{k_{eg}}{m}} = \sqrt{\frac{750}{10}} = 8.66 \text{ rad/s}$$

Same physical components, very different frequency
Allows some design flexibility in using off-the-shelf components

-

Free Vibration with Coulomb Damping

- Coulomb's law of dry friction states that, when two bodies are in contact, the force required to produce sliding is proportional to the normal force acting in the plane of contact. Thus, the friction force *F* is given by:

$$F = \mu N = \mu W = \mu mg \qquad (2.106)$$

where N is normal force,

μ is the coefficient of sliding or kinetic friction

μ is usu 0.1 for lubricated metal, 0.3 for nonlubricated metal on metal, 1.0 for rubber on metal

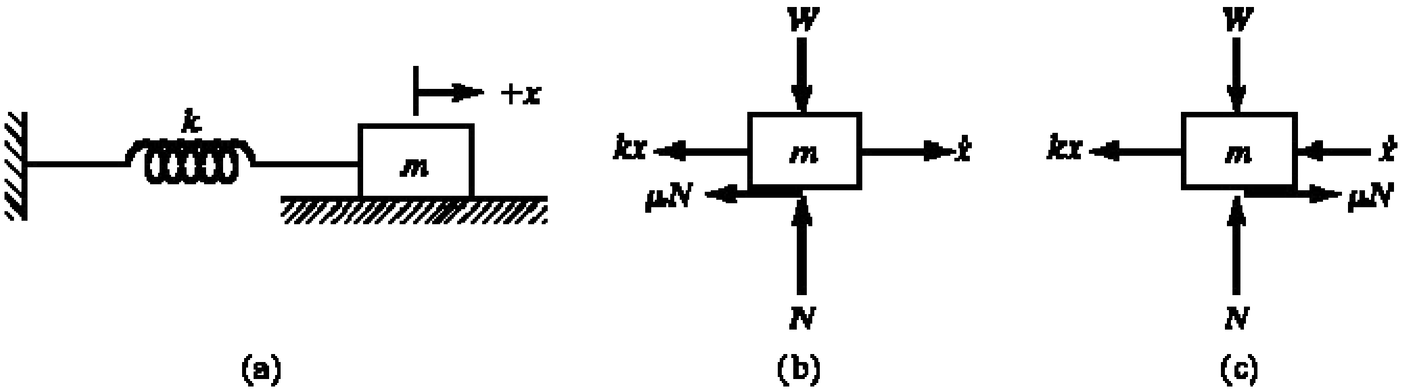
- Coulomb damping is sometimes called *constant damping*

-

Free Vibration with Coulomb Damping

- Equation of Motion:

Consider a single degree of freedom system with dry friction as shown in Fig.(a) below.



Since friction force varies with the direction of velocity, we need to consider two cases as indicated in Fig.(b) and (c).

-

Free Vibration with Coulomb Damping

Case 1. When x is positive and dx/dt is positive or when x is negative and dx/dt is positive (i.e., for the half cycle during which the mass moves from left to right) the equation of motion can be obtained using Newton's second law (Fig.b):

$$m\ddot{x} = -kx - \mu N \quad \text{or} \quad m\ddot{x} + kx = -\mu N \qquad (2.107)$$

Hence,

$$x(t) = A_1 \cos \omega_n t + A_2 \sin \omega_n t \boxed{-\frac{\mu N}{k}} \qquad (2.108)$$

where $\omega_n = \sqrt{k/m}$ is the frequency of vibration
A_1 & A_2 are constants

Free Vibration with Coulomb Damping

Case 2. When x is positive and *dx/dt* is negative or when x is negative and *dx/dt* is negative (i.e., for the half cycle during which the mass moves from right to left) the equation of motion can be derived from Fig. (c):

$$-kx + \mu N = m\ddot{x} \quad \text{or} \quad m\ddot{x} + kx = \mu N \qquad (2.109)$$

The solution of the equation is given by:

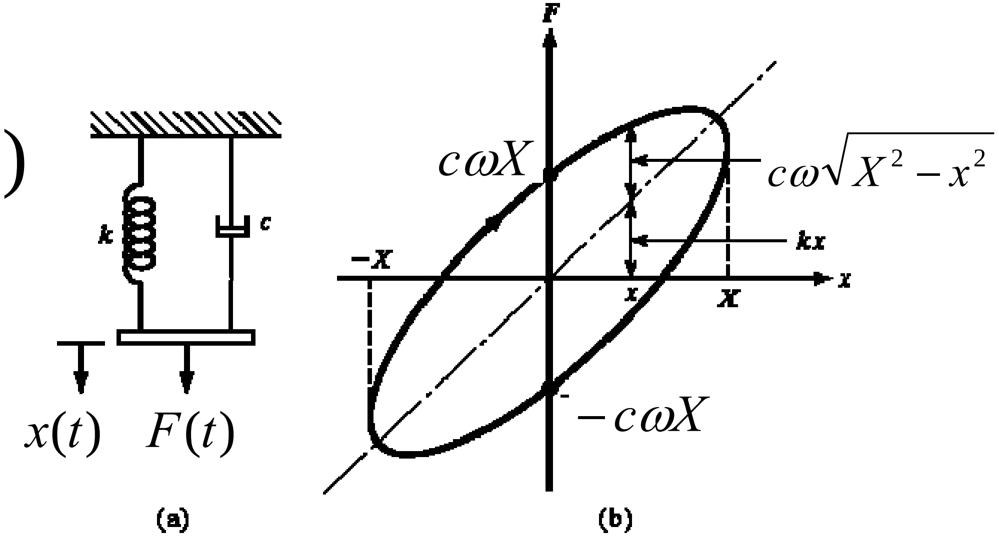
$$x(t) = A_3 \cos\omega_n t + A_4 \sin\omega_n t \boxed{+\frac{\mu N}{k}} \qquad (2.110)$$

where A_3 & A_4 are constants

Free Vibration with Hysteretic Damping

Consider the spring-viscous damper arrangement shown in the figure below. The force needed to cause a displacement:

$$F = kx + c\dot{x} \qquad (2.122)$$



For a harmonic motion of frequency ω and amplitude X,

$$x(t) = X \sin \omega t \quad (2.123)$$

$$\therefore \qquad F(t) = kX \sin \omega t + cX\omega \cos \omega t$$

$$= kx \pm c\omega\sqrt{X^2 - (X \sin \omega t)^2}$$

$$= kx \pm c\omega\sqrt{X^2 - x^2} \qquad (2.124)$$

Free Vibration with Hysteretic Damping

When F versus x is plotted, Eq.(2.124) represents a closed loop, as shown in Fig(b). The area of the loop denotes the energy dissipated by the damper in a cycle of motion and is given by:

$$\Delta W = \oint F dx = \int_0^{2\pi/\omega} (kX \sin \omega t + cX\omega \cos \omega t)(\omega X \cos \omega t) dt$$

$$= \pi \omega c X^2 \qquad (2.125)$$

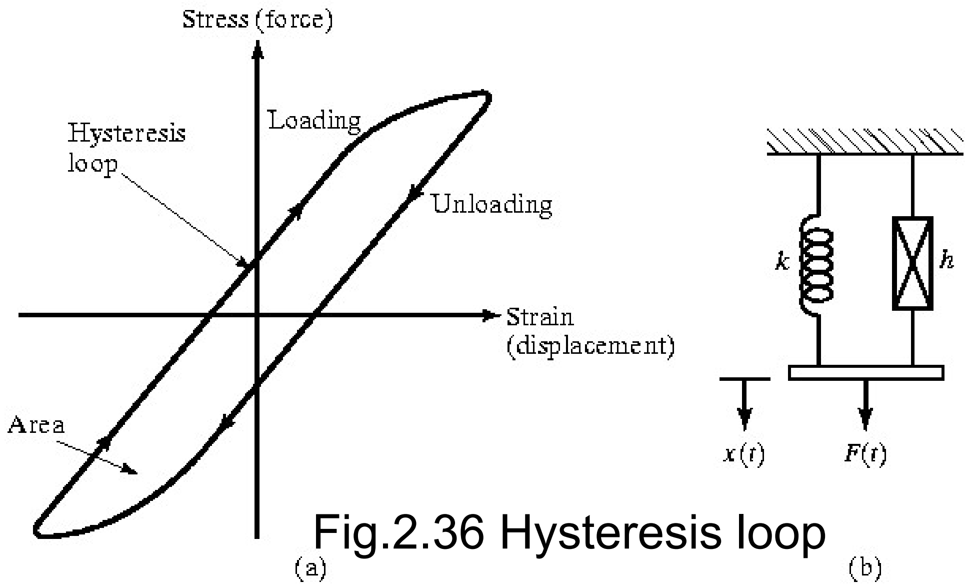


Fig.2.36 Hysteresis loop

Hence, the damping coefficient:

$$c = \frac{h}{\omega} \qquad (2.126)$$

where h is called the hysteresis damping constant.

Free Vibration with Hysteretic Damping

Eqs.(2.125) and (2.126) gives

$$\Delta W = \pi h X^2 \qquad (2.127)$$

Complex Stiffness.

For general harmonic motion, $x = Xe^{i\omega t}$, the force is given by

$$F = kXe^{i\omega t} + c\omega i X e^{i\omega t} = (k + i\omega c)x \qquad (2.128)$$

Thus, the force-displacement relation:

$$F = (k + ih)x \qquad (2.129)$$

where $$k + ih = k\left(1 + i\frac{h}{k}\right) = k(1 + i\beta) \qquad (2.130)$$

-

2.6.4 Energy dissipated in Viscous Damping:

In a viscously damped system, the rate of change of energy with time is given by:

$$\frac{dW}{dt} = \text{force} \times \text{velocity} = Fv = -cv^2 = -c\left(\frac{dx}{dt}\right)^2 \qquad (2.93)$$

The energy dissipated in a complete cycle is:

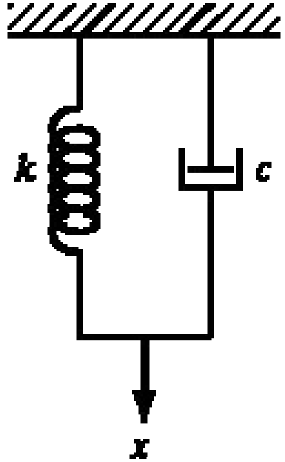
$$\Delta W = \int_{t=0}^{(2\pi/\omega_d)} c\left(\frac{dx}{dt}\right)^2 dt = \int_0^{2\pi} cX^2\omega_d \cos^2 \omega_d t \cdot d(\omega_d t)$$

$$= \pi c \omega_d X^2 \qquad (2.94)$$

Energy dissipation

Consider the system shown in the figure below.
The total force resisting the motion is:

$$F = -kx - cv = -kx - c\dot{x} \qquad (2.95)$$



If we assume simple harmonic motion:

$$x(t) = X \sin \omega_d t \qquad (2.96)$$

Thus, Eq.(2.95) becomes

$$F = -kX \sin \omega_d t - c\omega_d X \cos \omega_d t \qquad (2.97)$$

The energy dissipated in a complete cycle will be

$$\Delta W = \int_{t=0}^{2\pi/\omega_d} Fv dt$$

$$= \int_{t=0}^{2\pi/\omega_d} kX^2 \omega_d \sin \omega_d t \cdot \cos \omega_d t \cdot d(\omega_d t)$$

$$+ \int_{t=0}^{2\pi/\omega_d} c\omega_d X^2 \cos^2 \omega_d t \cdot d(\omega_d t) = \pi c \omega_d X^2 \qquad (2.98)$$

-

Energy dissipation and Loss Coefficient

Computing the fraction of the total energy of the vibrating system that is dissipated in each cycle of motion, Specific Damping Capacity

$$\frac{\Delta W}{W} = \frac{\pi c \omega_d X^2}{\frac{1}{2} m \omega_d^2 X^2} = 2\left(\frac{2\pi}{\omega_d}\right)\left(\frac{c}{2m}\right) = 2\delta \approx 4\pi\zeta = \text{constant} \qquad (2.99)$$

where W is either the max potential energy or the max kinetic energy.

The loss coefficient, defined as the ratio of the energy dissipated per radian and the total strain energy:

$$\text{loss coefficient} = \frac{(\Delta W / 2\pi)}{W} = \frac{\Delta W}{2\pi W} \qquad (2.100)$$