RAYLEIGH'S METHOD

Ref: Textbook: Sec. 2.5: Rayleigh's energy method, Sec. 7.3, 8.7: Rayleigh method

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Introduction

Dynamic systems can be characterized in terms of one or more natural frequencies. The natural frequency is the frequency at which the system would vibrate if it were given an initial disturbance and then allowed to vibrate freely.

There are many available methods for determining the natural frequency. Some examples are

1. Newton's Law of Motion
2. Rayleigh's Method
3. Energy Method
4. Lagrange's Equation

Not that the Rayleigh, Energy, and Lagrange methods are closely related.

Some of these methods directly yield the natural frequency. Others yield a governing equation of motion, from which the natural frequency may be determined.

This tutorial focuses on Rayleigh's method, which yields the natural frequency.

Rayleigh's method requires an assumed displacement function. The method thus reduces the dynamic system to a single-degree-of-freedom system. Furthermore, the assumed displacement function introduces additional constraints which increase the stiffness of the system. Thus, Rayleigh's method yields an upper limit of the true fundamental frequency.

Definition

Rayleigh's method can be summarized as

$$(\mathrm{KE})_{max} = (\mathrm{PE})_{max} = \text{total energy of the system.} \qquad \text{(A-1)}$$

where

KE = kinetic energy
PE = potential energy

Note that potential energy is also referred to as strain energy for the case of certain systems, such as beams.

Equation (A-1) can only be satisfied if the system is vibrating at its natural frequency.

Pendulum Example

Consider a conservative system. An example is the pendulum shown in Figure A-1.

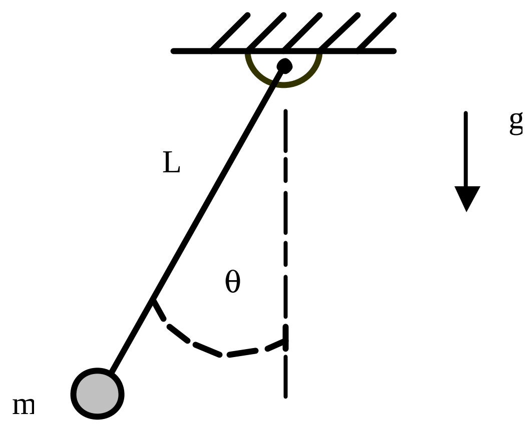

Figure A-1.

The kinetic energy becomes zero when the pendulum reaches its maximum angular displacement. The kinetic energy reaches its maximum value when the pendulum passes through $\theta = 0°$, which is also the "static equilibrium point."

On the other hand, the potential energy reaches its maximum level as the pendulum reaches its maximum angular displacement. The potential energy reaches its minimum value when the pendulum is at its static equilibrium point. For simplicity, the potential energy can be considered as zero at the static equilibrium point.

Let

m = pendulum mass,
L = length,
θ = angular displacement.

Assume a small angular displacement.

The potential energy is

$$PE = mgL\,(1-\cos\theta) \tag{A-2}$$

The kinetic energy is

$$KE = \frac{1}{2}m(L\dot{\theta})^2 \tag{A-3}$$

Assume a displacement equation of

$$\theta(t) = \alpha\sin\left(\omega_n t\right) \tag{A-4}$$

$$\theta_{max} = \alpha \tag{A-5}$$

The velocity equation is

$$\dot{\theta}(t) = \alpha\,\omega_n\cos\left(\omega_n t\right) \tag{A-6}$$

$$\dot{\theta}_{max} = \alpha\,\omega_n \tag{A-7}$$

$$\left(PE\right)_{max} = mgL\,(1-\cos\alpha) \tag{A-8}$$

Consider the expansion

$$\cos\alpha = \left(1-\frac{\alpha^2}{2!}+\frac{\alpha^4}{4!}-\cdots\right) \tag{A-9}$$

Consider the maximum potential energy. Substitute equation (A-9) into (A-8).

$$\left(PE\right)_{max} = mgL\left[1-\left(1-\frac{\alpha^2}{2!}+\frac{\alpha^4}{4!}-\cdots\right)\right] \tag{A-10}$$

$$\left(PE\right)_{max} \approx mgL\left[\frac{\alpha^2}{2!}+\frac{\alpha^4}{4!}\right] \tag{A-11}$$

$$(PE)_{max} \approx mgL\left[\frac{\alpha^2}{2}\right] \quad \text{(A-12)}$$

Consider the maximum kinetic energy. Substitute equation (A-7) into (A-3).

$$(KE)_{max} = \frac{1}{2}m(L\alpha\omega_n)^2 \quad \text{(A-13)}$$

$$\frac{1}{2}m(L\alpha\omega_n)^2 = mgL\left(\frac{\alpha^2}{2}\right) \quad \text{(A-14)}$$

Simplifying,

$$\omega_n{}^2 = \frac{g}{L} \quad \text{(A-15)}$$

The pendulum natural frequency is thus

$$\omega_n = \sqrt{\frac{g}{L}} \quad \text{(A-16)}$$

<u>Cantilever Beam with End Mass</u>

Consider a mass mounted on the end of a cantilever beam, as shown in Figure B-1. Assume that the end-mass is much greater than the mass of the beam.

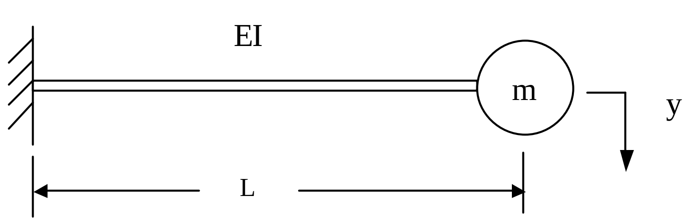

Figure B-1.

- E is the modulus of elasticity
- I is the area moment of inertia
- L is the length
- g is gravity
- m is the mass
- x is the displacement

The static stiffness at the end of the beam is

$$k = \frac{3EI}{L^3} \qquad \text{(B-1)}$$

Equation (B-1) is derived in Reference 1.

The potential energy is

$$PE = \frac{1}{2}\left[\frac{3EI}{L^3}\right]y^2 \qquad \text{(B-2)}$$

The kinetic energy is

$$KE = \frac{1}{2} m \dot{y}^2 \qquad \text{(B-3)}$$

Assume an end displacement of

$$y = A \sin \omega_n t \qquad \text{(B-4)}$$

The corresponding velocity is

$$\dot{y} = \omega_n A \cos \omega_n t \qquad \text{(B-5)}$$

The maximum displacement is A. The maximum velocity is $\omega_n A$. Thus,

$$KE_{max} = \frac{1}{2} m (\omega_n A)^2 \qquad \text{(B-6)}$$

$$PE_{max} = \frac{1}{2}\left[\frac{3EI}{L^3}\right]A^2 \qquad \text{(B-7)}$$

Applying Rayleigh’s method,

$$\frac{1}{2} m (\omega_n A)^2 = \frac{1}{2}\left[\frac{3EI}{L^3}\right]A^2 \qquad \text{(B-8)}$$

The natural frequency of the end mass supported by the cantilever beam is thus

$$\omega_n{}^2 = \left[\frac{3EI}{m L^3}\right] \tag{B-9}$$

$$\omega_n = \sqrt{\frac{3EI}{m L^3}} \tag{B-10}$$

Cantilever Beam with Internal Distributed Mass

Consider a cantilever beam with mass per length ρ. Assume that the beam has a uniform cross section. Determine the natural frequency.

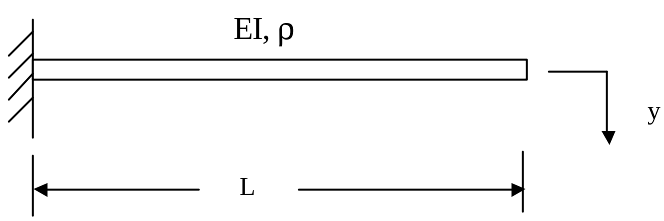

Figure C-1.

The governing differential equation is

$$-EI\frac{\partial^4 y}{\partial x^4} = \rho\frac{\partial^2 y}{\partial t^2} \tag{C-1}$$

The boundary conditions at the fixed end x = 0 are

$$y(0) = 0 \qquad \text{(zero displacement)} \tag{C-2}$$

$$\left.\frac{dy}{dx}\right|_{x=0} = 0 \qquad \text{(zero slope)} \tag{C-3}$$

The boundary conditions at the free end x = L are

$$\left.\frac{d^2y}{dx^2}\right|_{x=L} = 0 \qquad \text{(zero bending moment)} \tag{C-4}$$

$$\left.\frac{d^3y}{dx^3}\right|_{x=L} = 0 \qquad \text{(zero shear force)} \tag{C-5}$$

Propose a quarter cosine wave solution.

$$y(x) = y_o\left[1 - \cos\left(\frac{\pi x}{2L}\right)\right] \tag{C-6}$$

$$\frac{dy}{dx} = y_o\left(\frac{\pi}{2L}\right)\sin\left(\frac{\pi x}{2L}\right) \tag{C-7}$$

$$\frac{d^2y}{dx^2} = y_o\left(\frac{\pi}{2L}\right)^2 \cos\left(\frac{\pi x}{2L}\right) \tag{C-8}$$

$$\frac{d^3y}{dx^3} = -y_o\left(\frac{\pi x}{2L}\right)^3 \sin\left(\frac{\pi x}{2L}\right) \tag{C-9}$$

The proposed solution meets all of the boundary conditions expect for the zero shear force at the right end. The proposed solution is accepted as an approximate solution for the deflection shape, despite one deficiency.

Again, Rayleigh's method is used to find the natural frequency. The total potential energy and the total kinetic energy must be determined.

The total potential energy P in the beam is

$$P = \frac{EI}{2}\int_0^L \left(\frac{d^2y}{dx^2}\right)^2 dx \tag{C-10}$$

By substitution,

$$P = \frac{EI}{2}\int_0^L \left[y_o\left(\frac{\pi}{2L}\right)^2 \cos\left(\frac{\pi x}{2L}\right)\right]^2 dx \tag{C-11}$$

$$P = \frac{EI}{2}\left[y_o\left(\frac{\pi}{2L}\right)^2\right]^2 \int_0^L \left[\cos\left(\frac{\pi x}{2L}\right)\right]^2 dx \tag{C-12}$$

$$P = \frac{EI}{2}\left[y_o\left(\frac{\pi}{2L}\right)^2\right]^2 \int_0^L \left[\frac{1}{2}\right]\left[1+\cos\left(\frac{\pi x}{L}\right)\right] dx \tag{C-13}$$

$$P = \frac{EI}{2}\left[y_o\left(\frac{\pi}{2L}\right)^2\right]^2 \left[\frac{1}{2}\right]\left[x+\left(\frac{L}{\pi}\right)\sin\left(\frac{\pi x}{L}\right)\right]\Bigg|_0^L \tag{C-14}$$

$$P = \frac{EI}{2}\left[y_o\right]^2\left[\frac{\pi^4}{32\,L^4}\right]L \tag{C-15}$$

$$P = \frac{1}{64}\pi^4\left[\frac{EI}{L^3}\right]\left[y_o\right]^2 \tag{C-16}$$

The total kinetic energy T is

$$T = \frac{1}{2}\rho\,\omega_n^2 \int_0^L [y]^2\, dx \tag{C-17}$$

$$T = \frac{1}{2}\rho\,\omega_n^2 \int_0^L \left\{y_o\left[1-\cos\left(\frac{\pi x}{2L}\right)\right]\right\}^2 dx \tag{C-18}$$

$$T = \frac{1}{2}\rho\,\omega_n^2 \left[y_o\right]^2 \int_0^L \left[1-2\cos\left(\frac{\pi x}{2L}\right)+\cos^2\left(\frac{\pi x}{2L}\right)\right] dx \tag{C-19}$$

$$T = \frac{1}{2}\rho\,\omega_n^2 \left[y_o\right]^2 \int_0^L \left[1-2\cos\left(\frac{\pi x}{2L}\right)+\cos^2\left(\frac{\pi x}{2L}\right)\right] dx \tag{C-20}$$

$$T = \frac{1}{2}\rho\,\omega_n^2 \left[y_o\right]^2 \int_0^L \left[1-2\cos\left(\frac{\pi x}{2L}\right)+\frac{1}{2}+\frac{1}{2}\cos\left(\frac{\pi x}{L}\right)\right] dx \tag{C-21}$$

$$T = \frac{1}{2}\rho\,\omega_n^2 \left[y_o\right]^2 \int_0^L \left[\frac{3}{2}-2\cos\left(\frac{\pi x}{2L}\right)+\cos\left(\frac{\pi x}{L}\right)\right] dx \tag{C-22}$$

$$T = \frac{1}{2}\rho\ \omega_n^2 [y_o]^2 \left[\frac{3}{2}x - \left(\frac{4L}{\pi}\right)\sin\left(\frac{\pi x}{2L}\right) + \left(\frac{L}{\pi}\right)\sin\left(\frac{\pi x}{L}\right)\right]\Bigg|_0^L \tag{C-23}$$

$$T = \frac{1}{2}\rho\ \omega_n^2 [y_o]^2 \left[\frac{3}{2}L - \left(\frac{4L}{\pi}\right)\right] \tag{C-24}$$

$$T = \frac{1}{4}\rho\ \omega_n^2 [y_o]^2 L\left[3 - \left(\frac{8}{\pi}\right)\right] \tag{C-25}$$

Now equate the potential and the kinetic energy terms.

$$\frac{1}{4}\rho\ \omega_n^2 [y_o]^2 L\left[3 - \left(\frac{8}{\pi}\right)\right] = \frac{1}{64}\pi^4\left[\frac{EI}{L^3}\right][y_o]^2 \tag{C-26}$$

$$\rho\ \omega_n^2 L\left[3 - \left(\frac{8}{\pi}\right)\right] = \frac{1}{16}\pi^4\left[\frac{EI}{L^3}\right] \tag{C-27}$$

$$\omega_n^2 = \left\{\frac{\pi^4\left[\frac{EI}{L^3}\right]}{16\rho L\left[3 - \left(\frac{8}{\pi}\right)\right]}\right\} \tag{C-28}$$

$$\omega_n = \left\{\frac{\pi^4\left[\frac{EI}{L^3}\right]}{16\rho L\left[3 - \left(\frac{8}{\pi}\right)\right]}\right\}^{1/2} \tag{C-29}$$

$$f_n = \left\{\frac{1}{2\pi}\right\}\left\{\frac{\pi^4\left[\frac{EI}{L^4}\right]}{16\rho\left[3 - \left(\frac{8}{\pi}\right)\right]}\right\}^{1/2} \tag{C-30}$$

$$f_n = \left\{\frac{1}{2\pi}\right\}\left\{\frac{\pi^4\left[\frac{EI}{L^4}\right]}{16\rho\left[3-\left(\frac{8}{\pi}\right)\right]}\right\}^{1/2} \quad \text{(C-31)}$$

$$f_n = \left\{\frac{1}{2\pi}\right\}\left\{\frac{\pi^2}{4L^2}\right\}\left\{\frac{EI}{\rho\left[3-\left(\frac{8}{\pi}\right)\right]}\right\}^{1/2} \quad \text{(C-32)}$$

$$f_n \approx \left\{\frac{1}{2\pi}\right\}\left\{\frac{3.664}{L^2}\right\}\sqrt{\frac{EI}{\rho}} \quad \text{(C-33)}$$

<u>Rayleigh's Quotient</u>

Rayleigh's method can also be applied to multi-degree-freedom-systems, as follows.

$$\omega^2 = \frac{X^T K X}{X^T M X} \quad \text{(D-1)}$$

where

K is the stiffness matrix

M is the mass matrix

X is an assumed mode shape with arbitrary scale

Equation (D-1) is essentially a numerical approximation. It overestimates the true fundamental frequency. Thus, it should be used in a trial-and-error manner.

Note that the numerator in equation (D-1) is equal to twice the potential energy. The denominator is equal to twice the kinetic energy if first multiplied by ω^2.

As an example, consider the system defined in Figure D-1 and Table D-1.

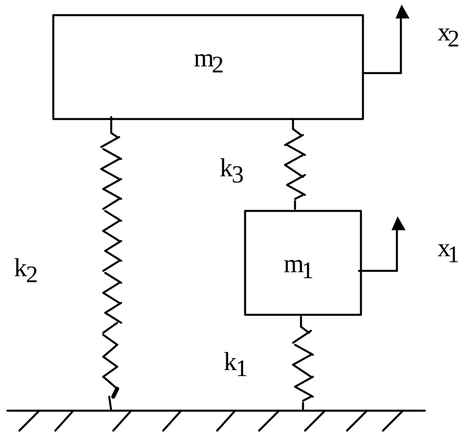

Figure D-1.

Table D-1. Parameters	
Variable	Value
m_1	2.0 kg
m_2	1.0 kg
k_1	1000 N/m
k_2	2000 N/m
k_3	3000 N/m

The homogeneous equation of motion is

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} k_1 + k_3 & -k_3 \\ -k_3 & k_2 + k_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \text{(D-2)}$$

The mass matrix is

$$M = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \text{kg} \qquad \text{(D-3)}$$

The stiffness matrix is

$$K = \begin{bmatrix} 4000 & -3000 \\ -3000 & 5000 \end{bmatrix} \text{N/m} \qquad \text{(D-4)}$$

The natural frequencies can be computed using the eigenvalue method. The eigenvalues are the roots of the following equation.

$$\det \left[K - \omega^2 M \right] = 0 \qquad \text{(D-5)}$$

Equation (D-5) can be solved exactly for systems with up to four degrees-of-freedom. The first natural frequency is thus

$$\omega_1 = 30.03 \;\; \text{rad/sec} \qquad \text{(D-6)}$$

The next task is to test the Rayleigh quotient method. Several candidate mode shape are evaluated as shown in Table D-2.

Note that

$$X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \qquad \text{(D-7)}$$

Table D-2. Rayleigh Quotient Trials	
X	ω (rad/sec)
$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$	31.62
$\begin{bmatrix} 1 \\ 1.5 \end{bmatrix}$	30.68
$\begin{bmatrix} 1 \\ 2 \end{bmatrix}$	31.62
$\begin{bmatrix} 1 \\ 0.5 \end{bmatrix}$	36.51
$\begin{bmatrix} 1 \\ -1 \end{bmatrix}$	54.77

Again, the Rayleigh quotient overestimates the true fundamental frequency.

Thus, the best estimate for the natural frequency after five trials is

$$\omega_1 = 30.68 \ \ \text{rad/sec} \qquad \text{(estimate)} \qquad \text{(D-8)}$$

The estimated value is 2.2% higher than the exact value.

The Rayleigh quotient method thus gives very good results for this example.
