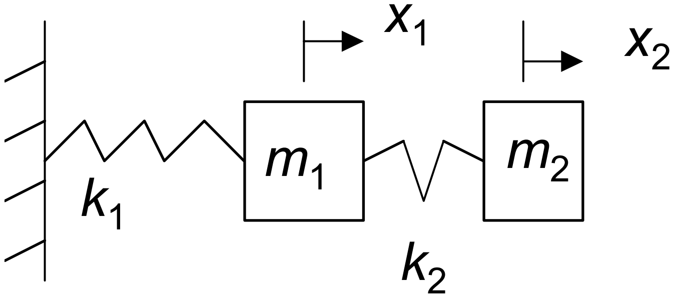
# Vibration 3

**2DOF and**
**Multiple DOF systems**



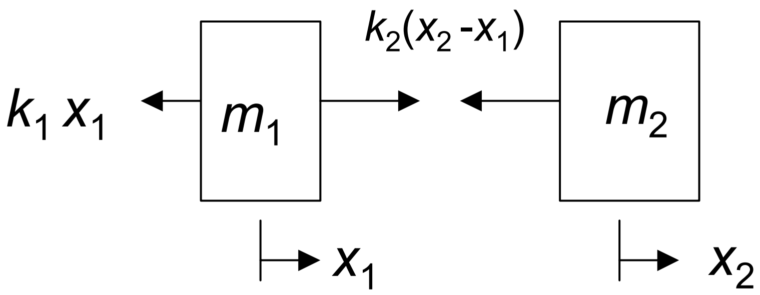


## Two Degrees of Freedom




Textbook: Chapter 5

# Free-Body Diagram

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# Equations of Motion

$$m_1\ddot{x}_1(t) = -k_1x_1(t) + k_2\left(x_2(t) - x_1(t)\right)$$
$$m_2\ddot{x}_2(t) = -k_2\left(x_2(t) - x_1(t)\right)$$

Rearranging :

$$m_1\ddot{x}_1(t) + (k_1 + k_2)x_1(t) - k_2x_2(t) = 0$$
$$m_2\ddot{x}_2(t) - k_2x_1(t) + k_2x_2(t) = 0$$

# Initial Conditions

- Two coupled, second -order, ordinary differential equations with constant coefficients
- Needs 4 constants of integration to solve
- Thus 4 initial conditions on positions and velocities

$$x_1(0)=x_{10}\ \dot{x}_1(0)=\dot{x}_{10}$$

$$x_2(0)=x_{20}\ \dot{x}_2(0)=\dot{x}_{20}$$



# Solution by Matrix Methods

$$\mathbf{x}(t)=\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}, \dot{\mathbf{x}}(t)=\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix}, \ddot{\mathbf{x}}(t)=\begin{bmatrix} \ddot{x}_1(t) \\ \ddot{x}_2(t) \end{bmatrix}$$

$$M=\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}, K=\begin{bmatrix} k_1+k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix}$$

$$M\ddot{\mathbf{x}}+K\mathbf{x}=\mathbf{0}$$

## Initial Conditions

$$\mathbf{x}(0) = \begin{bmatrix} x_{10} \\ x_{20} \end{bmatrix}, \quad \dot{\mathbf{x}}(0) = \begin{bmatrix} \dot{x}_{10} \\ \dot{x}_{20} \end{bmatrix}$$



## Solution:

Let $\mathbf{x}(t) = \mathbf{u}e^{j\omega t}$

$j = \sqrt{-1}, \ \mathbf{u} \neq \mathbf{0}, \ \omega$ unknown

$$\Rightarrow \left(-\omega^2 M + K\right)\mathbf{u}e^{j\omega t} = \mathbf{0}$$

$$\Rightarrow \left(-\omega^2 M + K\right)\mathbf{u} = \mathbf{0}$$

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## Changes ODE into algebraic equation:

$$\left(-\omega^2 M + K\right)\mathbf{u} = \mathbf{0} \Rightarrow$$

two algebraic equation in 3 uknowns

$$\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \text{ and } \omega$$



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## Condition for Solution:

$$\text{inv}\left(-\omega^2 M + K\right) \text{ exists } \Rightarrow \mathbf{u} = \mathbf{0}$$

$$\text{Require } \mathbf{u} \neq \mathbf{0} \quad \Rightarrow \left(-\omega^2 M + K\right)^{-1} \text{ does not exist}$$

$$\text{or} \quad \det\left(-\omega^2 M + K\right) = \mathbf{0}$$

**One equation in one unknown** ω

## Back to our specific system: the characteristic equation

$$\det\left(-\omega^2 M + K\right) = 0 \Rightarrow$$

$$\det\begin{bmatrix} -\omega^2 m_1 + k_1 + k_2 & -k_2 \\ -k_2 & -\omega^2 m_2 + k_2 \end{bmatrix} = 0 \Rightarrow$$

$$m_1 m_2 \omega^4 - (m_1 k_2 + m_2 k_1 + m_2 k_2)\omega^2 + k_1 k_2 = 0$$

Quadratic in $\omega^2$ so four solutions $\omega^2_1$ and $\omega^2_2$ or $\pm\omega_1$ and $\pm\omega_2$



## Calculating the corresponding vectors $\mathbf{u}_1$ and $\mathbf{u}_2$

A vector equation for each square frequency

$$(-\omega_1^2 M + K)\mathbf{u}_1 = \mathbf{0}$$

And:

$$(-\omega_2^2 M + K)\mathbf{u}_2 = \mathbf{0}$$

4 equations in the 4 unknowns (each vector has 2 components, but...

## Numerical example

- $m_1$=9 kg,$m_2$=1kg, $k_1$=24 N/m and $k_2$=3 N/m
- Characteristic equation becomes

$\omega^4$-6$\omega^2$+8=($\omega^2$-2)($\omega^2$-4)=0

$\omega^2$ =2 and $\omega^2$ =4 or

$$\omega_{1,3} = \pm\sqrt{2} \text{ rad/s}, \quad \omega_{2,4} = \pm 2 \text{ rad/s}$$

Each value of $\omega^2$ yields an expression or **u:**



## Computing the vectors **u**

For $\omega_1^2 = 2$, let $\mathbf{u}_1 = \begin{bmatrix} u_{11} \\ u_{12} \end{bmatrix}$ then we have

$$(-\omega_1^2 M + K)\mathbf{u} = \mathbf{0} \Rightarrow$$

$$\begin{bmatrix} 27-9(2) & -3 \\ -3 & 3-(2) \end{bmatrix} \begin{bmatrix} u_{11} \\ u_{12} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow$$

$$9u_{11} - 3u_{12} = 0 \text{ and } -3u_{11} + u_{12} = 0$$

2 equations, 2 unknowns but DEPENDENT!

# continued

$$\frac{u_{11}}{u_{12}} = \frac{1}{3} \Rightarrow u_{11} = \frac{1}{3}u_{12} \text{ from both equations :}$$

only the direction, not the magnitude can be determined!

This is because : $\det(-\omega_1^2 M + K) = 0.$

The magnitude is arbitrary.

Suppose $\mathbf{u}_1$ satisfies

$(-\omega_1^2 M + K)\mathbf{u}_1 = \mathbf{0}$, so does $a\mathbf{u}_1$, $a$ arbitrary :

$$(-\omega_1^2 M + K)a\mathbf{u}_1 = \mathbf{0} \Leftrightarrow (-\omega_1^2 M + K)\mathbf{u}_1 = \mathbf{0}$$



# Likewise for the second value of $\omega^2$:

For $\omega_2^2 = 4$, let $\mathbf{u}_2 = \begin{bmatrix} u_{21} \\ u_{22} \end{bmatrix}$ then we have

$$(\text{-}\omega_1^2 M + K)\mathbf{u} = \mathbf{0} \Rightarrow$$

$$\begin{bmatrix} 27 - 9(4) & -3 \\ -3 & 3 - (4) \end{bmatrix} \begin{bmatrix} u_{21} \\ u_{22} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow$$

$$-9u_{21} - 3u_{22} = 0 \quad \text{or} \quad u_{21} = -\frac{1}{3}u_{22}$$

Note that the other equation is the same

## What to do about the magnitude!

**Several possibilities, here we just fix one element:**

Choose:

$$u_{12} = 1 \Rightarrow \mathbf{u}_1 = \begin{bmatrix} 1/3 \\ 1 \end{bmatrix}$$

Choose:

$$u_{22} = 1 \Rightarrow \mathbf{u}_2 = \begin{bmatrix} -1/3 \\ 1 \end{bmatrix}$$



## Solution to the algebraic matrix equation :

$$\omega_{1,3} = \pm\sqrt{2}, \quad \mathbf{u}_1 = \begin{bmatrix} 1/3 \\ 1 \end{bmatrix}$$

$$\omega_{2,4} = \pm 2, \quad \mathbf{u}_2 = \begin{bmatrix} -1/3 \\ 1 \end{bmatrix}$$

# Return the time response:

We have four solutions:

$$\mathbf{x}(t) = \mathbf{u}_1 e^{-j\omega_1 t}, \mathbf{u}_1 e^{j\omega_1 t}, \mathbf{u}_2 e^{-j\omega_2 t}, \mathbf{u}_2 e^{j\omega_2 t} \Rightarrow$$

Since linear we can combine as:

$$\mathbf{x}(t) = a\mathbf{u}_1 e^{-j\omega_1 t} + b\mathbf{u}_1 e^{j\omega_1 t} + c\mathbf{u}_2 e^{-j\omega_2 t} + d\mathbf{u}_2 e^{j\omega_2 t}$$

$$\Rightarrow \mathbf{x}(t) = \left(ae^{-j\omega_1 t} + be^{j\omega_1 t}\right)\mathbf{u}_1 + \left(ce^{-j\omega_2 t} + de^{j\omega_2 t}\right)\mathbf{u}_2$$

$$= A_1 \sin(\omega_1 t + \phi_1)\mathbf{u}_1 + A_2 \sin(\omega_2 t + \phi_2)\mathbf{u}_2$$

where $A_1, A_2, \phi_1,$ and $\phi_2$ are constants of integration determined by initial conditions



# Physical interpretation

- Each of the TWO masses is oscillating at TWO natural frequencies $w_1$ and $w_2$
- The relative magnitude of each sine term, and hence of the magnitude of oscillation of $m_1$ and $m_2$ is determined by the value of $A_1\mathbf{u}_1$ and $A_2\mathbf{u}_2$
- The vectors $\mathbf{u}_1$ and $\mathbf{u}_2$ are called mode shapes

# What is a mode shape?

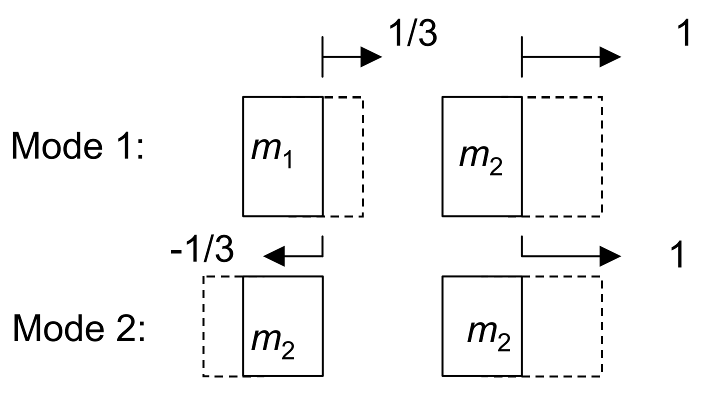
- First note that $A_1$,$A_2$, $f_1$ and $f_2$ are determined by the initial conditions
- Choose them so that $A_2 = f_1 = f_2 = 0$
- Then:

$$\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = A_1 \begin{bmatrix} u_{11} \\ u_{12} \end{bmatrix} \sin \omega_1 t$$

- Thus each mass oscillates at (one) frequency $w_1$ with magnitudes proportional to $\mathbf{u}_1$ the1st *mode shape*



# Mode shapes:



# Solution as a sum of modes

$$\mathbf{x}(t) = \mathbf{u}_1 \cos \omega_1 t + \mathbf{u}_2 \cos \omega_2 t$$

Determines how the first frequency contributes to the response ($\mathbf{u}_1$)

Determines how the second frequency contributes to the response ($\mathbf{u}_2$)



# Things to note

- Two degrees of freedom implies two natural frequencies
- Each mass oscillates at with these two frequencies present in the response
- Frequencies are not those of two component systems

$$\omega_1 = \sqrt{2} \neq \sqrt{\frac{k_1}{m_1}} = 1.63, \omega_2 = 2 \neq \sqrt{\frac{k_2}{m_2}} = 1.732$$