Vibration 3

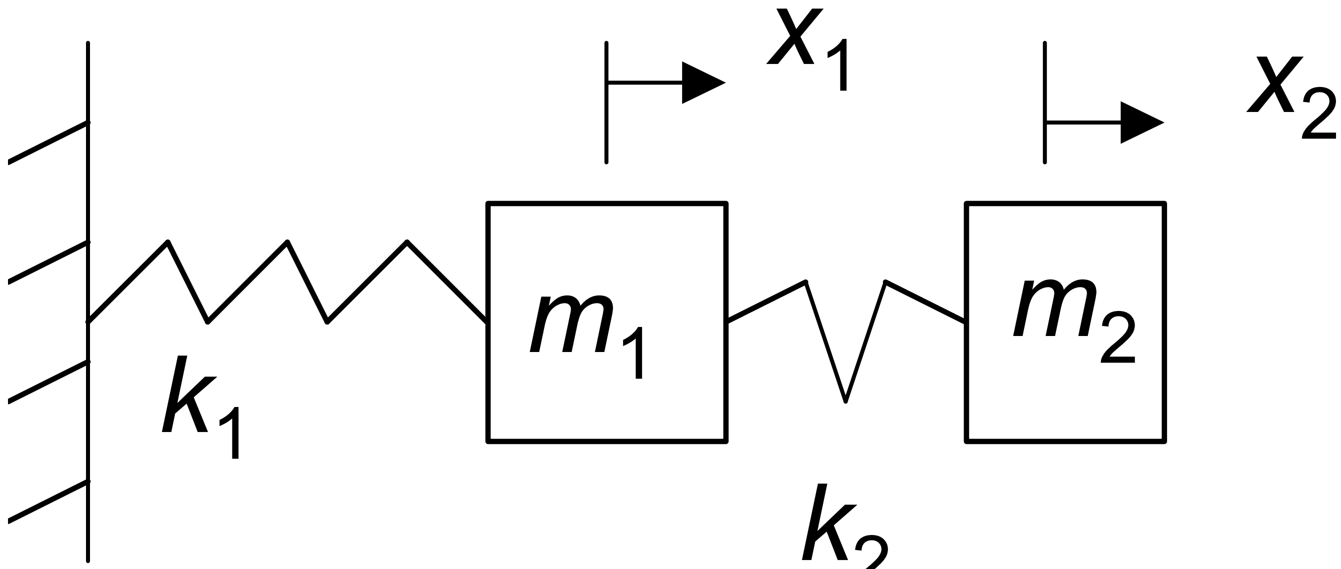
2DOF and Multiple DOF systems

Chapter 4 Multiple Degree of Freedom Systems

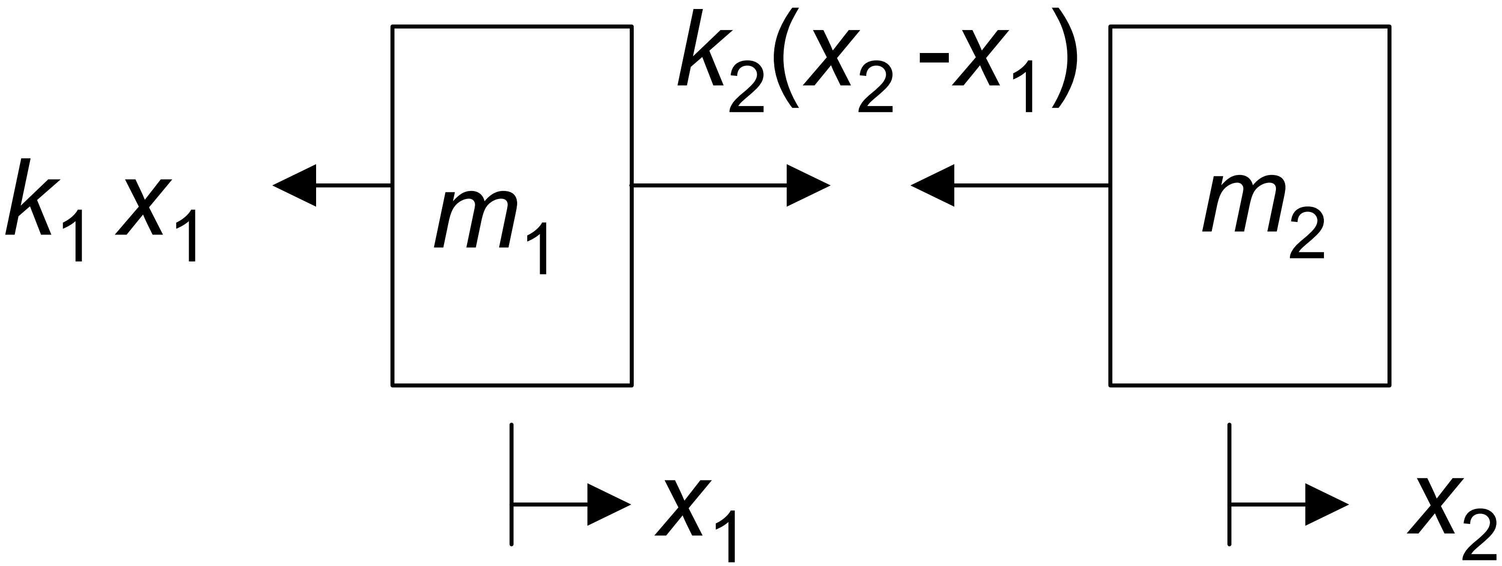
Extending the first 3 chapters to more then one degree of freedom

Rao's textbook: Chap 5

Two Degrees of Freedom (4.1)



Free-Body Diagram



Equations of Motion

$$m_1\ddot{x}_1(t) = -k_1x_1(t) + k_2\left(x_2(t) - x_1(t)\right)$$

$$m_2\ddot{x}_2(t) = -k_2\left(x_2(t) - x_1(t)\right)$$

Rearranging :

$$m_1\ddot{x}_1(t) + (k_1 + k_2)x_1(t) - k_2x_2(t) = 0$$

$$m_2\ddot{x}_2(t) - k_2x_1(t) + k_2x_2(t) = 0$$

Initial Conditions

- Two coupled, second -order, ordinary differential equations with constant coefficients
- Needs 4 constants of integration to solve
- Thus 4 initial conditions on positions and velocities

$$x_1(0)=x_{10}, \dot{x}_1(0)=\dot{x}_{10}, x_2(0)=x_{20}, \dot{x}_2(0)=\dot{x}_{20}$$

Solution by Matrix Methods

$$\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}, \dot{\mathbf{x}}(t) = \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix}, \ddot{\mathbf{x}}(t) = \begin{bmatrix} \ddot{x}_1(t) \\ \ddot{x}_2(t) \end{bmatrix}$$

$$\mathbf{M} = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}, \mathbf{K} = \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix}$$

$$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{K}\mathbf{x} = \mathbf{0}$$

Initial Conditions

$$\mathbf{x}(0) = \begin{bmatrix} x_{10} \\ x_{20} \end{bmatrix}, \quad \dot{\mathbf{x}}(0) = \begin{bmatrix} \dot{x}_{10} \\ \dot{x}_{20} \end{bmatrix}$$

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Solution:

$$\text{Let } \mathbf{x}(t) = \mathbf{u}e^{j\omega t}$$

$$j = \sqrt{-1}, \quad \mathbf{u} \neq \mathbf{0}, \quad \omega \text{ unknown}$$

$$\Rightarrow \left(-\omega^2 M + K\right)\mathbf{u}e^{j\omega t} = \mathbf{0}$$

$$\Rightarrow \left(-\omega^2 M + K\right)\mathbf{u} = \mathbf{0}$$

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Changes ODE into algebraic equation

$$\left(-\omega^2 M + K\right)\mathbf{u} = \mathbf{0} \Rightarrow$$

two algebraic equation in 3 uknowns

$$\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \text{ and } \omega$$

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Condition for Solution:

$$\text{inv}\left(-\omega^2 M + K\right) \text{ exists } \Rightarrow \mathbf{u} = \mathbf{0}$$

$$\text{Require } \mathbf{u} \neq \mathbf{0} \quad \Rightarrow \left(-\omega^2 M + K\right)^{-1} \text{ does not exist}$$

$$\text{or} \quad \det\left(-\omega^2 M + K\right) = \mathbf{0}$$

One equation in one unknown w

Back to our specific system: the characteristic equation

$$\det\left(-\omega^2 M + K\right) = 0 \Rightarrow$$

$$\det\begin{bmatrix} -\omega^2 m_1 + k_1 + k_2 & -k_2 \\ -k_2 & -\omega^2 m_2 + k_2 \end{bmatrix} = 0 \Rightarrow$$

$$m_1 m_2 \omega^4 - (m_1 k_2 + m_2 k_1 + m_2 k_2)\omega^2 + k_1 k_2 = 0$$

Quadratic in ω^2 so four solutions ω^2_1 and ω^2_2 or $\pm\omega_1$ and $\pm\omega_2$

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Calculating the corresponding vectors u_1 and u_2

A vector equation for each square frequency

$$(-\omega_1^2 M + K)\mathbf{u}_1 = \mathbf{0}$$

And:

$$(-\omega_2^2 M + K)\mathbf{u}_2 = \mathbf{0}$$

4 equations in the 4 unknowns (each vector has 2 components, but...

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Numerical examples

- m_1=9 kg,m_2=1kg, k_1=24 N/m and k_2=3 N/m
- Characteristic equation becomes

$$\omega^4-6\omega^2+8=(\omega^2-2)(\omega^2-4)=0$$

$$\omega^2=2 \text{ and } \omega^2=4 \text{ or}$$

$$\omega_{1,3}=\pm\sqrt{2}\text{ rad/s}, \quad \omega_{2,4}=\pm 2 \text{ rad/s}$$

Each value of ω^2 yields an expression or **u:**

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Computing the vectors **u**

For $\omega_1^2 = 2$, let $\mathbf{u}_1 = \begin{bmatrix} u_{11} \\ u_{12} \end{bmatrix}$ then we have

$$(-\omega_1^2 M + K)\mathbf{u} = \mathbf{0} \Rightarrow$$

$$\begin{bmatrix} 27-9(2) & -3 \\ -3 & 3-(2) \end{bmatrix}\begin{bmatrix} u_{11} \\ u_{12} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow$$

$$9u_{11} - 3u_{12} = 0 \quad \text{and} \quad -3u_{11} + u_{12} = 0$$

2 equations, 2 unknowns but DEPENDENT!

continued

$$\frac{u_{11}}{u_{12}} = \frac{1}{3} \Rightarrow u_{11} = \frac{1}{3} u_{12} \text{ from both equations:}$$

only the direction, not the magnitude can be determined!

This is because: $\det(-\omega_1^2 M + K) = 0.$

The magnitude is arbitrary. Suppose $\mathbf{u}_1$ satisfies

$(-\omega_1^2 M + K)\mathbf{u}_1 = \mathbf{0}$, so does $a\mathbf{u}_1$, a arbitrary:

$$(-\omega_1^2 M + K)a\mathbf{u}_1 = \mathbf{0} \Leftrightarrow (-\omega_1^2 M + K)\mathbf{u}_1 = \mathbf{0}$$

Likewise for the second value of ω^2:

For $\omega_2^2 = 4$, let $\mathbf{u}_2 = \begin{bmatrix} u_{21} \\ u_{22} \end{bmatrix}$ then we have

$$(-\omega_1^2 M + K)\mathbf{u} = \mathbf{0} \Rightarrow$$

$$\begin{bmatrix} 27 - 9(4) & -3 \\ -3 & 3 - (4) \end{bmatrix} \begin{bmatrix} u_{21} \\ u_{22} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow$$

$$-9u_{21} - 3u_{22} = 0 \quad \text{or} \quad u_{21} = -\frac{1}{3} u_{22}$$

Note that the other equation is the same

What to do about the magnitude!

Several possibilities, here we just fix one element:

Choose:

$$u_{12} = 1 \Rightarrow \mathbf{u}_1 = \begin{bmatrix} \frac{1}{3} \\ 1 \end{bmatrix}$$

Choose:

$$u_{22} = 1 \Rightarrow \mathbf{u}_2 = \begin{bmatrix} -\frac{1}{3} \\ 1 \end{bmatrix}$$

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Thus the solution to the algebraic matrix equation is:

$$\omega_{1,3} = \pm\sqrt{2}, \quad \mathbf{u}_1 = \begin{bmatrix} 1/3 \\ 1 \end{bmatrix}$$

$$\omega_{2,4} = \pm 2, \quad \mathbf{u}_2 = \begin{bmatrix} -1/3 \\ 1 \end{bmatrix}$$

Return now to the time response:

We have four solutions:

$$\mathbf{x}(t) = \mathbf{u}_1 e^{-j\omega_1 t}, \mathbf{u}_1 e^{j\omega_1 t}, \mathbf{u}_2 e^{-j\omega_2 t}, \mathbf{u}_2 e^{j\omega_2 t} \Rightarrow$$

Since linear we can combine as:

$$\mathbf{x}(t) = a\mathbf{u}_1 e^{-j\omega_1 t} + b\mathbf{u}_1 e^{j\omega_1 t} + c\mathbf{u}_2 e^{-j\omega_2 t} + d\mathbf{u}_2 e^{j\omega_2 t}$$

$$\Rightarrow \mathbf{x}(t) = \left(ae^{-j\omega_1 t} + be^{j\omega_1 t}\right)\mathbf{u}_1 + \left(ce^{-j\omega_2 t} + de^{j\omega_2 t}\right)\mathbf{u}_2$$

$$= A_1 \sin(\omega_1 t + \phi_1)\mathbf{u}_1 + A_2 \sin(\omega_2 t + \phi_2)\mathbf{u}_2$$

where $A_1, A_2, \phi_1,$ and ϕ_2 are constants of integration determined by initial conditions

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Physical interpretation of all that math!

- Each of the TWO masses is oscillating at TWO natural frequencies ω_1 and ω_2
- The relative magnitude of each sine term, and hence of the magnitude of oscillation of m_1 and m_2 is determined by the value of $A_1\mathbf{u}_1$ and $A_2\mathbf{u}_2$
- The vectors $\mathbf{u}_1$ and $\mathbf{u}_2$ are called mode shapes

What is a mode shape?

- First note that A_1,A_2, ϕ_1 and ϕ_2 are determined by the initial conditions
- Choose them so that $A_2 = \phi_1 = \phi_2 = 0$
- Then:

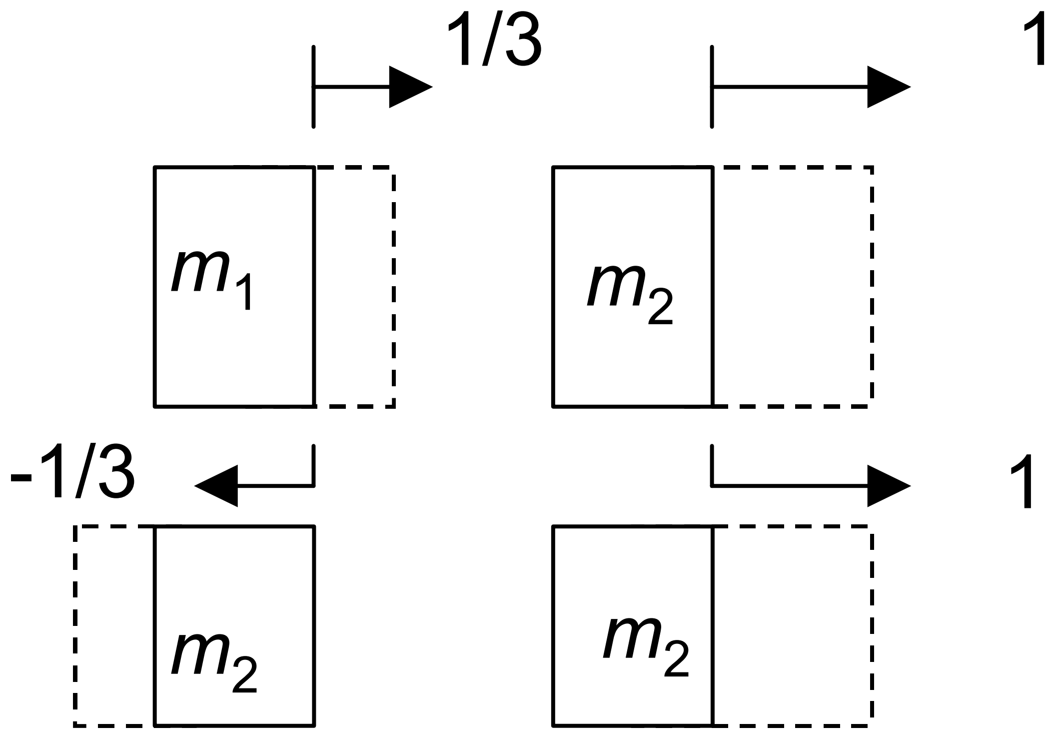
$$\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = A_1 \begin{bmatrix} u_{11} \\ u_{12} \end{bmatrix} \sin \omega_1 t$$

- Thus each mass oscillates at (one) frequency ω_1 with magnitudes proportional to $\mathbf{u}_1$ the 1st *mode shape*

Mode shapes:

Mode 1:

Mode 2:



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Example (continue on)

$$\text{consider } \mathbf{x}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \text{mm}, \dot{\mathbf{x}}(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} \frac{A_1}{3}\sin\left(\sqrt{2}t + \phi_1\right) - \frac{A_2}{3}\sin\left(2t + \phi_2\right) \\ A_1 \sin\left(\sqrt{2}t + \phi_1\right) + A_2 \sin\left(2t + \phi_2\right) \end{bmatrix}$$

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} \frac{A_1}{3}\sqrt{2}\cos\left(\sqrt{2}t + \phi_1\right) - \frac{A_2}{3}2\cos\left(2t + \phi_2\right) \\ A_1\sqrt{2}\cos\left(\sqrt{2}t + \phi_1\right) + A_2 2\cos\left(2t + \phi_2\right) \end{bmatrix}$$

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At t=0 we have

$$\begin{bmatrix} 1\,\mathrm{mm} \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{A_1}{3}\sin(\phi_1) - \frac{A_2}{3}\sin(\phi_2) \\ A_1\sin(\phi_1) + A_2\sin(\phi_2) \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{A_1}{3}\sqrt{2}\sin(\phi_1) - 2\frac{A_2}{3}\sin(\phi_2) \\ A_1\sqrt{2}\sin(\phi_1) + 2A_2\sin(\phi_2) \end{bmatrix}$$

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4 equations in 4 unknowns:

$$3 = A_1 \sin(\phi_1) - A_2 \sin(\phi_2)$$

$$0 = A_1 \sin(\phi_1) + A_2 \sin(\phi_2)$$

$$0 = A_1 \sqrt{2} \cos(\phi_1) - A_2 2 \cos(\phi_2)$$

$$0 = A_1 \sqrt{2} \cos(\phi_1) + A_2 2 \cos(\phi_2)$$

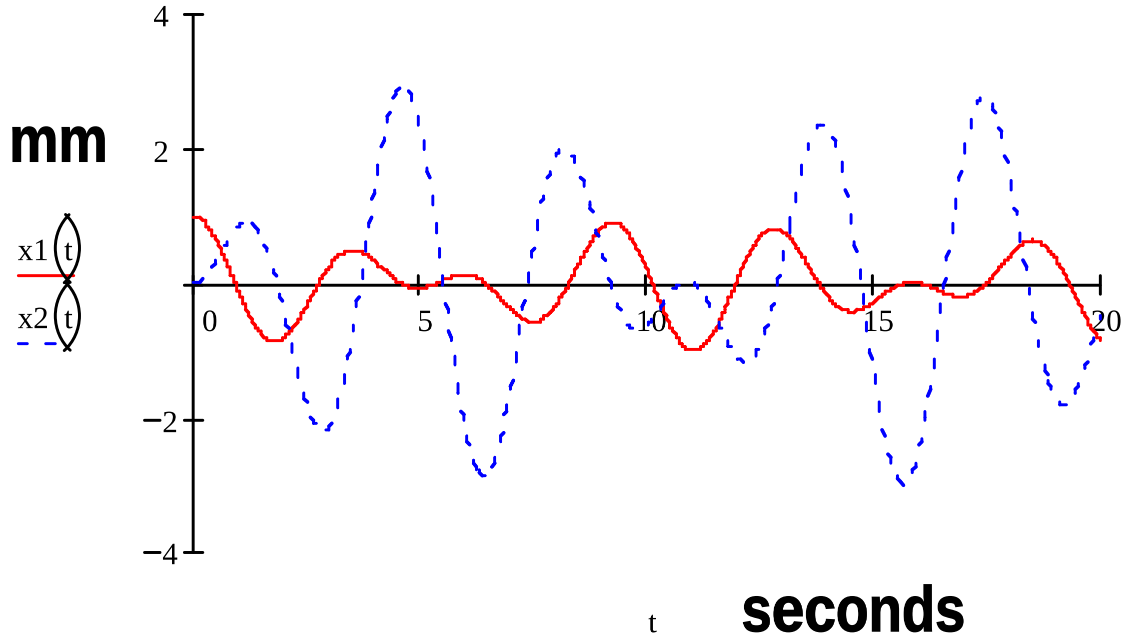
Yields:

$$A_1 = 1.5 \text{ mm}, A_2 = -1.5 \text{ mm}, \phi_1 = \phi_2 = \frac{\pi}{2} \text{ rad}$$

Solution:

$$x_1(t) = 0.5\cos\sqrt{2}t + 0.5\cos 2t$$

$$x_2(t) = 1.5\cos\sqrt{2}t - 1.5\cos 2t$$



Solution as a sum of modes

$$\mathbf{x}(t) = \mathbf{u}_1 \cos\omega_1 t + \mathbf{u}_2 \cos\omega_2 t$$

Determines how the first frequency contributes to the response

Determines how the second frequency contributes to the response

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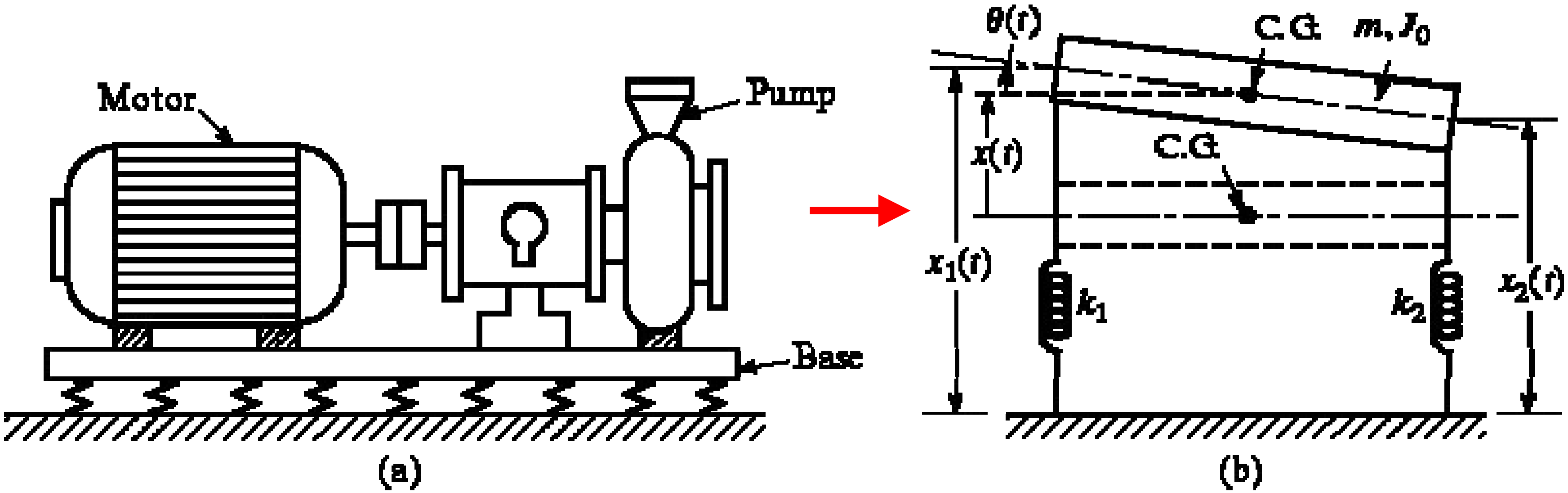
Things to note

- Two degrees of freedom implies two natural frequencies
- Each mass oscillates at with these two frequencies present in the response
- Frequencies are not those of two component systems

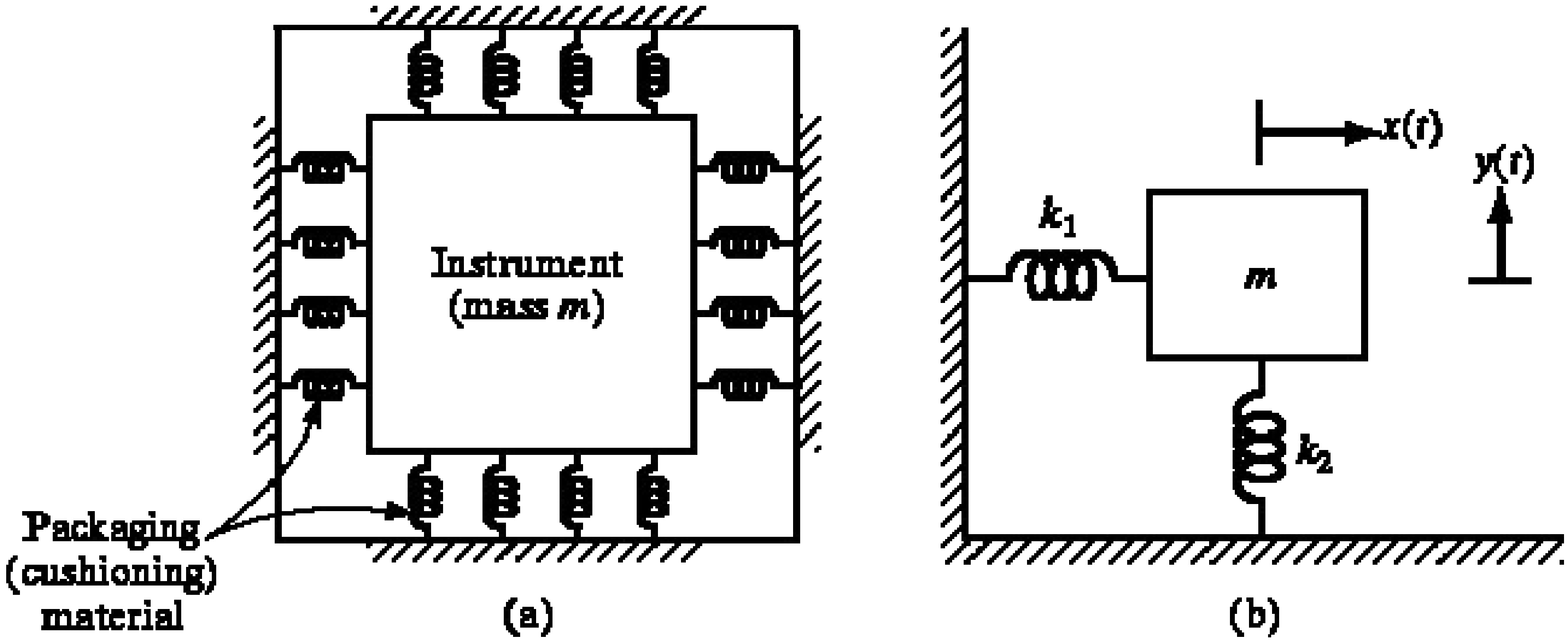
$$\omega_1 = \sqrt{2} \neq \sqrt{\frac{k_1}{m_1}} = 1.63, \omega_2 = 2 \neq \sqrt{\frac{k_2}{m_2}} = 1.732$$

Motor-pump system on springs



Figs.5.1

Packaging of an instrument (portable electronics)



Figs.5.2

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Principle coordinates

- As is evident from the systems shown in Figs.5.1 and 5.2, the configuration of a system can be specified by a set of independent coordinates termed as generalized coordinates, such as length, angle, or some other physical parameters.
- *Principle coordinates* is defined as any set of coordinates that leads a coupled equation of motion to an uncoupled system of equations.

Equations of Motion for 2DOF System

$$m_1\ddot{x}_1 + (c_1 + c_2)\dot{x}_1 - c_2\dot{x}_2 + (k_1 + k_2)x_1 - k_2 x_2 = F_1 \qquad (5.1)$$

$$m_2\ddot{x}_2 - c_2\dot{x}_1 + (c_2 + c_3)\dot{x}_2 - k_2 x_1 + (k_2 + k_3)x_2 = F_2 \qquad (5.2)$$

$$[m]\ddot{\vec{x}}(t) + [c]\dot{\vec{x}}(t) + [k]\vec{x}(t) = \vec{F}(t) \qquad (5.3)$$

where [m], [c], and [k] are called the mass, damping, and stiffness matrices, respectively, and are given by

Properties of M & K Matrices

$$[m] = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \qquad [k] = \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 + k_3 \end{bmatrix}$$

$$[m]^T = [m], \qquad [k]^T = [k]$$

where the superscript T denotes the transpose of the matrix.

Matrices [m] and [k] are symmetric:

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Ex 5.3 Free Vibration Response of a Two Degree of Freedom System

Find the free vibration response of the system shown in Fig.5.3(a) with k_1 = 30, k_2 = 5, k_3 = 0, m_1 = 10, m_2 = 1 and c_1 = c_2 = c_3 = 0 for the initial conditions $x_1(0) = 1,\ \dot{x}_1(0) = x_2(0) = \dot{x}_2(0).$

Solution: For the given data, the eigenvalue problem, Eq.(5.8), becomes

$$\begin{bmatrix} -m_1\omega^2 + k_1 + k_2 & -k_2 \\ -k_2 & -m_2\omega^2 + k_2 + k_3 \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

or

$$\begin{bmatrix} -10\omega^2 + 35 & -5 \\ -5 & -\omega^2 + 5 \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \qquad \text{(E.1)}$$

Example 5.3 Solution

By setting the determinant of the coefficient matrix in Eq.(E.1) to zero, we obtain the frequency equation,

$$10\omega^4 - 85\omega^2 + 150 = 0 \qquad \text{(E.2)}$$

from which the natural frequencies can be found as

$$\omega_1^2 = 2.5, \qquad \omega_2^2 = 6.0$$

$$\omega_1 = 1.5811, \qquad \omega_2 = 2.4495 \qquad \text{(E.3)}$$

The normal modes (or eigenvectors) are given by

$$X^{(1)} \quad \begin{matrix} X_1^{(1)} \\ X_2^{(1)} \end{matrix} \quad \begin{matrix} 1 \\ 2 \end{matrix} \; X_1^{(1)} \qquad \text{(E.4)}$$

$$X^{(2)} \quad \begin{Bmatrix} X_1^{(2)} \\ X_2^{(2)} \end{Bmatrix} = \begin{Bmatrix} 1 \\ -5 \end{Bmatrix} X_1^{(2)} \qquad \text{(E.5)}$$

-

Example 5.3 Solution

The free vibration responses of the masses m_1 and m_2 are given by (see Eq.5.15):

$$x_1(t) = X_1^{(1)} \cos(1.5811t + \phi_1) + X_1^{(2)} \cos(2.4495t + \phi_2) \qquad (E.6)$$

$$x_2(t) = 2X_1^{(1)} \cos(1.5811t + \phi_1) - 5X_1^{(2)} \cos(2.4495t + \phi_2) \qquad (E.7)$$

By using the given initial conditions in Eqs.(E.6) and (E.7), we obtain

$$x_1(t=0) = 1 = X_1^{(1)} \cos\phi_1 + X_1^{(2)} \cos\phi_2 \qquad (E.8)$$

$$x_2(t=0) = 0 = 2X_1^{(1)} \cos\phi_1 - 5X_1^{(2)} \cos\phi_2 \qquad (E.9)$$

$$\dot{x}_1(t=0) = 0 = -1.5811X_1^{(1)} \sin\phi_1 - 2.4495X_1^{(2)} \sin\phi_2 \qquad (E.10)$$

$$\dot{x}_2(t=0) = -3.1622X_1^{(1)} + 12.2475X_1^{(2)} \sin\phi_2 \qquad (E.11)$$

Example 5.3 Solution

The solution of Eqs.(E.8) and (E.9) yields

$$X_1^{(1)} \cos \phi_1 = \frac{5}{7}; \qquad X_1^{(2)} \cos \phi_2 = \frac{2}{7} \qquad \text{(E.12)}$$

while the solution of Eqs.(E.10) and (E.11) leads to

$$X_1^{(1)} \sin \phi_1 = 0, \quad X_1^{(2)} \sin \phi_2 = 0 \qquad \text{(E.13)}$$

Equations (E.12) and (E.13) give

$$X_1^{(1)} = \frac{5}{7}, \qquad X_1^{(2)} = \frac{2}{7}, \qquad \phi_1 = 0, \quad \phi_2 = 0 \qquad \text{(E.14)}$$

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Example 5.3 Solution

Thus the free vibration responses of m_1 and m_2 are given by

$$x_1(t) = \frac{5}{7}\cos 1.5811t + \frac{2}{7}\cos 2.4495t \qquad \text{(E.15)}$$

$$x_2(t) = \frac{10}{7}\cos 1.5811t - \frac{10}{7}\cos 2.4495t \qquad \text{(E.16)}$$

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5.4 Torsional System

Consider a torsional system as shown in Fig.5.6. The differential equations of rotational motion for the discs can be derived as

$$J_1\ddot{\theta}_1 = -k_{t1}\theta_1 + k_{t2}(\theta_2 - \theta_1) + M_{t1}$$
$$J_2\ddot{\theta}_2 = -k_{t2}(\theta_2 - \theta_1) - k_{t3}\theta_2 + M_{t2}$$

which upon rearrangement become

$$J_1\ddot{\theta}_1 + (k_{t1} + k_{t2})\theta_1 - k_{t2}\theta_2 = M_{t1}$$
$$J_2\ddot{\theta}_2 - k_{t2}\theta_1 + (k_{t2} + k_{t3})\theta_2 = M_{t2} \qquad (5.19)$$

For the free vibration analysis of the system, Eq.(5.19) reduces to

5.4 Torsional System

$$J_1\ddot{\theta}_1 + (k_{t1} + k_{t2})\theta_1 - k_{t2}\theta_2 = 0$$

$$J_2\ddot{\theta}_2 - k_{t2}\theta_1 + (k_{t2} + k_{t3})\theta_2 = 0 \qquad (5.20)$$

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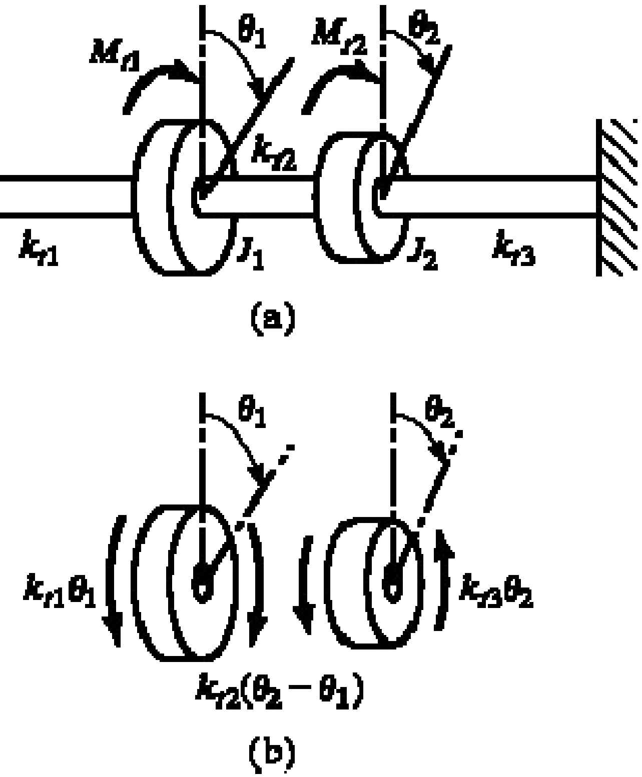

Figure 5.6: Torsional system with discs mounted on a shaft

Ex 5.4 Natural Frequencies of a Torsional System

Find the natural frequencies and mode shapes for the torsional system shown in Fig.5.7 for $J_1 = J_0$, $J_2 = 2J_0$ and $k_{t1} = k_{t2} = k_t$.

Solution:

The differential equations of motion, Eq.(5.20), reduce to (with $k_{t3} = 0$, $k_{t1} = k_{t2} = k_t$, $J_1 = J_0$ and $J_2 = 2J_0$):

$$J_0\ddot{\theta}_1 + 2k_t\theta_1 - k_t\theta_2 = 0$$

$$2J_0\ddot{\theta}_2 - k_t\theta_1 + k_t\theta_2 = 0 \qquad \text{(E.1)}$$

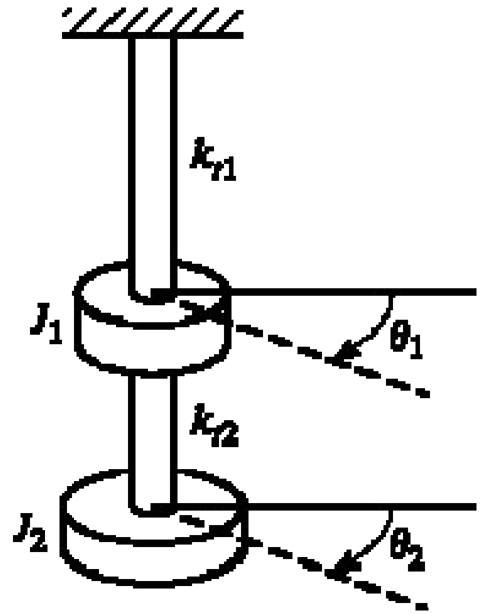

Fig.5.7: Torsional system

Example 5.4 Solution

Rearranging and substituting the harmonic solution:

$$\theta_i(t) = \Theta_i \cos(\omega t + \phi); \; i = 1,2 \qquad \text{(E.2)}$$

gives the frequency equation:

$$2\omega^4 J_0^2 - 5\omega^2 J_0 k_t + k_t^2 = 0 \qquad \text{(E.3)}$$

The solution of Eq.(E.3) gives the natural frequencies

$$\omega_1 = \sqrt{\frac{k_t}{4J_0}(5-\sqrt{17})} \quad \text{and} \quad \omega_2 = \sqrt{\frac{k_t}{4J_0}(5+\sqrt{17})} \quad \text{(E.4)}$$

$$= 0.4682\sqrt{\mathrm{k_t} / J_0} \qquad = 1.5102\sqrt{\mathrm{k_t} / J_0}$$

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Example 5.4 Solution

The amplitude ratios are given by

$$r_1 = \frac{\Theta_2^{(1)}}{\Theta_1^{(1)}} = 2 - \frac{(5-\sqrt{17})}{4} = 1.7808$$

$$r_2 = \frac{\Theta_2^{(2)}}{\Theta_1^{(2)}} = 2 - \frac{(5+\sqrt{17})}{4} = -0.2808 \qquad \text{(E.5)}$$

Equations (E.4) and (E.5) can also be obtained by substituting the following in Eqs.(5.10) and (5.11).

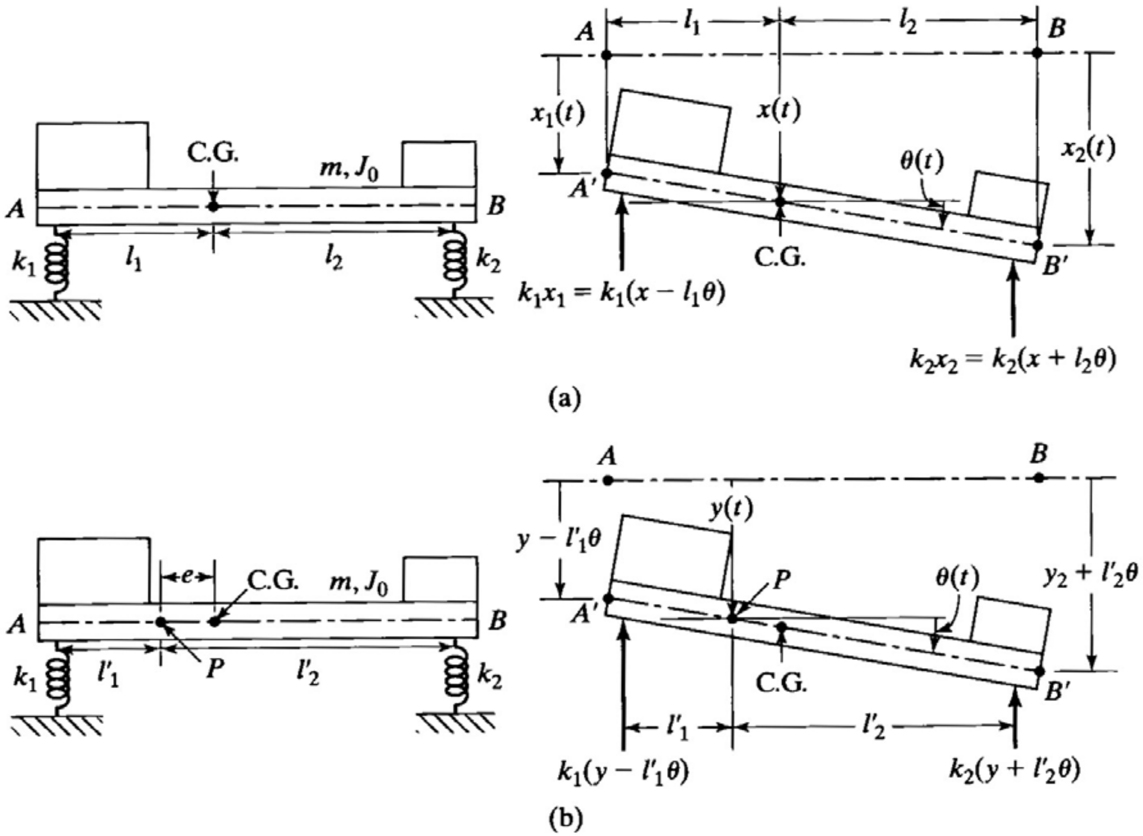
$$k_1 = k_{t1} = k_t, \quad k_2 = k_{t2} = k_t,$$

$$m_1 = J_1 = J_0, \quad m_2 = J_2 = 2J_0 \text{ and } \quad k_3 = 0$$

5.5 Coordinate Coupling and Principal Coordinates

Generalized coordinates are sets of n coordinates used to describe the configuration of the system.

Equations of motion for a lathe Using x(t) and θ(t)



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5.5 Coordinate Coupling and Principal Coordinates

From the free-body diagram shown in Fig.5.10a, with the positive values of the motion variables as indicated, the force equilibrium equation in the vertical direction can be written as

$$m\ddot{x} = -k_1(x - l_1\theta) - k_2(x + l_2\theta) \qquad (5.21)$$

and the moment equation about C.G. can be expressed as

$$J_0\ddot{\theta} = k_1(x - l_1\theta)l_1 - k_2(x + l_2\theta)l_2 \qquad (5.22)$$

Eqs.(5.21) and (5.22) can be rearranged and written in matrix form as

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5.5 Coordinate Coupling and Principal Coordinates

$$\begin{bmatrix} m & 0 \\ 0 & J_0 \end{bmatrix} \begin{Bmatrix} \ddot{x} \\ \ddot{\theta} \end{Bmatrix} + \begin{bmatrix} (k_1 + k_2) & -(k_1 l_1 - k_2 l_2) \\ -(k_1 l_1 - k_2 l_2) & (k_1 l_1^2 + k_2 l_2^2) \end{bmatrix} \begin{Bmatrix} x \\ \theta \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \qquad (5.23)$$

The lathe rotates in the vertical plane and has vertical motion as well, unless $k_1 l_1 = k_2 l_2$. This is known as *elastic or static coupling*.

•***Equations of motion Using y(t) and θ(t).***

From Fig.5.10b, the equations of motion for translation and rotation can be written as

$$m\ddot{y} = -k_1(y - l_1'\theta) - k_2(y + l_2'\theta) - me\ddot{\theta}$$

5.5 Coordinate Coupling and Principal Coordinates

$$J_P\ddot{\theta} = k_1(y - l_1'\theta)l_1' - k_2(y + l_2'\theta)l_2' - me\ddot{y} \qquad (5.24)$$

These equations can be rearranged and written in matrix form as

$$\begin{bmatrix} m & me \\ me & J_P \end{bmatrix}\begin{Bmatrix} \ddot{y} \\ \ddot{\theta} \end{Bmatrix} + \begin{bmatrix} (k_1 + k_2) & (k_2 l_2' - k_1 l_1') \\ (-k_1 l_1' + k_2 l_2') & (k_1 l_1'^2 + k_2 l_2'^2) \end{bmatrix}\begin{Bmatrix} y \\ \theta \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \qquad (5.25)$$

If $k_1 l_1' = k_2 l_2'$, the system will have *dynamic or inertia coupling only*.

Note the following characteristics of these systems:

5.5 Coordinate Coupling and Principal Coordinates

1. In the most general case, a viscously damped 2DOF system has the equations of motions in the form:

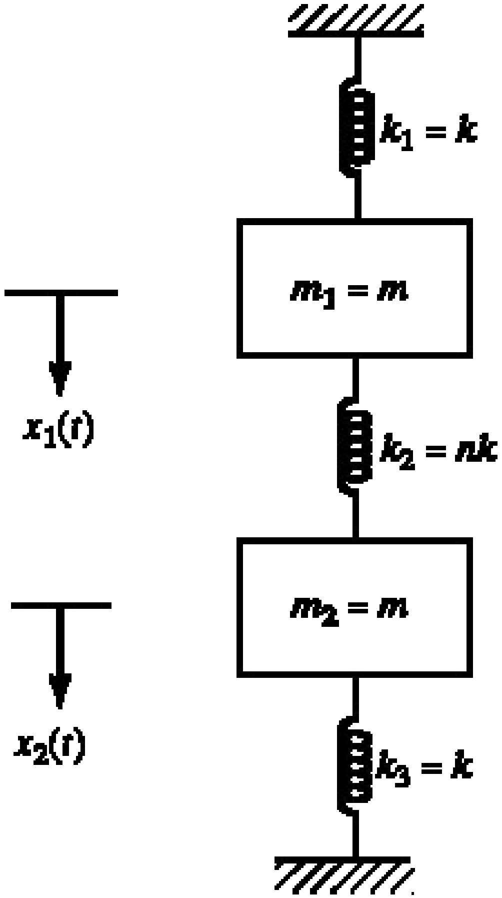
$$\begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} \begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{Bmatrix} + \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \qquad (5.26)$$

2. The system vibrates in its own natural way regardless of the coordinates used. The choice of the coordinates is a mere convenience.

3. *Principal* or *natural coordinates* are defined as system of coordinates which give equations of motion that are uncoupled both statically and dynamically.

Ex 5.6 Principal Coordinates of Spring-Mass System

Determine the principal coordinates for the spring-mass system shown in Fig.5.4.



Example 5.6 Solution

Approach: Define two independent solutions as principal coordinates and express them in terms of the solutions $x_1(t)$ and $x_2(t)$.

The general motion of the system shown is

$$x_1(t) = B_1 \cos\left(\sqrt{\frac{k}{m}}t + \phi_1\right) + B_2 \cos\left(\sqrt{\frac{3k}{m}}t + \phi_2\right)$$

$$x_2(t) = B_1 \cos\left(\sqrt{\frac{k}{m}}t + \phi_1\right) - B_2 \cos\left(\sqrt{\frac{3k}{m}}t + \phi_2\right) \qquad \text{(E.1)}$$

We define a new set of coordinates such that

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Example 5.6 Solution

$$q_1(t) = B_1 \cos\left(\sqrt{\frac{k}{m}}t + \phi_1\right)$$

$$q_2(t) = B_2 \cos\left(\sqrt{\frac{3k}{m}}t + \phi_2\right) \qquad \text{(E.2)}$$

Since the coordinates are harmonic functions, their corresponding equations of motion can be written as

$$\ddot{q}_1 + \left(\frac{k}{m}\right)q_1 = 0$$

$$\ddot{q}_2 + \left(\frac{3k}{m}\right)q_2 = 0 \qquad \text{(E.3)}$$

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Example 5.6 Solution

From Eqs.(E.1) and (E.2), we can write

$$x_1(t) = q_1(t) + q_2(t)$$
$$x_2(t) = q_1(t) - q_2(t) \qquad \text{(E.4)}$$

The solution of Eqs.(E.4) gives the principal coordinates:

$$q_1(t) = \frac{1}{2}[x_1(t) + x_2(t)]$$
$$q_2(t) = \frac{1}{2}[x_1(t) - x_2(t)] \qquad \text{(E.5)}$$

Forced Vibration Analysis

The equations of motion of a general 2DOF system under external forces can be written as

$$\begin{bmatrix} m_{11} & m_{12} \\ m_{12} & m_{22} \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} \begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{Bmatrix} + \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix} \qquad (5.27)$$

Consider the external forces to be harmonic:

$$F_j(t) = F_{j0} e^{i\omega t}, \qquad j = 1,2 \qquad (5.28)$$

where ω is the forcing frequency. We can write the steady-state solutions as

$$x_j(t) = X_j e^{i\omega t}, \qquad j = 1,2 \qquad (5.29)$$

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Forced Vibration Analysis

Substitution of Eqs.(5.28) and (5.29) into Eq.(5.27) leads to

$$\begin{bmatrix} (-\omega^2 m_{11} + i\omega c_{11} + k_{11}) & (-\omega^2 m_{12} + i\omega c_{12} + k_{12}) \\ (-\omega^2 m_{12} + i\omega c_{12} + k_{12}) & (-\omega_2 m_{22} + i\omega c_{22} + k_{22}) \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix}$$

$$= \begin{Bmatrix} F_{10} \\ F_{20} \end{Bmatrix} \qquad (5.30)$$

We defined as in Section 3.5 the mechanical impedance $Z_{re}(i\omega)$ as

$$Z_{rs}(i\omega) = -\omega^2 m_{rs} + i\omega c_{rs} + k_{rs}, r, s = 1,2 \qquad (5.31)$$

Forced Vibration Analysis

And write Eq.(5.30) as:

$$\left[Z(i\omega)\right]\vec{X} = \vec{F}_0 \qquad (5.32)$$

where

$$\left[Z(i\omega)\right] = \begin{bmatrix} Z_{11}(i\omega) & Z_{12}(i\omega) \\ Z_{12}(i\omega) & Z_{22}(i\omega) \end{bmatrix} = \text{Impedance matrix}$$

$$\vec{X} = \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix}$$

$$\vec{F}_0 = \begin{Bmatrix} F_{10} \\ F_{20} \end{Bmatrix}$$

Eq.(5.32) can be solved to obtain:

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Forced Vibration Analysis

$$\vec{X} = \left[Z(i\omega)\right]^{-1} \vec{F}_0 \qquad (5.33)$$

where the inverse of the impedance matrix is given

$$\left[Z(i\omega)\right]^{-1} = \frac{1}{Z_{11}(i\omega)Z_{22}(i\omega) - Z_{12}^2(i\omega)} \begin{bmatrix} Z_{22}(i\omega) & -Z_{12}(i\omega) \\ -Z_{12}(i\omega) & Z_{11}(i\omega) \end{bmatrix} \qquad (5.34)$$

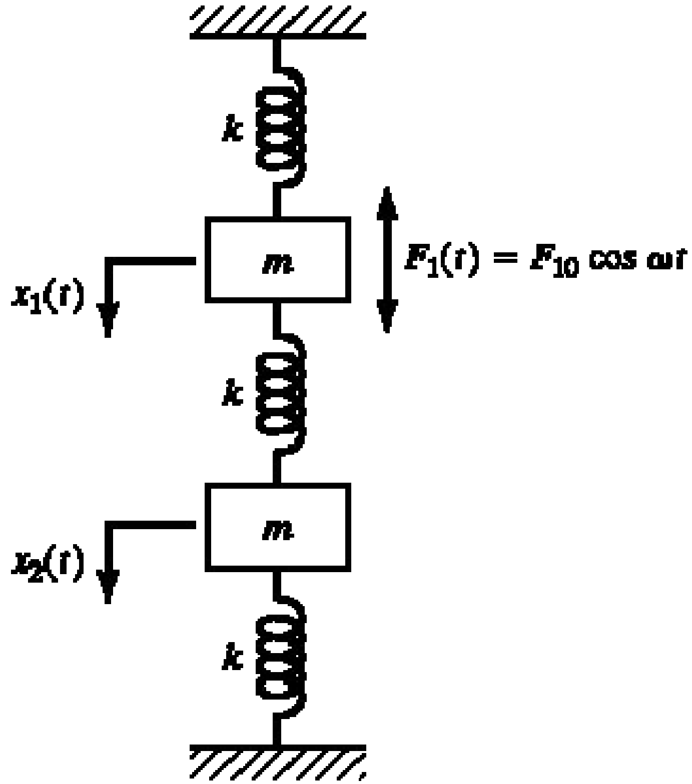
Eqs.(5.33) and (5.34) lead to the solution

$$X_1(i\omega) = \frac{Z_{22}(i\omega)F_{10} - Z_{12}(i\omega)F_{20}}{Z_{11}(i\omega)Z_{22}(i\omega) - Z_{12}^2(i\omega)}$$

$$X_2(i\omega) = \frac{-Z_{12}(i\omega)F_{10} + Z_{11}(i\omega)F_{20}}{Z_{11}(i\omega)Z_{22}(i\omega) - Z_{12}^2(i\omega)} \qquad (5.35)$$

Ex 5.8 Steady-State Response of Spring-Mass System

Find the steady-state response of system shown in Fig.5.13 when the mass m_1 is excited by the force $F_1(t) = F_{10} \cos \omega t$. Also, plot its frequency response curve.



Example 5.8 Solution

The equations of motion of the system can be expressed as

$$\begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} 2k & -k \\ -k & 2k \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} F_{10} \cos \omega t \\ 0 \end{Bmatrix} \qquad \text{(E.1)}$$

We assume the solution to be as follows.

$$x_j(t) = X_j \cos \omega t; \qquad j = 1,2 \qquad \text{(E.2)}$$

Eq.(5.31) gives

$$Z_{11}(\omega) = Z_{22}(\omega) = -m\omega^2 + 2k, \qquad Z_{12}(\omega) = -k \qquad \text{(E.3)}$$

Example 5.8 Solution

Hence,

$$X_1(\omega) = \frac{(-\omega^2 m + 2k)F_{10}}{(-\omega^2 m + 2k)^2 - k^2} = \frac{(-\omega^2 m + 2k)F_{10}}{(-m\omega^2 + 3k)(-m\omega^2 + k)} \qquad \text{(E.4)}$$

$$X_2(\omega) = \frac{kF_{10}}{(-m\omega^2 + 2k)^2 - k^2} = \frac{kF_{10}}{(-m\omega^2 + 3k)(-m\omega^2 + k)} \qquad \text{(E.5)}$$

Eqs.(E.4) and (E.5) can be expressed as

$$X_1(\omega) = \frac{\left\{2 - \left(\frac{\omega}{\omega_1}\right)^2\right\}F_{10}}{k\left[\left(\frac{\omega_2}{\omega_1}\right)^2 - \left(\frac{\omega}{\omega_1}\right)^2\right]\left[1 - \left(\frac{\omega}{\omega_1}\right)^2\right]} \qquad \text{(E.6)}$$

Example 5.8 Solution

$$X_2(\omega) = \frac{F_{10}}{k\left[\left(\frac{\omega_2}{\omega_1}\right)^2 - \left(\frac{\omega}{\omega_1}\right)^2\right]\left[1-\left(\frac{\omega}{\omega_1}\right)^2\right]} \qquad \text{(E.7)}$$

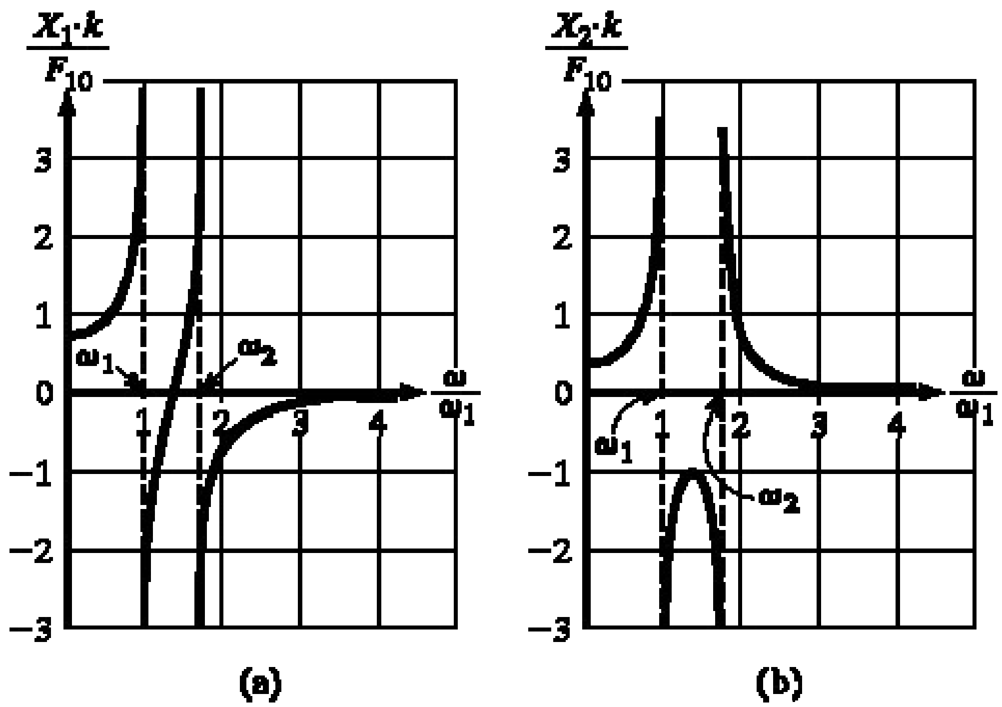


Fig.5.14: Frequency response curves

Semidefinite Systems

Semidefinite systems are also known as *unrestrained* or *degenerate systems*. Two examples of such systems are shown in Fig.5.15. For Fig.5.15a, the equations of motion can be written as

$$\begin{aligned} m_1\ddot{x}_1 + k(x_1 - x_2) &= 0 \\ m_2\ddot{x}_2 + k(x_2 - x_1) &= 0 \end{aligned} \qquad (5.36)$$

For free vibration, we assume the motion to be harmonic:

$$x_j(t) = X_j \cos(\omega t + \phi_j), \qquad j = 1,2 \qquad (5.37)$$

Semidefinite Systems

Substituting Eq.(5.37) into Eq.(5.36) gives

$$(-m_1\omega^2 - k)X_1 - kX_2 = 0$$

$$-kX_1 + (-m_2\omega^2 + k)X_2 = 0 \qquad (5.38)$$

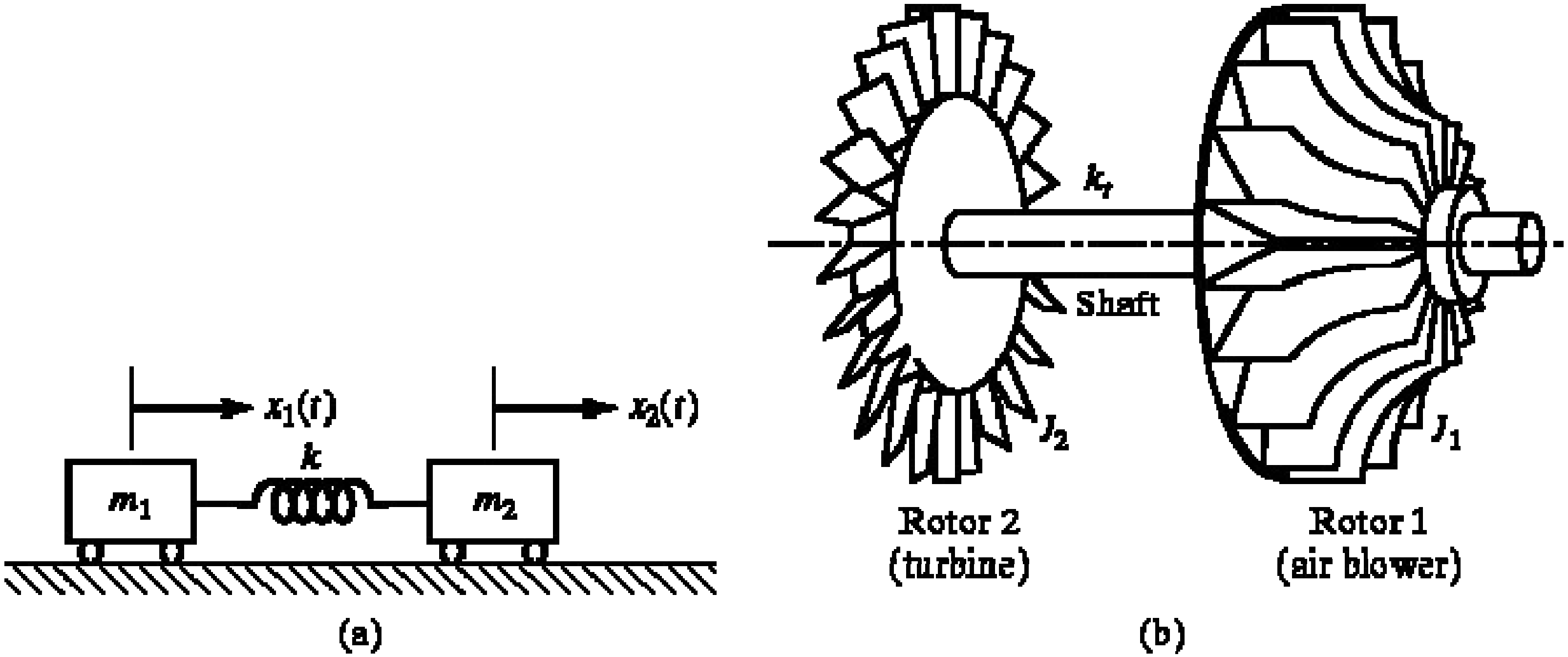


Fig.5.15: Semidefinite Systems

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Semidefinite Systems

We obtain the frequency equation as

$$\omega^2[m_1 m_2 \omega^2 - k(m_1 + m_2)] = 0 \qquad (5.39)$$

From which the natural frequencies can be obtained:

$$\omega_1 = 0 \quad \text{and} \quad \omega_2 = \sqrt{\frac{k(m_1 + m_2)}{m_1 m_2}} \qquad (5.40)$$

Such systems, which have one of the natural frequencies equal to zero, are called *semidefinite systems*.

Frequency (characteristic) equation for 2DOF mass-spring system

$$(m_1 m_2)\omega^4 - \{(k_1 + k_2)m_2 + (k_2 + k_3)m_1\}\omega$$
$$+\{(k_1 + k_2)(k_2 + k_3) - k_2^2\} = 0 \qquad (5.9)$$

$$\omega_1^2, \omega_2^2 = \frac{1}{2}\left\{\frac{(k_1 + k_2)m_2 + (k_2 + k_3)m_1}{m_1 m_2}\right\}$$
$$\mp \frac{1}{2}\left[\left\{\frac{(k_1 + k_2)m_2 + (k_2 + k_3)m_1}{m_1 m_2}\right\}^2\right.$$
$$\left. - 4\left\{\frac{(k_1 + k_2)(k_2 + k_3) - k_2^2}{m_1 m_2}\right\}\right]^{1/2} \qquad (5.10)$$