THE GENERALIZED COORDINATE METHOD FOR DISCRETE SYSTEMS

Two-degree-of-freedom System

The method of generalized coordinates is demonstrated by an example. Consider the system in Figure 1.

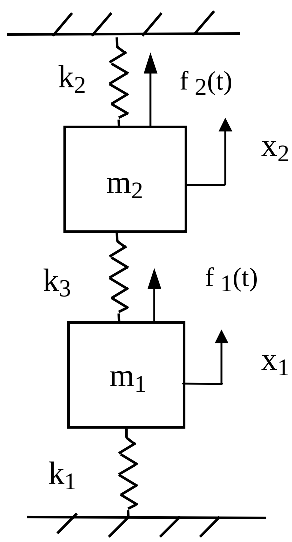

Figure 1.

A free-body diagram of mass 1 is given in Figure 2. A free-body diagram of mass 2 is given in Figure 3.

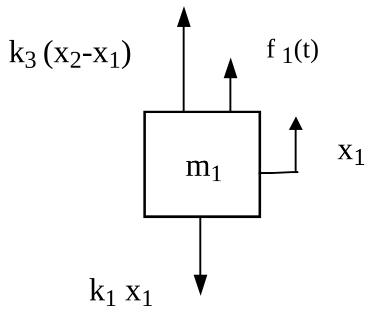

Figure 2.

Determine the equation of motion for mass 1.

$$\Sigma F = m_1 \ddot{x}_1 \tag{1}$$

$$m_1 \ddot{x}_1 = f_1(t) + k_3 (x_2 - x_1) - k_1 x_1 \tag{2}$$

$$m_1 \ddot{x}_1 + k_1 x_1 - k_3 (x_2 - x_1) = f_1(t) \tag{3}$$

$$m_1 \ddot{x}_1 + k_1 x_1 + k_3 (x_1 - x_2) = f_1(t) \tag{4}$$

$$m_1 \ddot{x}_1 + (k_1 + k_3) x_1 - k_3 x_2 = f_1(t) \tag{5}$$

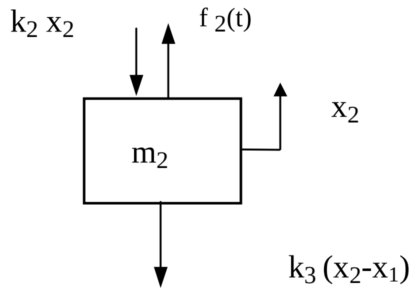

Figure 3.

Derive the equation of motion for mass 2.

$$\Sigma F = m_2 \ddot{x}_2 \tag{6}$$

$$m_2 \ddot{x}_2 = f_2(t) - k_3(x_2 - x_1) - k_2 x_2 \tag{7}$$

$$m_2 \ddot{x}_2 + k_2 x_2 + k_3(x_2 - x_1) = f_2(t) \tag{8}$$

$$m_2 \ddot{x}_2 + (k_2 + k_3)x_2 - k_3 x_1 = f_2(t) \tag{9}$$

Assemble the equations in matrix form.

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} k_1 + k_3 & -k_3 \\ -k_3 & k_2 + k_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} f_1(t) \\ f_2(t) \end{bmatrix} \tag{10}$$

<u>Decoupling</u>

Equation (10) is coupled via the stiffness matrix. An intermediate goal is to decouple the equation.

Simplify,

$$M \ddot{\overline{x}} + K \overline{x} = \overline{F} \tag{11}$$

where

$$M = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \tag{12}$$

$$K = \begin{bmatrix} k_1 + k_3 & -k_3 \\ -k_3 & k_2 + k_3 \end{bmatrix} \tag{13}$$

$$\bar{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{14}$$

$$\bar{F} = \begin{bmatrix} f_1(t) \\ f_2(t) \end{bmatrix} \tag{15}$$

Consider the homogeneous form of equation (11).

$$M\,\ddot{\bar{x}} + K\,\bar{x} = \bar{0} \tag{16}$$

Seek a solution of the form

$$\bar{x} = \bar{q}\exp(\,j\omega t) \tag{17}$$

The q vector is the generalized coordinate vector.

Note that

$$\dot{\bar{x}} = j\omega\,\bar{q}\exp(\,j\omega t) \tag{18}$$

$$\ddot{\bar{x}} = -\omega^2\;\bar{q}\exp(\,j\omega t) \tag{19}$$

Substitute equations (17) through (19) into equation (16).

$$-\omega^2 M\;\bar{q}\exp(\,j\omega t) + K\,\bar{q}\exp(\,j\omega t) = \bar{0} \tag{20}$$

$$\left\{-\omega^2 M\;\bar{q} + K\,\bar{q}\right\}\exp(\,j\omega t) = \bar{0} \tag{21}$$

$$-\omega_n{}^2 M\;\bar{q} + K\,\bar{q} = \bar{0} \tag{22}$$

$$\left\{-\omega^2 M\;+K\right\}\bar{q} = \bar{0} \tag{23}$$

$$\left\{ K - \omega^2 M \right\} \bar{q} = \bar{0} \tag{24}$$

Equation (24) is an example of a generalized eigenvalue problem. The eigenvalues can be found by setting the determinant equal to zero.

$$\det \left\{ K - \omega^2 M \right\} = 0 \tag{25}$$

$$\det\left\{ \begin{bmatrix} k_1 + k_3 & -k_3 \\ -k_3 & k_2 + k_3 \end{bmatrix} - \omega^2 \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \right\} = 0 \tag{26}$$

$$\det \begin{bmatrix} (k_1 + k_3) - \omega^2 m_1 & -k_3 \\ -k_3 & (k_2 + k_3) - \omega^2 m_2 \end{bmatrix} = 0 \tag{27}$$

$$\left[(k_1 + k_3) - \omega^2 m_1 \right]\left[(k_2 + k_3) - \omega^2 m_2 \right] - {k_3}^2 = 0 \tag{28}$$

$$\omega^4 m_1 m_2 - \omega^2 \left[m_1 (k_2 + k_3) + m_2 (k_1 + k_3) \right] - {k_3}^2 = 0 \tag{29}$$

The eigenvalues are the roots of the polynomial.

$${\omega_1}^2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a} \tag{30}$$

$${\omega_2}^2 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \tag{31}$$

where

$$a = m_1 m_2 \tag{32}$$

$$b = -\left[m_1 (k_2 + k_3) + m_2 (k_1 + k_3) \right] \tag{33}$$

$$c = -{k_3}^2 \tag{34}$$

The eigenvectors are found via the following equations.

$$\left\{ K - \omega_1{}^2 M \right\} \bar{q}_1 = \bar{0} \tag{35}$$

$$\left\{ K - \omega_2{}^2 M \right\} \bar{q}_2 = \bar{0} \tag{36}$$

where

$$\bar{q}_1 = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \tag{37}$$

$$\bar{q}_2 = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \tag{38}$$

An eigenvector matrix Q can be formed. The eigenvectors are inserted in column format.

$$Q = [\ \bar{q}_1 \quad | \quad \bar{q}_2\ ] \tag{39}$$

$$Q = \begin{bmatrix} v_1 & w_1 \\ v_2 & w_2 \end{bmatrix} \tag{40}$$

The eigenvectors represent orthogonal mode shapes.

Each eigenvector can be multiplied by an arbitrary scale factor. A mass-normalized eigenvector matrix $\hat{Q}$ can be obtained such that the following orthogonality relations are obtained.

$$\hat{Q}^T M \hat{Q} = I \tag{41}$$

and

$$\hat{Q}^T K \hat{Q} = \Omega \tag{42}$$

were

superscript T represents transpose,
I is the identity matrix,
Ω is a diagonal matrix of eigenvalues.

Note that

$$\hat{Q} = \begin{bmatrix} \hat{v}_1 & \hat{w}_1 \\ \hat{v}_2 & \hat{w}_2 \end{bmatrix} \tag{43a}$$

$$\hat{Q}^T = \begin{bmatrix} \hat{v}_1 & \hat{v}_2 \\ \hat{w}_1 & \hat{w}_2 \end{bmatrix} \tag{43b}$$

Rigorous proof of the orthogonality relationships is beyond the scope of this tutorial. Further discussion is given in References 1 and 2.

Nevertheless, the orthogonality relationships are demonstrated by an example in this tutorial.

Now define a modal coordinate $\eta(t)$ such that

$$\bar{x} = \hat{Q}\ \bar{\eta} \tag{44}$$

Substitute equation (43) into the equation of motion, equation (11).

$$M\hat{Q}\ \bar{\ddot{\eta}} + K\hat{Q}\ \bar{\eta} = \bar{F} \tag{45}$$

Premultiply by the transpose of the normalized eigenvector matrix.

$$\hat{Q}^T\ M\hat{Q}\ \bar{\ddot{\eta}}\ +\ \hat{Q}^T K\hat{Q}\ \bar{\eta} =\ \hat{Q}^T\bar{F} \tag{46}$$

The orthogonality relationships yield

$$I\ \bar{\ddot{\eta}}\ +\ \Omega\ \bar{\eta} =\ \hat{Q}^T\bar{F} \tag{47}$$

For the sample problem, equation (46) becomes

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \ddot{\eta}_1 \\ \ddot{\eta}_2 \end{bmatrix} + \begin{bmatrix} \omega_1^2 & 0 \\ 0 & \omega_2^2 \end{bmatrix} \begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix} = \begin{bmatrix} \hat{v}_1 & \hat{v}_2 \\ \hat{w}_1 & \hat{w}_2 \end{bmatrix} \begin{bmatrix} f_1(t) \\ f_2(t) \end{bmatrix} \tag{48}$$

Note that the two equations are decoupled in terms of the modal coordinate.

Equation (48) yields two equations

$$\ddot{\eta}_1 + \omega_1^{\,2}\,\eta_1 = \hat{v}_1 f_1(t) + \hat{v}_2 f_2(t) \tag{49}$$

$$\ddot{\eta}_2 + \omega_2^{\,2}\,\eta_2 = \hat{w}_1 f_1(t) + \hat{w}_2 f_2(t) \tag{50}$$

The equations can be solved in terms of Laplace transforms, or some other differential equation solution method.

Now consider the initial conditions. Recall

$$\bar{x} = \hat{Q}\;\bar{\eta} \tag{51}$$

Thus,

$$\bar{x}(0) = \hat{Q}\;\bar{\eta}(0) \tag{52}$$

Premultiply by $\hat{Q}^T\;M$.

$$\hat{Q}^T\;M\,\bar{x}(0) = \hat{Q}^T\;M\,\hat{Q}\,\eta(0) \tag{53}$$

Recall

$$\hat{Q}^T M\,\hat{Q} = I \tag{54}$$

$$\hat{Q}^T\;M\,\bar{x}(0) = I\;\;\eta(0) \tag{55}$$

$$\hat{Q}^T\;M\,\bar{x}(0) = \;\eta(0) \tag{56}$$

Finally, the transformed initial displacement is

$$\eta(0) = \hat{Q}^T\;M\,\bar{x}(0) \tag{57}$$

Similarly, the transformed initial velocity is

$$\dot{\eta}(0) = \hat{Q}^T\;M\,\bar{\dot{x}}(0) \tag{58}$$

A basis for a solution is thus derived.

Harmonic Force

Now consider the special case of harmonic forcing functions.

$$f_1(t) = B_1 \sin(\alpha t) \tag{59}$$

$$f_2(t) = B_2 \sin(\beta t) \tag{60}$$

Thus,

$$\ddot{\eta}_1 + \omega_1^2 \eta_1 = \hat{v}_1\{B_1 \sin(\alpha t)\} + \hat{v}_1\{B_2 \sin(\beta t)\} \tag{61}$$

$$\ddot{\eta}_2 + \omega_2^2 \eta_2 = \hat{w}_1\{B_1 \sin(\alpha t)\} + \hat{w}_2\{B_2 \sin(\beta t)\} \tag{62}$$

Take the Laplace transform of equation (61).

$$L\{\ddot{\eta}_1 + \omega_1^2 \eta_1\} = L\{\ \hat{v}_1 B_1 \sin(\alpha t) + \hat{v}_2 B_2 \sin(\beta t)\ \} \tag{63}$$

$$s^2 I_1(s) - s\eta_1(0) - \dot{\eta}_1(0) + \omega_1^2 I_1(s) = + \hat{v}_1 B_1 \left\{ \frac{\alpha}{s^2 + \alpha^2} \right\} + \hat{v}_2 B_2 \left\{ \frac{\beta}{s^2 + \beta^2} \right\} \tag{64}$$

$$\left\{ s^2 + \omega_1^2 \right\} I_1(s) - s\eta_1(0) - \dot{\eta}_1(0) = + \hat{v}_1 B_1 \left\{ \frac{\alpha}{s^2 + \alpha^2} \right\} + \hat{v}_2 B_2 \left\{ \frac{\beta}{s^2 + \beta^2} \right\} \tag{65}$$

$$\left\{ s^2 + \omega_1^2 \right\} I_1(s) = + s\eta_1(0) + \dot{\eta}_1(0) + \hat{v}_1 B_1 \left\{ \frac{\alpha}{s^2 + \alpha^2} \right\} + \hat{v}_2 B_2 \left\{ \frac{\beta}{s^2 + \beta^2} \right\} \tag{66}$$

$$I_1(s) = \frac{+s\eta_1(0) + \dot{\eta}_1(0)}{\left\{s^2 + \omega_1^2\right\}} + \hat{v}_1 B_1 \left\{ \frac{\alpha}{\left[s^2 + \omega_1^2\right]\left[s^2 + \alpha^2\right]} \right\} + \hat{v}_2 B_2 \left\{ \frac{\beta}{\left[s^2 + \omega_1^2\right]\left[s^2 + \beta^2\right]} \right\}$$

(67)

Partial fraction expansion can be used to simplify the terms on the right-hand side of equation (69) as shown in Appendix A.

The inverse Laplace transform is performed using Reference 3. The resulting modal displacement is

$$\eta_1(t) = \left\{ \frac{\dot{\eta}_1(0)}{\omega_1} \right\} \sin(\omega_1 t) + \eta_1(0) \cos(\omega_1 t)$$

$$+ \left\{ \frac{\hat{v}_1 B_1 \alpha}{\omega_1^2 - \alpha^2} \right\} \left\{ \frac{1}{\alpha} \sin(\alpha t) - \frac{1}{\omega_1} \sin(\omega_1 t) \right\} + \left\{ \frac{\hat{v}_2 B_2 \beta}{\omega_1^2 - \beta^2} \right\} \left\{ \frac{1}{\beta} \sin(\beta t) - \frac{1}{\omega_1} \sin(\omega_1 t) \right\}$$

(68)

Similarly,

$$\eta_2(t) = \left\{ \frac{\dot{\eta}_2(0)}{\omega_2} \right\} \sin(\omega_2 t) + \eta_2(0) \cos(\omega_2 t)$$

$$+ \left\{ \frac{\hat{v}_1 B_1 \alpha}{\omega_2^2 - \alpha^2} \right\} \left\{ \frac{1}{\alpha} \sin(\alpha t) - \frac{1}{\omega_2} \sin(\omega_2 t) \right\} + \left\{ \frac{\hat{v}_2 B_2 \beta}{\omega_2^2 - \beta^2} \right\} \left\{ \frac{1}{\beta} \sin(\beta t) - \frac{1}{\omega_2} \sin(\omega_2 t) \right\}$$

(69)

The physical displacements are then found as

$$\bar{x} = \hat{Q} \ \bar{\eta} \qquad (70a)$$

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} \hat{v}_1 & \hat{w}_1 \\ \hat{v}_2 & \hat{w}_2 \end{bmatrix} \begin{bmatrix} \eta_1(t) \\ \eta_2(t) \end{bmatrix} \qquad (70b)$$

Return to Example

Recall the system in Figure 1. Consider the values in Table 1.

Table 1.	
Variable	Value
m_1	1 kg
m_2	2 kg
k_1	2000 N/m
k_2	3000 N/m
k_3	2000 N/m
B_1	10 N
B_2	20 N
α	50 rad/sec
β	100 rad/sec
$x_1(0)$	0.001 m
$\dot{x}_1(0)$	0 m/sec
$x_2(0)$	0.002 m
$\dot{x}_2(0)$	0 m/sec

Solve for the displacement response time histories. The complete problem in matrix form is

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} k_1 + k_3 & -k_3 \\ -k_3 & k_2 + k_3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} f_1(t) \\ f_2(t) \end{bmatrix} \quad (71a)$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} 4000 & -2000 \\ -2000 & 5000 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 10 \sin(100\, t) \\ 20 \sin(200\, t) \end{bmatrix} \quad (71b)$$

The homogeneous problem is

$$\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} 4000 & -2000 \\ -2000 & 5000 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{72}$$

The eigenvalue problem is

$$\begin{bmatrix} 4000-\omega^2 & -2000 \\ -2000 & 5000-2\omega^2 \end{bmatrix}\begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{73}$$

Set the determinant equal to zero

$$\det \begin{bmatrix} 4000-\omega^2 & -2000 \\ -2000 & 5000-2\omega^2 \end{bmatrix} = 0 \tag{74}$$

$$\left(4000-\omega^2\right)\left(5000-2\omega^2\right) - 2000^2 = 0 \tag{75}$$

$$2\omega^4 - 13{,}000\,\omega^2 + 2.00(10^7) - 4.00(10^6) = 0 \tag{76}$$

$$2\omega^4 - 13{,}000\,\omega^2 + 1.60(10^7) = 0 \tag{77}$$

The roots of the polynomial are

$$\omega_1{}^2 = [1649.2 \text{ rad/sec}]^2 \tag{78a}$$

$$\omega_1 = 40.61 \text{ rad/sec} \tag{78b}$$

$$\omega_2{}^2 = [4850.8 \text{ rad/sec}]^2 \tag{79a}$$

$$\omega_2 = 69.65 \text{ rad/sec} \tag{79b}$$

Solve for the first eigenvector.

$$\begin{bmatrix} 4000-\omega_1^2 & -2000 \\ -2000 & 5000-2\omega_1^2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{80}$$

$$\begin{bmatrix} 4000-(1649.2) & -2000 \\ -2000 & 5000-2(1649.2) \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{81}$$

$$\begin{bmatrix} 2350.8 & -2000 \\ -2000 & 1701.6 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{82}$$

Scale each row.

$$\begin{bmatrix} 1 & -0.8508 \\ 1 & -0.8508 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{83}$$

Note that the equations are not linearly independent. Thus, the eigenvector may be multiplied by an arbitrary scale factor u.

The eigenvector corresponding to the first root is thus

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = u \begin{bmatrix} 0.8508 \\ 1 \end{bmatrix} \tag{84}$$

Solve for the second eigenvector.

$$\begin{bmatrix} 4000-\omega_2^2 & -2000 \\ -2000 & 5000-2\omega_2^2 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{85}$$

$$\begin{bmatrix} 4000-(4850.8) & -2000 \\ -2000 & 5000-2(4850.8) \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{86}$$

$$\begin{bmatrix} -850.8 & -2000 \\ -2000 & -4701.6 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{87}$$

Scale each row.

$$\begin{bmatrix} 1 & 2.3507 \\ 1 & 2.3507 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{88}$$

Note that the equations are not linearly independent. Thus, the eigenvector may be multiplied by an arbitrary scale factor w.

The eigenvector corresponding to the second root is thus

$$\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = w \begin{bmatrix} -2.3507 \\ 1 \end{bmatrix} \tag{89}$$

The eigenvector matrix is thus

$$Q = \begin{bmatrix} v_1 & w_1 \\ v_2 & w_2 \end{bmatrix} \tag{90}$$

$$Q = \begin{bmatrix} 0.8508\,u & -2.3507\,w \\ 1u & 1w \end{bmatrix} \tag{91}$$

The next goal is to obtain a normalized eigenvector matrix $\hat{Q}$ such that

$$\hat{Q}^T M \hat{Q} = I \tag{92}$$

$$\begin{bmatrix} 0.8508\,u & 1u \\ -2.3507\,z & 1z \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 0.8508\,u & -2.3507\,z \\ 1u & 1z \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{93}$$

$$\begin{bmatrix} 0.8508\,u & 1u \\ -2.3507\,z & 1z \end{bmatrix} \begin{bmatrix} 0.8508\,u & -2.3507\,z \\ 2u & 2z \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{94}$$

$$\begin{bmatrix} 2.7239\,u^2 & 0 \\ 0 & 7.5259\,z^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{95}$$

Finally, the normalization factors are

$$\begin{bmatrix} u \\ z \end{bmatrix} = \begin{bmatrix} 0.6059 \\ 0.3645 \end{bmatrix} \tag{96}$$

The normalized eigenvector matrix is thus

$$\hat{Q} = \begin{bmatrix} 0.5155 & -0.8569 \\ 0.6059 & 0.3645 \end{bmatrix} \tag{97}$$

Note that the eigenvector in the first column has a uniform polarity. Thus, the two masses vibrate in phase for the first mode.

The eigenvector in the second column has two components with opposite polarity. The two masses vibration 180 degrees out of phase for the second mode.

Verify

$$\hat{Q}^T M \hat{Q} = I \tag{98}$$

$$\begin{bmatrix} 0.5155 & 0.6059 \\ -0.8569 & 0.3645 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 0.5155 & -0.8569 \\ 0.6059 & 0.3645 \end{bmatrix} \overset{?}{=} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{99}$$

$$\begin{bmatrix} 0.5155 & 0.6059 \\ -0.8569 & 0.3645 \end{bmatrix} \begin{bmatrix} 0.5155 & 0.8569 \\ -1.2118 & 0.7290 \end{bmatrix} \overset{?}{=} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{100}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{101}$$

The normalized eigenvectors are thus verified with respect to the mass matrix.

Now verify

$$\hat{Q}^T K \hat{Q} = \Omega \tag{102}$$

$$\begin{bmatrix} 0.5155 & 0.6059 \\ -0.8569 & 0.3645 \end{bmatrix} \begin{bmatrix} 4000 & -2000 \\ -2000 & 5000 \end{bmatrix} \begin{bmatrix} 0.5155 & -0.8569 \\ 0.6059 & 0.3645 \end{bmatrix} \overset{?}{=} \begin{bmatrix} 1649.2 & 0 \\ 0 & 4850.8 \end{bmatrix} \tag{103}$$

$$\begin{bmatrix} 0.5155 & 0.6059 \\ -0.8569 & 0.3645 \end{bmatrix} \begin{bmatrix} 850.2 & -4156. \\ 1998.5 & 3536.3 \end{bmatrix} \overset{?}{=} \begin{bmatrix} 1649.2 & 0 \\ 0 & 4850.8 \end{bmatrix} \tag{104}$$

$$\begin{bmatrix} 1649.2 & -0.1 \\ -0.1 & 4850.8 \end{bmatrix} \approx \begin{bmatrix} 1649.2 & 0 \\ 0 & 4850.8 \end{bmatrix} \tag{105}$$

The normalized eigenvectors are thus verified with respect to the stiffness matrix, within reasonable numerical precision.

Now transform the initial conditions.

The transformed initial displacement is

$$\eta(0) = \hat{Q}^T \ M\,\bar{x}(0) \tag{106}$$

$$\begin{bmatrix} \eta_1(0) \\ \eta_2(0) \end{bmatrix} = \begin{bmatrix} 0.5155 & 0.6059 \\ -0.8569 & 0.3645 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 0.001 \\ 0.002 \end{bmatrix} \tag{107}$$

$$\begin{bmatrix} \eta_1(0) \\ \eta_2(0) \end{bmatrix} = \begin{bmatrix} 0.5155 & 0.6059 \\ -0.8569 & 0.3645 \end{bmatrix} \begin{bmatrix} 0.001 \\ 0.004 \end{bmatrix} \tag{108}$$

$$\begin{bmatrix} \eta_1(0) \\ \eta_2(0) \end{bmatrix} = \begin{bmatrix} 0.0029 \\ 0.0006 \end{bmatrix} \tag{109}$$

Similarly, the transformed initial velocity is

$$\dot{\eta}(0) = \hat{Q}^T \ M\,\bar{\dot{x}}(0) \tag{110}$$

The initial velocity for both physical coordinates is zero. Thus

$$\begin{bmatrix} \dot{\eta}_1(0) \\ \dot{\eta}_2(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{111}$$

Recall

$$\eta_1(t) = \left\{ \frac{\dot{\eta}_1(0)}{\omega_1} \right\} \sin(\omega_1 t) + \eta_1(0) \cos(\omega_1 t)$$

$$+ \left\{ \frac{\hat{v}_1 B_1 \alpha}{\omega_1^2 - \alpha^2} \right\} \left\{ \frac{1}{\alpha} \sin(\alpha t) - \frac{1}{\omega_1} \sin(\omega_1 t) \right\} + \left\{ \frac{\hat{v}_2 B_2 \beta}{\omega_1^2 - \beta^2} \right\} \left\{ \frac{1}{\beta} \sin(\beta t) - \frac{1}{\omega_1} \sin(\omega_1 t) \right\}$$

$$(112)$$

and

$$\eta_2(t) = \left\{\frac{\dot{\eta}_2(0)}{\omega_2}\right\} \sin(\omega_2 t) + \eta_2(0) \cos(\omega_2 t)$$

$$+\left\{\frac{\hat{w}_1 B_1 \alpha}{\omega_2{}^2 - \alpha^2}\right\}\left\{\frac{1}{\alpha}\sin(\alpha t) - \frac{1}{\omega_2}\sin(\omega_2 t)\right\} + \left\{\frac{\hat{w}_2 B_2 \beta}{\omega_2{}^2 - \beta^2}\right\}\left\{\frac{1}{\beta}\sin(\beta t) - \frac{1}{\omega_2}\sin(\omega_2 t)\right\}$$

(113)

Substitute the variables.

$$\eta_1(t) = +[0.0029 \text{ m}] \cos\left(\left[40.61 \frac{\text{rad}}{\text{sec}}\right] t\right)$$

$$+\left\{\frac{[0.5155][10 \text{ N}]\left[50 \frac{\text{rad}}{\text{sec}}\right]}{\left[40.61 \frac{\text{rad}}{\text{sec}}\right]^2 - \left[50 \frac{\text{rad}}{\text{sec}}\right]^2}\right\}\left\{\frac{1}{\left[50 \frac{\text{rad}}{\text{sec}}\right]}\sin\left(\left[50 \frac{\text{rad}}{\text{sec}}\right]t\right) - \frac{1}{\left[40.61 \frac{\text{rad}}{\text{sec}}\right]}\sin\left(\left[40.61 \frac{\text{rad}}{\text{sec}}\right]t\right)\right\}$$

$$+\left\{\frac{[0.6059][20 \text{ N}]\left[100 \frac{\text{rad}}{\text{sec}}\right]}{\left[40.61 \frac{\text{rad}}{\text{sec}}\right]^2 - \left[100 \frac{\text{rad}}{\text{sec}}\right]^2}\right\}\left\{\frac{1}{\left[100 \frac{\text{rad}}{\text{sec}}\right]}\sin\left(\left[100 \frac{\text{rad}}{\text{sec}}\right]t\right) - \frac{1}{\left[40.61 \frac{\text{rad}}{\text{sec}}\right]}\sin\left(\left[40.61 \frac{\text{rad}}{\text{sec}}\right]t\right)\right\}$$

(114)

and

$$\eta_2(t) = [0.0006\ \mathrm{m}]\cos(\omega_2 t)$$

$$+\left\{\frac{[0.3645][10\ \mathrm{N}]\left[50\ \frac{\mathrm{rad}}{\mathrm{sec}}\right]}{\left[69.65\frac{\mathrm{rad}}{\mathrm{sec}}\right]^2-\left[50\ \frac{\mathrm{rad}}{\mathrm{sec}}\right]^2}\right\}\left\{\frac{1}{\left[50\ \frac{\mathrm{rad}}{\mathrm{sec}}\right]}\sin\left(\left[50\ \frac{\mathrm{rad}}{\mathrm{sec}}\right]t\right)-\frac{1}{\left[69.65\frac{\mathrm{rad}}{\mathrm{sec}}\right]}\sin\left(\left[69.65\frac{\mathrm{rad}}{\mathrm{sec}}\right]t\right)\right\}$$

$$+\left\{\frac{[-0.8569][20\ \mathrm{N}]\left[100\ \frac{\mathrm{rad}}{\mathrm{sec}}\right]}{\left[69.65\frac{\mathrm{rad}}{\mathrm{sec}}\right]^2-\left[100\ \frac{\mathrm{rad}}{\mathrm{sec}}\right]^2}\right\}\left\{\frac{1}{\left[100\ \frac{\mathrm{rad}}{\mathrm{sec}}\right]}\sin\left(\left[100\ \frac{\mathrm{rad}}{\mathrm{sec}}\right]t\right)-\frac{1}{\left[100\ \frac{\mathrm{rad}}{\mathrm{sec}}\right]}\sin\left(\left[69.65\frac{\mathrm{rad}}{\mathrm{sec}}\right]t\right)\right\}$$

$$(115)$$

The final solution is thus

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} \hat{v}_1 & \hat{w}_1 \\ \hat{v}_2 & \hat{w}_2 \end{bmatrix}\begin{bmatrix} \eta_1(t) \\ \eta_2(t) \end{bmatrix} \tag{116}$$

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} 0.5155 & -0.8569 \\ 0.6059 & 0.3645 \end{bmatrix}\begin{bmatrix} \eta_1(t) \\ \eta_2(t) \end{bmatrix} \tag{117}$$

<u>References</u>

1. Bathe, Finite Element Procedures in Engineering Analysis, Prentice-Hall, New Jersey, 1982. Section 12.3.1.

2. Weaver and Johnston, Structural Dynamics by Finite Elements, Prentice-Hall, New Jersey, 1987. Chapter 4.

3. T. Irvine, Table of Laplace Transforms, Vibrationdata.com Publications, 2000.

APPENDIX A

<u>Partial Fraction Expansion</u>

$$\left\{\frac{1}{s^2+\omega^2}\right\}\left\{\frac{1}{s^2+\alpha^2}\right\} = \left\{\frac{\lambda s+\rho}{s^2+\omega^2}\right\}+\left\{\frac{\sigma s+\phi}{s^2+\alpha^2}\right\} \tag{A-1}$$

Multiply through by the denominator.

$$1=\{\lambda s+\rho\}\left\{s^2+\alpha^2\right\}+\{\sigma s+\phi\}\left\{s^2+\omega^2\right\} \tag{A-2}$$

$$\begin{aligned}1 = \quad & \lambda s^3+\rho s^2+\lambda\alpha^2 s+\rho\alpha^2 \\ & +\sigma s^3+\phi s^2+\sigma\omega^2 s+\phi\omega^2\end{aligned} \tag{A-3}$$

$$\begin{aligned}1 = \quad & [\lambda+\sigma]s^3 \\ & +[\rho+\phi]s^2 \\ & +\left[\lambda\alpha^2+\sigma\omega^2\right]s \\ & +\left[\rho\alpha^2+\phi\omega^2\right]\end{aligned}$$

(A-4)

$$\lambda + \sigma = 0 \tag{A-5}$$

$$\rho + \phi = 0 \tag{A-6}$$

$$\lambda\alpha^2 + \sigma\omega^2 = 0 \tag{A-7}$$

$$\rho\alpha^2 + \phi\omega^2 = 1 \tag{A-8}$$

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ \alpha^2 & \omega^2 & 0 & 0 \\ 0 & 0 & \alpha^2 & \omega^2 \end{bmatrix} \begin{bmatrix} \lambda \\ \sigma \\ \rho \\ \phi \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \tag{A-9}$$

Multiply the first row by $-\alpha^2$ and add to the third row.

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & \omega^2 - \alpha^2 & 0 & 0 \\ 0 & 0 & \alpha^2 & \omega^2 \end{bmatrix} \begin{bmatrix} \lambda \\ \sigma \\ \rho \\ \phi \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \tag{A-10}$$

Multiply the second row by $-\alpha^2$ and add to the fourth row.

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & \omega^2 - \alpha^2 & 0 & 0 \\ 0 & 0 & 0 & \omega^2 - \alpha^2 \end{bmatrix} \begin{bmatrix} \lambda \\ \sigma \\ \rho \\ \phi \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \tag{A-11}$$

Divide the third and fourth rows by $\omega^2 - \alpha^2$.

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \lambda \\ \sigma \\ \rho \\ \phi \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1/\left(\omega^2 - \alpha^2\right) \end{bmatrix} \quad \text{(A-12)}$$

Multiply the third row by -1 and add to the first row.

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \lambda \\ \sigma \\ \rho \\ \phi \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1/\left(\omega^2 - \alpha^2\right) \end{bmatrix} \quad \text{(A-13)}$$

Multiply the fourth row by -1 and add to the second row.

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \lambda \\ \sigma \\ \rho \\ \phi \end{bmatrix} = \begin{bmatrix} 0 \\ -1/\left(\omega^2 - \alpha^2\right) \\ 0 \\ 1/\left(\omega^2 - \alpha^2\right) \end{bmatrix} \quad \text{(A-14)}$$

Interchange the second and third rows.

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \lambda \\ \sigma \\ \rho \\ \phi \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -1/\left(\omega^2 - \alpha^2\right) \\ 1/\left(\omega^2 - \alpha^2\right) \end{bmatrix} \quad \text{(A-15)}$$

Thus,

$$\begin{bmatrix} \lambda \\ \sigma \\ \rho \\ \phi \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -1/\left(\omega^2-\alpha^2\right) \\ 1/\left(\omega^2-\alpha^2\right) \end{bmatrix} \tag{A-16}$$

$$\left\{\frac{1}{s^2+\omega^2}\right\}\left\{\frac{1}{s^2+\alpha^2}\right\} = \left\{\frac{1}{\omega^2-\alpha^2}\right\}\left\{\frac{-1}{s^2+\omega^2}\right\} + \left\{\frac{1}{\omega^2-\alpha^2}\right\}\left\{\frac{1}{s^2+\alpha^2}\right\} \tag{A-17}$$

The inverse Laplace Transform f(t) is

$$f(t) = \left\{\frac{-1}{\omega^2-\alpha^2}\right\}\left\{\frac{1}{\omega}\right\}\sin(\omega t) + \left\{\frac{1}{\omega^2-\alpha^2}\right\}\left\{\frac{1}{\alpha}\right\}\sin(\alpha t) \tag{A-18}$$

$$f(t) = \left\{\frac{1}{\omega^2-\alpha^2}\right\}\left\{\frac{1}{\alpha}\sin(\alpha t) - \frac{1}{\omega}\sin(\omega t)\right\} \tag{A-19}$$