Second Order Circuits

EEENG 223

Circuit

Theory I

Second Order Circuits

Order of a circuit	=	Order of the differential equation (DE) required to describe the circuit	=	The number of independent* energy storage elements (C's and L's)

* C's and L's are independent if they cannot be combined with other C's and L's (in series or parallel, for example)

Second Order Circuits

- 2^{nd}-order circuits have 2 independent energy storage elements (inductors and/or capacitors)
- Analysis of a 2^{nd}-order circuit yields a 2^{nd}-order differential equation (DE)
- A 2nd-order differential equation has the form:

$$\frac{d^2x}{dt^2} + a_1 \frac{dx}{dt} + a_o x(t) = f(t)$$

- Solution of a 2^{nd}-order differential equation requires two initial conditions: x(0) and x'(0)
- All higher order circuits (3^{rd}, 4^{th}, etc) have the same types of responses as seen in 1^{st}-order and 2^{nd}-order circuits
- Since 2nd-order circuits have two energy-storage types, the circuits can have the following forms:
    1) Two capacitors
    2) Two inductors
    3) One capacitor and one inductor
        A) Series RLC circuit
        B) Parallel RLC circuit
        C) Others

Form of the solution to differential equations

As seen with 1st-order circuits in Chapter 7, the general solution to a differential equation has two parts:

$$x(t) = x_h + x_p = \text{homogeneous solution} + \text{particular solution}$$

or $$x(t) = x_n + x_f = \text{natural solution} + \text{forced solution}$$

where x_h or x_n is due to the initial conditions in the circuit

and x_p or x_f is due to the forcing functions (independent voltage and current sources for $t > 0$).

The forced response

The forced response is due to the independent sources in the circuit for $t > 0$. Since the natural response will die out once the circuit reaches steady-state (under dc conditions), the forced response can be found by analyzing the circuit at $t = \infty$. In particular,

$$\boxed{x_f = x(\infty)}$$

The natural response

A 2nd-order differential equation has the form:

$$\frac{d^2x}{dt^2} + a_1\frac{dx}{dt} + a_o x(t) = f(t)$$

where x(t) is a voltage v(t) or a current i(t).

To find the natural response, set the forcing function f(t) (the right-hand side of the DE) to zero.

$$\frac{d^2x}{dt^2} + a_1\frac{dx}{dt} + a_o x(t) = 0$$

Substituting the general form of the solution Ae^{st} yields the ***characteristic equation***:

$$s^2 + a_1 s + a_o = 0$$

Finding the roots of this quadratic (called the ***characteristic roots*** or ***natural frequencies***) yields:

$$s_1\ ,\ s_2 = \frac{-a_1 \pm \sqrt{(a_1)^2 - 4a_o}}{2}$$

The roots of the quadratic equation above may be real and distinct, repeated, or complex. Thus, the natural response to a 2nd-order circuit has 3 possible forms:

1) **Overdamped response**

Roots are real and distinct [$(a_1)^2 > 4a_o$]

Solution has the form:

$$x_n = A_1 e^{s_1 t} + A_2 e^{s_2 t}$$

2) <u>Critically damped response</u>

Roots are repeated [$(a_1)^2 = 4a_o$] so $s_1 = s_2 = s = -a_1/2$

Solution has the form:

$$x_n = \left(A_1 t + A_2\right)e^{st}$$

3) **Underdamped response**

Roots are complex [$(a_1)^2 < 4a_o$] so $s_1, s_2 = \alpha \pm j\beta$

Show that the solution has the form:

$$x_n = e^{\alpha t}\left[A_1\cos(\beta t) + A_2\sin(\beta t)\right]$$

Series and Parallel RLC Circuits

Two common second-order circuits are now considered:

- ***series RLC circuits***
- ***parallel RLC circuits***.

Relationships for these circuits can be easily developed such that the characteristic equation can be determined directly from component values without writing a differential equation for each example.

A general 2nd-order characteristic equation has the form:

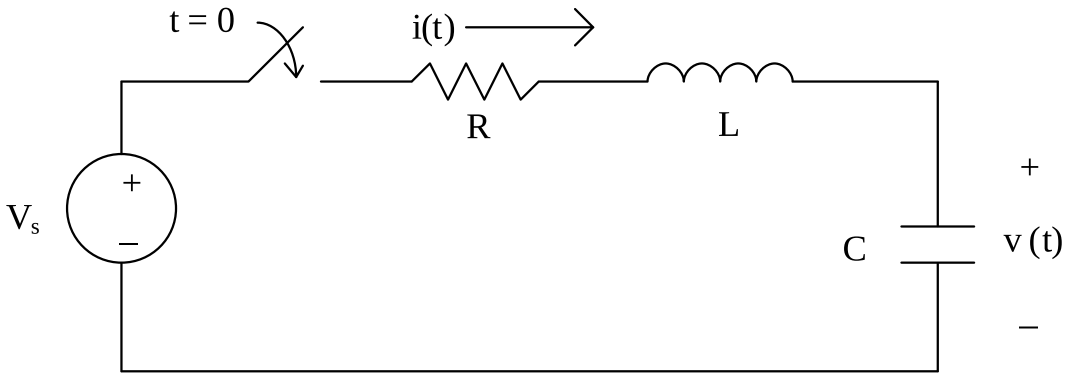
A general 2nd-order characteristic equation has the form: $s^2 + 2\alpha s + w_o^2 = 0$

where

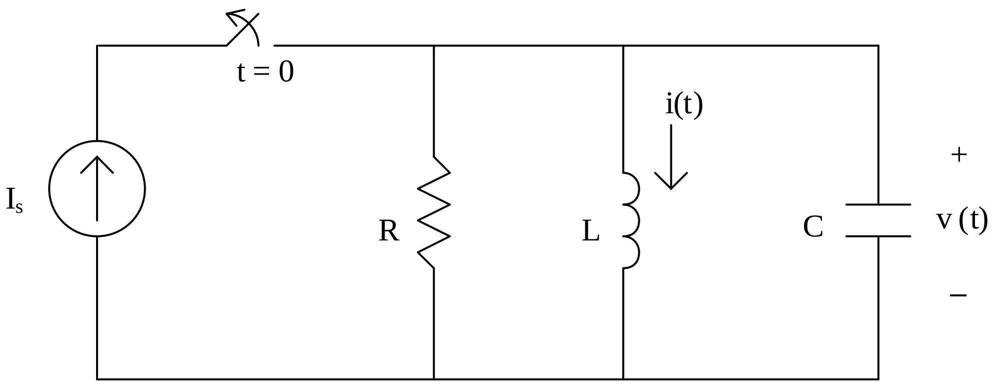
α = damping coefficient

w_o = resonant frequency

Series RLC Circuit - develop expressions for α and w_o



Parallel RLC Circuit - develop expressions for α and w_o



Procedure for analyzing 2nd-order circuits

1. ***Find the characteristic equation and the natural response***

   A) Determine if the circuit is a series RLC or parallel RLC (for t > 0 with independent sources killed). If the circuit is not series RLC or parallel RLC determine the describing equation of capacitor voltage or inductor current.

   B) Obtain the characteristic equation. Use the standard formulas for α and w_o for a series RLC circuit or a parallel RLC circuit. Use these values of α and w_o in the characteristic equation as: $s^2 + 2\alpha s + w_o^2$.

   C) Find the roots of the characteristic equation (characteristic roots).

   D) Determine the form of the natural response based on the type of characteristic roots:

      A) Real distinct roots: $x_n = A_1 e^{s_1 t} + A_2 e^{s_2 t}$

      B) Complex roots: $x_n = e^{\alpha t}\left[A_1 \cos(\beta\, t) + A_2 \sin(\beta\, t)\right]$

      C) Repeated roots: $x_n = \left(A_1 t + A_2\right)e^{st}$

2. ***Find the forced response -*** Analyze the circuit at $t = \infty$ to find $x_f = x(\infty)$.

3. ***Find the initial conditions***, x(0) and x'(0).

   A) Find x(0) by analyzing the circuit at $t = 0^-$ (find all capacitor voltages and inductor currents)

   B) Analyze the circuit at $t = 0^+$ (use the values for v_C and i_L found at $t = 0^-$ in the circuit) and find $dv_C(0^+)/dt = i_C(0^+)/C$ or $di_L(0^+)/dt = v_L(0^+)/L$.

3. ***Find the complete response***

   A) Find the total response, $x(t) = x_n + x_f$.

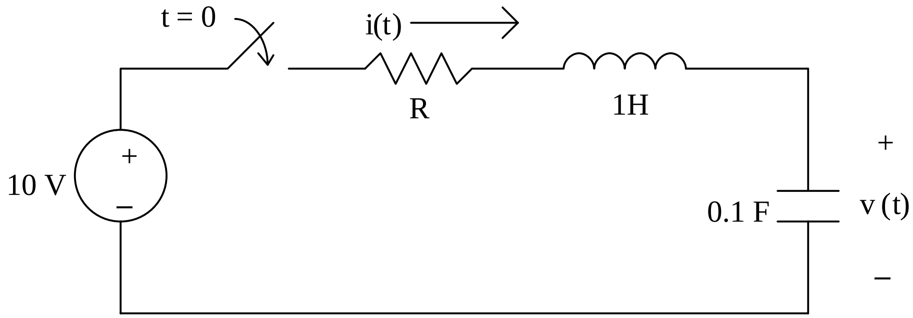
   B) Use the two initial conditions to solve for the two unknowns in the total response.

Example: Determine v(t) in the circuit shown below for t > 0 if :

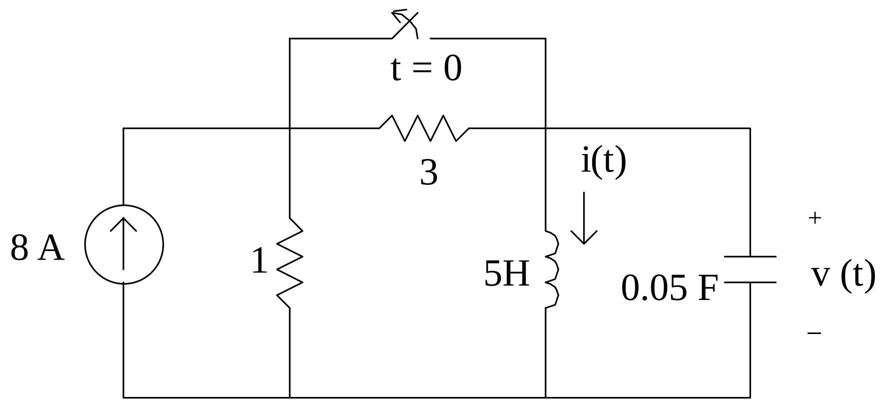
A) R = 7

B) R = 2

C) R = 2 √10



Example: Determine i(t) in the circuit shown below for t > 0.



Example: Determine v(t) in the circuit shown below for t > 0.