Chapter 4:
The Finite Volume Method for Diffusion Problems

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Introduction

General transport equation is

$$\frac{\partial (\rho \phi)}{\partial t} + \text{div}(\rho \phi u) = \text{div}(\Gamma \text{grad}\phi) + S_\phi$$

For steady diffusion:

$$\text{div}(\Gamma \text{grad}\phi) + S_\phi = 0$$

Control volume integration gives

$$\int_{CV} \text{div}(\Gamma \text{grad}\phi)dV + \int_{CV} S_\phi dV = \int_A n \cdot (\Gamma \text{grad}\phi)dA + \int_{CV} S_\phi dV$$
Finite Volume Method for One-dimensional Steady State Diffusion

Steady-state diffusion of a general property \( \phi \) in one-dimensional domain is

\[
\frac{d}{dx} \left( \Gamma \frac{d\phi}{dx} \right) + S_\phi = 0
\]

Step 1: Grid generation:

Step 2: Discretisation

Integration of the diffusion equation over the CV gives

\[
\int_{\Delta V} \frac{d}{dx} \left( \Gamma \frac{d\phi}{dx} \right) dV + \int_{\Delta V} S_\phi dV = \left( \Gamma A \frac{d\phi}{dx} \right)_e - \left( \Gamma A \frac{d\phi}{dx} \right)_w + \bar{S} \Delta V = 0
\]

To find expressions at the east and west faces, use Taylor series approximations

\[
\phi(x + \Delta x) = \phi(x) + \left( \frac{\partial \phi}{\partial x} \right)_x \Delta x + \left( \frac{\partial^2 \phi}{\partial x^2} \right)_x \frac{\Delta x^2}{2} + \ldots
\]
\[ \phi(x + \Delta x) = \phi(x) + \left( \frac{\partial \phi}{\partial x} \right)_x \Delta x + \frac{\left( \frac{\partial^2 \phi}{\partial x^2} \right)_x}{2} \Delta x^2 + \cdots \]

\[ \phi_e = \phi_p + \left( \frac{\partial \phi}{\partial x} \right)_p \Delta x + \frac{\left( \frac{\partial^2 \phi}{\partial x^2} \right)_p}{2} \Delta x^2 + \cdots \]

\[ \phi_w = \phi_p - \left( \frac{\partial \phi}{\partial x} \right)_p \Delta x + \frac{\left( \frac{\partial^2 \phi}{\partial x^2} \right)_p}{2} \Delta x^2 + \cdots \]

Neglect

Adding and subtracting \[ \left( \frac{\partial \phi}{\partial x} \right)_p \]

At the east face \[ \left( \frac{\partial \phi}{\partial x} \right)_e \]

Rewriting the diffusion equation for an interior point \( P \):

\[ \int \frac{d}{dx} \left( \Gamma \frac{d \phi}{dx} \right) dV + \int S_d dV = 0 \]

\[ \left( \Gamma_A \frac{d \phi}{dx} \right)_e - \left( \Gamma_A \frac{d \phi}{dx} \right)_w + \vec{S}_d \Delta V = 0 \] (4.4)

On a uniform grid linear interpolation of \( \Gamma \) is

\[ \Gamma_w = \frac{\Gamma_w + \Gamma_p}{2} \quad \Gamma_e = \frac{\Gamma_p + \Gamma_E}{2} \] (4.5)

Diffusive flux terms are

\[ \left( \Gamma_A \frac{d \phi}{dx} \right)_e = \Gamma_e A_e \left( \frac{\phi_E - \phi_p}{\delta x_{PE}} \right) \] (4.6)

\[ \left( \Gamma_A \frac{d \phi}{dx} \right)_w = \Gamma_w A_w \left( \frac{\phi_p - \phi_W}{\delta x_{WP}} \right) \] (4.7)
The source term $S$ may be a function of $\phi \Rightarrow$ express $S$ in linear form as:

$$S_\phi = S_u + S_p \phi_p$$  \hfill (4.8)

Substituting (4.6), (4.7) and (4.8) into (4.4)

$$-\Gamma_e A_e \left( \frac{\phi_e - \phi_p}{\delta x_{pe}} \right) + \Gamma_w A_w \left( \frac{\phi_p - \phi_w}{\delta x_{wp}} \right) = (S_u + S_p \phi_p) \Delta V$$  \hfill (4.9)

Rearranging,

$$\left[ \frac{\Gamma_e}{\delta x_{pe}} A_e + \frac{\Gamma_w}{\delta x_{wp}} A_w - S_p \Delta V \right] \phi_p - \left( \frac{\Gamma_w}{\delta x_{wp}} A_w \right) \phi_w - \left( \frac{\Gamma_e}{\delta x_{pe}} A_e \right) \phi_E = S_u \Delta V$$  \hfill (4.10)

or,

$$a_p \phi_p + a_w \phi_w + a_e \phi_E = S$$  \hfill (4.11)

where,

<table>
<thead>
<tr>
<th></th>
<th>$a_w$</th>
<th>$a_e$</th>
<th>$a_p$</th>
<th>$S$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Gamma_w A_w / \delta x_{wp}$</td>
<td>$\Gamma_e A_e / \delta x_{pe}$</td>
<td>$-a_w - a_e - S_p \Delta V$</td>
<td>$S_u \Delta V$</td>
<td></td>
</tr>
</tbody>
</table>

Special case: no source terms ($S_\phi = 0$), boundary values $\phi_A, \phi_B$ specified.

For the point near a west boundary (point 2):

$$\Gamma_e A_e \left( \frac{\phi_p - \phi_w}{\delta x_{pe}} \right) - \Gamma_w A_w \left( \frac{\phi_p - \phi_A}{\delta x_{wp}} \right) = 0$$

$$\left[ \frac{\Gamma_e}{\delta x_{pe}} A_e + \frac{\Gamma_w}{\delta x_{wp}} A_w \right] \phi_p - 0 \phi_w - \left( \frac{\Gamma_e}{\delta x_{pe}} A_e \right) \phi_E = \left( \frac{\Gamma_w}{\delta x_{wp}} A_w \right) \phi_A$$

or,

$$a_p \phi_p + a_w \phi_w + a_e \phi_E = S$$

<table>
<thead>
<tr>
<th></th>
<th>$a_w$</th>
<th>$a_e$</th>
<th>$a_p$</th>
<th>$S$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$0$</td>
<td>$-\Gamma_e A_e / \delta x_{pe}$</td>
<td>$-(a_w + a_e)_{int}$</td>
<td>$(a_w)_{int} \phi_A$</td>
<td></td>
</tr>
</tbody>
</table>
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For the point near a west boundary (point 2):

\[
\Gamma_e A_e \left( \frac{\phi_e - \phi_p}{\delta x_{PE}} \right) + q_w A_w = 0
\]

or,

\[
a_p \phi_p = a_w \phi_w + a_e \phi_e + S_u
\]

Summary of Boundary Conditions

For a one-dimensional CV of width \( \Delta x \) near boundary B:

1) Set coefficient \( a_B(i) = 0 \) \( (i \rightarrow P) \)

2) Add source contributions

   (a) Fixed value \( \phi_B \):

   Add:

   \[
   S_u = \frac{k_B A_B}{\Delta x / 2} \phi_B
   \]

   \[
   S_p = -\frac{k_B A_B}{\Delta x / 2}
   \]

   to the source terms \( S_u \) and \( S_p \)

   (b) Fixed flux \( q_B \):

   Add \( q_B A_B \) in the form of \( S_u + S_p \phi_p \) to the source terms \( S_u \) and \( S_p \).
Step 3: Solution of equations

Discretised equations of the form (4.11)

\[ a_P \phi_P = a_W \phi_W + a_E \phi_E + a_S \phi_S + a_N \phi_N + S_u \]  \hspace{1cm} (4-11)

must be set up at each nodal point.

\( \rightarrow \) We obtain a system of linear algebraic equations

Solve the system for \( \phi \) values

\( \rightarrow \) Use any matrix solution method.

\( e.g. \) Tri-diagonal matrix algorithm (see textbook)
or: Gauss Seidel iteration method.

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**Gauss Seidel Iteration Method**

For a 2-D problem consider the equation for point P in the form

\[ a_P \phi_P = a_W \phi_W + a_E \phi_E + a_S \phi_S + a_N \phi_N + S_u \]

\( \phi_P \) at iteration \( n+1 \) is calculated from

\[ \phi_P^{n+1} = \frac{1}{a_P} \left( a_W \phi_W^n + a_E \phi_E^n + a_S \phi_S^n + a_N \phi_N^n + S_u \right) \]

where superscript \( n \) refers to the latest available values; i.e., some are at the previous iteration \( n \) and some at \( n+1 \).

Iterations are repeated until the changes in all \( \phi_P \) values fall below a prescribed tolerance.

For one-dimensional problems the iteration equation is

\[ \phi_P^{n+1} = \frac{1}{a_P} \left( a_W \phi_W^n + a_E \phi_E^n + S_u \right) \]
Consider the problem of source-free heat conduction in an insulated rod whose ends are maintained at constant temperatures of 100 °C and 500 °C respectively. Calculate the steady state temperature in the rod. Take thermal conductivity $k = 1000 \text{ W/mK}$, cross-sectional area $A = 10 \times 10^{-3}$.

In this case, $S = 0$

Solution: Let us divide the rod into 5 equal control volumes (CV’s).

Rules for grid generation:
1) Locations of the CV faces are defined first.
2) Then nodal points are placed at the centers of the CV’s.
3) Numbering starts from the boundary node at left.
4) All CV’s have a volume of $\delta x \cdot A$
5) Inter-nodal distances are equal to $\delta x$, ($\delta x_{WP} = \delta x_{PE} = \delta x$)
6) Near west boundary (node 2), $\delta x_{WP} = \delta x/2$
7) Near east boundary (node 6), $\delta x_{PE} = \delta x/2$
\[ \Gamma_x A_e \left( \frac{\phi_e - \phi_p}{\delta x_{PE}} \right) - \Gamma_w A_w \left( \frac{\phi_p - \phi_w}{\delta x_{WP}} \right) = 0 \]  
\[ \left( \frac{\Gamma_x}{\delta x_{PE}} A_e + \frac{\Gamma_w}{\delta x_{WP}} A_w \right) T_p = \left( \frac{\Gamma_w}{\delta x_{WP}} A_w \right) T_w + \left( \frac{\Gamma_x}{\delta x_{PE}} A_e \right) T_E \]

For interior nodes (nodes 3-5):

\[ a_p T_p = a_w T_w + a_e T_E \]  

where,

<table>
<thead>
<tr>
<th></th>
<th>( a_w )</th>
<th>( a_e )</th>
<th>( a_p )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \frac{\Gamma_x A_e}{\delta x_{WP}} )</td>
<td>( \frac{\Gamma_x A_e}{\delta x_{PE}} )</td>
<td>( a_w + a_e - S_p )</td>
<td></td>
</tr>
</tbody>
</table>

General equation:

\[ a_p \phi_p = a_w \phi_w + a_e \phi_E + S_u \]

Interior nodes (nodes 3-5):

\[ S_u = 0, \ S_p = 0, \ \Gamma = k \]

\[ a_p \]

<table>
<thead>
<tr>
<th></th>
<th>( a_w )</th>
<th>( a_e )</th>
<th>( a_p )</th>
<th>( S_p )</th>
<th>( S_u )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \frac{\Gamma_x A_e}{\delta x_{WP}} )</td>
<td>( \frac{\Gamma_x A_e}{\delta x_{PE}} )</td>
<td>( a_w + a_e - S_p )</td>
<td>( 0 )</td>
<td>( 0 )</td>
<td></td>
</tr>
</tbody>
</table>

For boundary node 2:

\[ 0 \]

<table>
<thead>
<tr>
<th></th>
<th>( a_w )</th>
<th>( a_e )</th>
<th>( a_p )</th>
<th>( S_p )</th>
<th>( S_u )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \frac{\Gamma_x A_e}{\delta x_{WP}} )</td>
<td>0</td>
<td>( a_w + a_e - S_p )</td>
<td>( -kA )</td>
<td>( kA )</td>
<td>( T_A )</td>
</tr>
</tbody>
</table>

For boundary node 6:

\[ 0 \]

<table>
<thead>
<tr>
<th></th>
<th>( a_w )</th>
<th>( a_e )</th>
<th>( a_p )</th>
<th>( S_p )</th>
<th>( S_u )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \frac{\Gamma_x A_e}{\delta x_{WP}} )</td>
<td>0</td>
<td>( a_w + a_e - S_p )</td>
<td>( -kA )</td>
<td>( kA )</td>
<td>( T_B )</td>
</tr>
</tbody>
</table>
The resulting system of equations are

\[
\begin{bmatrix}
-a_{p_1} & a_{E_2} & 0 & 0 & 0 & 0 & 0 \\
 a_{W_2} & -a_{p_2} & a_{E_3} & 0 & 0 & 0 & 0 \\
 0 & a_{W_3} & -a_{p_3} & a_{E_4} & 0 & 0 & 0 \\
 0 & 0 & a_{W_4} & -a_{p_4} & a_{E_5} & 0 & 0 \\
 0 & 0 & 0 & a_{W_5} & -a_{p_5} & a_{E_6} & 0 \\
 0 & 0 & 0 & 0 & a_{W_6} & -a_{p_6} & a_{E_7} \\
 0 & 0 & 0 & 0 & 0 & a_{W_7} & -a_{p_7}
\end{bmatrix}
\begin{bmatrix}
T_1 \\
T_2 \\
T_3 \\
T_4 \\
T_5 \\
T_6 \\
T_7
\end{bmatrix}
= \begin{bmatrix}
-Su_2 \\
-Su_3 \\
-Su_4 \\
-Su_5 \\
-Su_6 \\
-Su_7 \\
-Su_8
\end{bmatrix}
\]

Solve the system of equations using **Tri-diagonal matrix algorithm (TDMA)** for \(T_2, T_3, \ldots T_{n-1}\), where \((n = 7)\). Or use Gauss-Seidel method

The solution is:

\[
\begin{bmatrix}
T_1 \\
T_2 \\
T_3 \\
T_4 \\
T_5 \\
T_6 \\
T_7
\end{bmatrix}
= \begin{bmatrix}
100 \\
140 \\
220 \\
300 \\
380 \\
460 \\
500
\end{bmatrix}
\]

Exact solution is:

\[T = 800x + 100\]
Homework 1:

Write a computer program to find the temperature distribution in the problem given in Example 1. Use 5 control volumes.

Use the algorithm given in the “Pseudo program to find the $a_E$ coefficients and source terms in 1-D Diffusion Problems” given in page 34 of the lecture notes.

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Example 4.2 Now we discuss a problem that includes sources other than those arising from boundary conditions.

Figure 4.6 shows a large plate of thickness $L = 2$ cm with constant thermal conductivity $k = 0.5$ W/m/K and uniform heat generation $q = 1000$ kW/m$^3$. The faces A and B are at temperatures of 100 ºC and 200 ºC respectively. Assuming that the dimensions in the y- and z-directions are so large that temperature gradients are significant in the x-direction only, calculate the steady state temperature distribution. Compare the numerical result with the analytical solution. The governing equation is

$$
\frac{d}{dx} \left( k \frac{dT}{dx} \right) + q = 0
$$

(4.25)
The governing equation is:

\[ \frac{d}{dx} \left( k \frac{dT}{dx} \right) + \dot{q} = 0 \]

The general equation is:

\[ \frac{d}{dx} \left( \Gamma \frac{d\phi}{dx} \right) + S = 0 \]

Comparing the above equations, where \( S\Delta V = S_u + S_p\phi_p \)

\[ \phi = T, \ \Gamma = k, \ S_u = q\Delta V \quad S_p = 0 \quad \text{where,} \ \Delta V = A\Delta x \]

Take area \( A = 1 \) in the \( y-z \) plane

Solution is similar to the previous example.

\[ \begin{array}{ccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 \\
\phi_A & \circ & \circ & \circ & W & P & \circ & \circ & \phi_B \\
\end{array} \]

\[ \delta x_{pw} \]

\[ \delta x_p \]

General equation:

\[ a_p \phi_p = a_w \phi_W + a_E \phi_E + S_u \]

Interior nodes (nodes 3-5):

\[ S_u = qA\Delta x, \ S_p = 0, \ \Gamma = k \]

\[ \begin{array}{cccc}
a_w & a_E & a_p & S_p & S_u \\
\frac{\Gamma A_w}{\delta x_{pw}} & \frac{\Gamma A_E}{\delta x_{pw}} & a_w + a_E - S_p & 0 & qA\Delta x \\
\end{array} \]

For boundary node 2:

\[ \begin{array}{cccc}
a_w & a_E & a_p & S_p & S_u \\
0 & \frac{\Gamma A_w}{\delta x_{pw}} & a_w + a_E - S_p & \frac{k_A A_w}{\delta x_{pw}} & \dot{q}A\Delta x + \frac{k_A A_w}{\delta x_{pw}} T_A \\
\end{array} \]

For boundary node 6:

\[ \begin{array}{cccc}
a_w & a_E & a_p & S_p & S_u \\
\frac{\Gamma A_w}{\delta x_{pw}} & 0 & a_w + a_E - S_p & -\frac{k_A A_E}{\delta x_{pw}} & \dot{q}A\Delta x + \frac{k_A A_E}{\delta x_{pw}} T_B \\
\end{array} \]
The solution is:

\[
\begin{bmatrix}
T_1 \\
T_2 \\
T_3 \\
T_4 \\
T_5 \\
T_6 \\
T_7
\end{bmatrix} =
\begin{bmatrix}
100 \\
150 \\
218 \\
254 \\
258 \\
230 \\
200
\end{bmatrix}
\]

Comparison of the numerical result with the analytical solution.

Exact solution is:

\[
T = \left[ \frac{T_B - T_A}{L} + \frac{q}{2k} (L-x) \right] x + T_A
\]

Shown in Figure 4.9 is a cylindrical fin with uniform cross-sectional area \(A\). The base is at a temperature of 100 °C \(T_B\) and the end is insulated. The fin is exposed to an ambient temperature of 20 °C. One-dimensional heat transfer in this situation is governed by

\[
\frac{d}{dx} \left( kA \frac{dT}{dx} \right) - hP(T - T_\infty) = 0
\]

(4.40)

where \(h\) is the convective heat transfer coefficient, \(P\) the perimeter, \(k\) the thermal conductivity of the material and \(T_\infty\) the ambient temperature. Calculate the temperature distribution along the fin and compare the results with the analytical solution given by

\[
\frac{T - T_\infty}{T_B - T_\infty} = \frac{\cosh(n(L-x))}{\cosh(nL)}
\]

(4.41)

where \(n^2 = hP/(kA)\), \(L\) is the length of the fin and \(x\) the distance along the fin. Data:

\(L = 1\) m, \(hP/(kA) = 25\) m\(^{-2}\) (note \(kA\) is constant).
The governing equation is:
\[ \frac{d}{dx} \left( kA \frac{dT}{dx} \right) - hP(T - T_\infty) = 0 \] or
\[ \frac{d}{dx} \left( \frac{dT}{dx} \right) - n^2(T - T_\infty) = 0 \]

The general equation is:
\[ \frac{d}{dx} \left( \Gamma \frac{d\phi}{dx} \right) + S_\phi = 0 \]

where
\[ n^2 = \frac{hP}{kA} \]
\[ S\Delta V = S_a + S_p\phi_p \]
\[ \Delta V = A\Delta x \]

Comparing the above equations,
\[ \phi = T, \Gamma = 1, \quad S_u = n^2 T \Delta V, \quad S_p = -n^2 \Delta V \]

Solution is similar to the previous example. Find coefficients of
\[ a_p\phi_p = a_w\phi_w + a_E\phi_E + S_u \]

General equation:
\[ a_p\phi_p = a_w\phi_w + a_E\phi_E + S_u \]

Interior nodes (nodes 3-5):
\[ S_u = n^2 \Delta VT_x, \quad S_p = -n^2 \Delta V, \quad \Gamma = k \]

For boundary node 2:
\[ a_w \quad a_E \quad a_p \quad S_p \quad S_u \]
\[ \Gamma_x A_x \quad \Gamma_x A_x \quad a_w + a_E - S_p \quad -n^2 \Delta V - \frac{\Gamma_x A_x}{\Delta x / 2} \]
\[ n^2 \Delta VT_x + \frac{\Gamma_x A_x}{\Delta x / 2} T_B \]

For boundary node 6:
\[ \Gamma_x A_x \quad 0 \quad a_w + a_E - S_p \quad -n^2 \Delta V \]
\[ n^2 \Delta VT_x \]
The solution is

\[
\begin{bmatrix}
T_1 \\
T_2 \\
T_3 \\
T_4 \\
T_5 \\
T_6 \\
T_7
\end{bmatrix}
= \begin{bmatrix}
100 \\
64.22 \\
36.91 \\
26.50 \\
22.60 \\
21.30 \\
21.30
\end{bmatrix}
\]

Comparison with the analytical solution

<table>
<thead>
<tr>
<th>Node</th>
<th>Distance</th>
<th>Finite volume solution</th>
<th>Analytical solution</th>
<th>Difference</th>
<th>Percentage Error</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>0.1</td>
<td>64.22</td>
<td>68.52</td>
<td>4.30</td>
<td>6.27</td>
</tr>
<tr>
<td>3</td>
<td>0.3</td>
<td>36.91</td>
<td>37.86</td>
<td>0.95</td>
<td>2.51</td>
</tr>
<tr>
<td>4</td>
<td>0.5</td>
<td>26.50</td>
<td>26.61</td>
<td>0.11</td>
<td>0.41</td>
</tr>
<tr>
<td>5</td>
<td>0.7</td>
<td>22.60</td>
<td>22.53</td>
<td>-0.07</td>
<td>-0.31</td>
</tr>
<tr>
<td>6</td>
<td>0.9</td>
<td>21.30</td>
<td>21.21</td>
<td>-0.09</td>
<td>-0.42</td>
</tr>
</tbody>
</table>

Maximum error: 6.27%

The numerical solution can be improved by employing a finer grid.

Consider the same problem, but use 10 control volumes.

Comparison of the results is given as follows
Homework 2:

Write a computer program to find the temperature distribution in the rod in example 4.3. Compare the results obtained using 10 and 50 points on a graph. Use the algorithm given in the pseudo program appearing in the following slide.

The geometry of example 4.3.
Consider the two-dimensional steady state diffusion equation

\[
\frac{\partial}{\partial x} \left( \Gamma_x \frac{\partial \phi}{\partial x} \right) + \frac{\partial}{\partial y} \left( \Gamma_y \frac{\partial \phi}{\partial y} \right) + S = 0
\]

Integrating the above equation over the CV,

\[
\int \frac{\partial}{\partial x} \left( \Gamma_x \frac{\partial \phi}{\partial x} \right) dx \cdot dy + \int \frac{\partial}{\partial y} \left( \Gamma_y \frac{\partial \phi}{\partial y} \right) dx \cdot dy + \int S \cdot dV = 0
\]

Noting that \( A_e = A_w = \Delta y \) and \( A_n = A_s = \Delta x \), we obtain:

\[
\left[ \Gamma_x A_e \left( \frac{\partial \phi}{\partial x} \right)_w - \Gamma_w A_e \left( \frac{\partial \phi}{\partial x} \right)_e \right] + \left[ \Gamma_y A_s \left( \frac{\partial \phi}{\partial y} \right)_n - \Gamma_n A_s \left( \frac{\partial \phi}{\partial y} \right)_s \right] + \bar{S} \Delta V = 0
\]  

Equation (4.53) represents a balance of the generation of \( \phi \) in a CV and the fluxes through its cell faces.

Flux across the west face = \( \Gamma_w A_e \left( \frac{\partial \phi}{\partial x} \right)_w = \Gamma_w A_e \frac{\phi_w - \phi_e}{\Delta x_{WP}} \)

Flux across the east face = \( \Gamma_e A_e \left( \frac{\partial \phi}{\partial x} \right)_e = \Gamma_e A_e \frac{\phi_e - \phi_w}{\Delta x_{PE}} \)

Flux across the south face = \( \Gamma_s A_s \left( \frac{\partial \phi}{\partial y} \right)_n = \Gamma_s A_s \frac{\phi_n - \phi_s}{\Delta y_{SP}} \)

Flux across the north face = \( \Gamma_n A_s \left( \frac{\partial \phi}{\partial y} \right)_s = \Gamma_n A_s \frac{\phi_s - \phi_n}{\Delta y_{PN}} \)
By substitution of the above expressions into eqn. (4.53) we obtain

$$\Gamma A \phi_w - \phi_p - \Gamma A_a \phi_p - \phi_e + \Gamma A \phi_s - \phi_n - \Gamma A a \phi_p - \phi_s + \delta \Delta V = 0$$

Substituting the linearised form of the source term $\delta \Delta V = S_a + S_p \phi_p$

$$\left( \frac{\Gamma A a}{\delta x_{WP}} + \frac{\Gamma A e}{\delta x_{PE}} + \frac{\Gamma A s}{\delta y_{SP}} \right) \phi_p = \left( \frac{\Gamma A a}{\delta x_{WP}} \right) \phi_w + \left( \frac{\Gamma A e}{\delta x_{PE}} \right) \phi_e + \left( \frac{\Gamma A s}{\delta y_{SP}} \right) \phi_s + \left( \frac{\Gamma A a}{\delta y_{PN}} \right) \phi_n + S_a$$

This eqn can be written in the form:

$$a \phi_p = a_w \phi_w + a_e \phi_e + a_s \phi_s + a_n \phi_n + S_a$$

where

$$a_w = a_e = a_s = a_n = a_p$$

$$A_a = A_e = \Delta y$$

$$A_s = A_n = \Delta x$$

$$\frac{\Gamma A a}{\delta x_{WP}} = \frac{\Gamma A e}{\delta x_{PE}} = \frac{\Gamma A s}{\delta y_{SP}} = \frac{\Gamma A a}{\delta y_{PN}} = a_w + a_e + a_s + a_n + S_p$$

**Finite Volume Method for Three-dimensional Diffusion Problems**

Steady state diffusion in a 3D situation is governed by

$$\frac{\partial}{\partial x} \left( \Gamma \frac{\partial \phi}{\partial x} \right) + \frac{\partial}{\partial y} \left( \Gamma \frac{\partial \phi}{\partial y} \right) + \frac{\partial}{\partial z} \left( \Gamma \frac{\partial \phi}{\partial z} \right) + S = 0 \quad (4.58)$$

A typical control volume is shown below.

---

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**ME555 : Computational Fluid Dynamics**

Page 33
Integration of eqn (4.58) over the control volume gives

\[
\begin{align*}
&\left[ \frac{\partial}{\partial x_j} \left( \Gamma \phi \frac{\partial}{\partial x_j} \right) \right] - \Gamma_w A_w \left( \frac{\partial}{\partial x_j} \phi \right) \bigg|_{j} + \left[ \frac{\partial}{\partial y} \left( \Gamma \phi \frac{\partial}{\partial y} \right) \right] - \Gamma_w A_w \left( \frac{\partial}{\partial y} \phi \right) \bigg|_{b} \\
&+ \left[ \frac{\partial}{\partial z} \left( \Gamma \phi \frac{\partial}{\partial z} \right) \right] - \Gamma_w A_w \left( \frac{\partial}{\partial z} \phi \right) \bigg|_{t} = 0
\end{align*}
\]

which can be discretized as

\[
\begin{align*}
\Gamma_w A_w \frac{\phi_j - \phi_w}{\Delta x_{wp}} &= \Gamma_w A_w \frac{\phi_b - \phi_p}{\Delta x_{wp}} + \Gamma_w A_w \frac{\phi_e - \phi_p}{\Delta x_{wp}} \\
+ \Gamma_w A_w \frac{\phi_s - \phi_p}{\Delta z_{wp}} - \Gamma_w A_w \frac{\phi_s - \phi_p}{\Delta z_{wp}} + (S_e + S_p) \phi_p = 0
\end{align*}
\]

Rearranging

\[
a_p \phi_p = a_w \phi_w + a_e \phi_e + a_s \phi_s + a_n \phi_n + a_b \phi_b + a_t \phi_t + S_p
\]

Summary of Discretized Equations for Diffusion Problems

\[
a_p \phi_p = \sum a_{ab} \phi_{ab} + S_p
\]

\[
a_p = \sum a_{ab} - S_p
\]

source terms: \(S \Delta V = S_p \phi_p\)
Example:

Consider a 2D plate

Thickness = 1cm, $k = 1000$W/m/K

Calculate the temperature distribution

\[ T = 100 \, ^\circ C \]

$\dot{q}_w = 500$ kW/m$^2$

First draw control volumes, with equal spacings

Then, place nodes at the center of the control volumes.

\[ \Delta x = L_x/N - 2 = 0.3(5 - 2) = 0.1, \Delta y = L_y/M - 2 = 0.4(6 - 2) = 0.1 \]
The governing equation is

\[ \frac{\partial}{\partial x} \left( \Gamma \frac{\partial T}{\partial x} \right) + \frac{\partial}{\partial y} \left( \Gamma \frac{\partial T}{\partial y} \right) = 0 \quad (\Gamma = k) \]

which can be discretised as

\[ a_T T_p = a_y T_y + a_x T_x + a_s T_s + a_n T_n + S_n \]

This equation is written for each node \((i, j)\) in the domain

\[ a_T(i, j)T(i, j) = a_y(i, j)T(i-1, j) + a_x(i, j)T(i+1, j) + a_s(i, j)T(i, j-1) + a_n(i, j)T(i, j+1) + S_n(i, j) \]

For interior points: \((i = 2 - 4, j = 2 - 6)\)

\[ a_w = \frac{\Gamma A_w}{\delta x_u}; \quad a_e = \frac{\Gamma A_e}{\delta x_u}; \quad a_s = \frac{\Gamma A_s}{\delta x_u}; \quad a_n = \frac{\Gamma A_n}{\delta x_u} \]

\[ a_p = a_w + a_e + a_s + a_n - S_p \quad S_p = 0, \quad S_n = 0 \]

After finding \(a_p, a_e, a_w, a_s, a_n\) coefficients solve for all \(\phi_p\) values using the Gauss-Seidel iteration method.
Repeat iterations until scaled residual norm becomes \( R \leq \varepsilon \) where \( \varepsilon = \) tolerance (use \( \varepsilon = 1.E-6 \))

\[
R = \sum_{j} \sum_{i} |a_r(i, j)T_r(i, j)| \]

where \( r \) is the residual norm defined as

\[
r = \sum_{j} \sum_{i} |r(i, j)|\] (Note the absolute value sign)

and

\[
r(i, j) = a_w(i, j)T(i - 1, j) + a_E(i, j)T(i + 1, j) + a_S(i, j)T(i, j - 1) + a_N(i, j)T(i, j + 1) + S_u(i, j) - a_p(i, j)T(i, j)\]

**Fast Iterative Solvers for Linear Systems of Equations**

Apart from TDMA, there are other iterative methods for solving the system of equations which are faster. Unlike TDMA, which solves the problem line by line, these iterative methods solve all equations simultaneously. As a result these methods are faster than TDMA. Some of the fast iterative methods are

1) **SIP** (strongly implicit procedure)
2) **MSIP** (modified SIP)
3) **CG** (Conjugate gradient method)
4) **BiCGSTAB** (bi-conjugate gradient stabilized method)

CG method is used for solving linear systems of equations which have a **symmetric** coefficient matrix. All other methods mentioned above are used for systems of equations involving **non-symmetric** coefficient matrices.
Homework 3:

Write a computer program to find the temperature distribution in the 2-D plate problem given in the previous example. Use the algorithm given in the pseudo program on the next page. (a) Use 5x6 grids (b) 51x51 grids and plot the temperature contours.

Thickness = 1cm, $k = 1000\text{W/m/K}$

$T = 100\, ^\circ\text{C}$

$q_w = 500\, \text{kW/m}^2$

$0.3\, \text{m}$

insulated

$0.4\, \text{m}$

insulated