5.2 The method of matrix invariants

The dynamic behavior of many systems studied in engineering can be described by differential equations or algebraic equations. It would be nice if we could describe and analyze completely the dynamic behavior of Petri nets by some equations. In this spirit, we present matrix equations that govern the dynamic behavior of concurrent systems modeled by Petri nets. However, the solvability of these equations is somewhat limited, partly because of the nondeterministic nature inherent in Petri-net models and because of the constraint that solutions must be found as non-negative integers. Whenever matrix equations are discussed in this paper, it is assumed that a Petri net is pure or is made pure by adding a dummy pair of a transition and a place.

For a Petri net with n transitions and m places, the incidence matrix $A=[a_{ij}]$ is an $n \times m$ matrix of integers and its typically entry is given by $a_{ij}=a_{ij}^{+}-a_{ij}^{-}$ where $a_{ij}^{+}=w(i,j)$ is the weight of the arc from the transition i to its output place j and $a_{ij}^{-}=w(j,i)$ is the weight of the arc to transition i from its input place j. It can be easily seen that a_{ij}^{-}, a_{ij}^{+} and a_{ij}, respectively, represent the number of removed, added and changed tokens in place j when transition i fires. Notice that a transition i is enable at marking M if and only if $a_{ij}^{-} \le M(j)$ for $j=1,2,\ldots,m$.

In writing matrix equations, we write a marking M_k as an $m \times 1$ column vector. The jth entry of M_k denotes the number of tokens in place j immediately after the kth firing in some firing sequence. The kth firing or control vector u_k is an $n \times 1$ column vector of $n-1$ 0's and one nonzero entry, a 1 in the ith position indicating that transition i fires at the kth firing. Since the ith row of the incidence matrix A denotes the change of the marking as the result of firing transition i, the state equation for a Petri net is expressed as $A^T \cdot x = \Delta M$ where $\Delta M = M_d - M_0$ and x is $n \times 1$ column vector of nonnegative integers called the *firing vector*. The ith entry of x denotes the number of times transition i must fire to tranform M_0 to M_d.

It has been shown that the existence of a nonnegative integer solution x satisfying the state equation is a necessary but, in general, not sufficient condition for M_d to be reachable from M_0. For acyclic P/T-nets the above condition is also sufficient. For general P/T-nets the state equation $A^T \cdot x = \Delta M$ can have a solution corresponding to the unreachable marking, while for acyclic

P/T-nets the feasibility of aforementioned condition is exact characterization of the reachable markings.

As an immediate consequence of the above theorem, we can now use the state equation to verify the reachability in acyclic P/T-nets. Given acyclic P/T-net and two markings M_0 and M_d, we only need to write related state equation and find a nonnegative integer solution x of the state equation. If such x exists then M_d is reachable from M_0. Otherwise, it is not.

Example. Consider acyclic Petri net shown in Fig. 38. Assuming that we want to check whether or not the destination marking $M_d = (001010)^T$ is reachable from the initial marking $M_0 = (110000)^T$, the state equation is illustrated below:

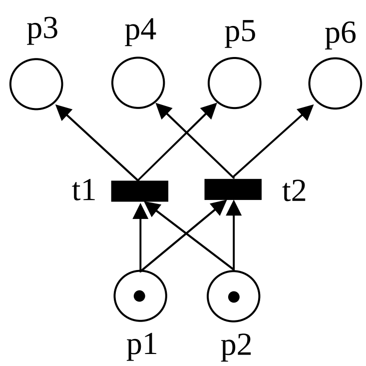

Fig. 38. A Petri net.

$$\begin{pmatrix} -1 & -1 \\ -1 & -1 \\ 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \\ 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}.$$

The above matrix equation has a unique solution $x_1 = 1;\ x_2 = 0$ and therefore M_d is reachable from M_0 through occurrence of t_1.