Questions related to SECTION 4.7

1. Evaluate the following limits.

   (a) $\lim_{x\to 0} \frac{3\sin\,4x}{5x}$

$$\begin{aligned}\lim_{x\to 0} \frac{3\sin\,4x}{5x} &= \left[\frac{0}{0}\right] \\ &= \lim_{x\to 0} \frac{12\cos\,4x}{5} = \frac{12}{5}\end{aligned}$$

   Recall also that $\lim_{x\to 0} \frac{\sin\,x}{x} = 1.$ Thus we can evaluate the limit as:

$$\lim_{x\to 0} \frac{3\sin\,4x}{5x} = \frac{3}{5}\lim_{x\to 0} \frac{\sin\,4x}{x} = \frac{12}{5}\lim_{x\to 0} \frac{\sin\,4x}{4x} = \frac{12}{5}$$

   (b) $\lim_{z\to 0} \frac{\tan 4z}{\tan 7z}$

$$\begin{aligned}\lim_{z\to 0} \frac{\tan 4z}{\tan 7z} &= \left[\frac{0}{0}\right] \\ &= \lim_{z\to 0} \frac{4\sec^2 4z}{7\sec^2 7z} = \frac{4}{7}\end{aligned}$$

   (c) $\lim_{x\to 0} \frac{1-\cos\,3x}{8x^2}$

$$\begin{aligned}\lim_{x\to 0} \frac{1-\cos\,3x}{8x^2} &= \left[\frac{0}{0}\right] \\ &= \lim_{x\to 0} \frac{3\sin\,3x}{16x} \quad \left[\frac{0}{0}\right] \\ &= \lim_{x\to 0} \frac{9\cos\,3x}{16} = \frac{9}{16}\end{aligned}$$

   (d) $\lim_{x\to 0} \frac{\sin^2 3x}{x^2}$

$$\lim_{x\to 0} \frac{\sin^2 3x}{x^2} = \left(\lim_{x\to 0} \frac{\sin\,3x}{x}\right)^2$$

   Then by applying L'Hopital's rule we get:

$$\begin{aligned}\lim_{x\to 0} \frac{\sin\,3x}{x} &= \left[\frac{0}{0}\right] \\ &= \lim_{x\to 0} \frac{3\cos\,3x}{1} = 3\end{aligned}$$

Thus the limit becomes:

$$\lim_{x\to 0}\frac{\sin^2 3x}{x^2} = \left(\lim_{x\to 0}\frac{\sin\ 3x}{x}\right)^2 = 3^2 = 9$$

(e) $\lim_{h\to 0}\frac{\sin(x+h)-\sin x}{h}$, x is a real number.

$$\begin{aligned}\lim_{h\to 0}\frac{\sin(x+h)-\sin\ x}{h} &= \left[\frac{0}{0}\right]\\ &= \lim_{h\to 0}\frac{\cos(x+h)}{1} = \cos\ x\end{aligned}$$

(f) $\lim_{x\to 2}\frac{\sqrt[3]{3x+2}-2}{x-2}$

$$\begin{aligned}\lim_{x\to 2}\frac{\sqrt[3]{3x+2}-2}{x-2} &= \left[\frac{0}{0}\right]\\ &= \lim_{x\to 2}\frac{(3x+2)^{-2/3}}{1} = 8^{-2/3} = \frac{1}{4}\end{aligned}$$

(g) $\lim_{x\to\infty}\frac{3x^4-x^2}{6x^4+12}$

$$\begin{aligned}\lim_{x\to\infty}\frac{3x^4-x^2}{6x^4+12} &= \left[\frac{\infty}{\infty}\right]\\ &= \lim_{x\to\infty}\frac{12x^3-2x}{24x^3}\quad\left[\frac{\infty}{\infty}\right]\\ &= \lim_{x\to\infty}\frac{36x^2-2}{72x^2}\quad\left[\frac{\infty}{\infty}\right]\\ &= \lim_{x\to\infty}\frac{72x}{144x} = \frac{1}{2}\end{aligned}$$

(h) $\lim_{x\to\infty}\frac{4x^3-2x^2+6}{\pi x^3+4}$

$$\begin{aligned}\lim_{x\to\infty}\frac{4x^3-2x^2+6}{\pi x^3+4} &= \left[\frac{\infty}{\infty}\right]\\ &= \lim_{x\to\infty}\frac{12x^2-4x}{3\pi x^2}\quad\left[\frac{\infty}{\infty}\right]\\ &= \lim_{x\to\infty}\frac{24x-4}{6\pi x}\quad\left[\frac{\infty}{\infty}\right]\\ &= \lim_{x\to\infty}\frac{24}{6\pi} = \frac{4}{\pi}\end{aligned}$$

(i) $\lim_{x\to\pi/2} \frac{2\tan x}{\sec^2 x}$

$$\begin{aligned}
\lim_{x\to\pi/2} \frac{2\tan x}{\sec^2 x} &= \left[\frac{\infty}{\infty}\right] \\
&= \lim_{x\to\pi/2} \frac{2\sec^2 x}{2\sec x \sec x \tan x} \\
&= \lim_{x\to\pi/2} \frac{1}{\tan x} \\
&= \lim_{x\to\pi/2} \cot x = 0
\end{aligned}$$

(j) $\lim_{x\to 0} x \csc x$

$$\begin{aligned}
\lim_{x\to 0} x \csc x &= [0\cdot\infty] \\
&= \lim_{x\to 0} \frac{x}{\sin x} \quad \left[\frac{0}{0}\right] \\
&= \lim_{x\to 0} \frac{1}{\cos x} = 1
\end{aligned}$$

(k) $\lim_{x\to(\pi/2)^-} \left(\frac{\pi}{2} - x\right)\sec x$

$$\begin{aligned}
\lim_{x\to(\pi/2)^-} \left(\frac{\pi}{2} - x\right)\sec x &= [0\cdot\infty] \\
&= \lim_{x\to(\pi/2)^-} \frac{\frac{\pi}{2} - x}{\cos x} \quad \left[\frac{0}{0}\right] \\
&= \lim_{x\to(\pi/2)^-} \frac{-1}{-\sin x} = 1
\end{aligned}$$

(l) $\lim_{x\to 0^+} \sqrt{\frac{1-x}{x}} \sin x$

$$\begin{aligned}
\lim_{x\to 0^+} \sqrt{\frac{1-x}{x}} \sin x &= [\infty\cdot 0] \\
&= \lim_{x\to 0^+} \sqrt{\frac{x(1-x)}{x^2}} \sin x \\
&= \lim_{x\to 0^+} \sqrt{x(1-x)} \cdot \lim_{x\to 0^+} \frac{\sin x}{x} = 0\cdot 1 = 0
\end{aligned}$$

where the second limit is evaluated by L'Hopital's rule.

(m) $\lim_{x\to 0^+}\left(\cot\,x-\frac{1}{x}\right)$

$$\begin{aligned}
\lim_{x\to 0^+}\left(\cot\,x-\frac{1}{x}\right) &= [\infty-\infty] \\
&= \lim_{x\to 0^+}\left(\frac{\cos\,x}{\sin\,x}-\frac{1}{x}\right) \\
&= \lim_{x\to 0^+}\frac{x\cos\,x-\sin\,x}{x\sin\,x} \quad \left[\frac{0}{0}\right] \\
&= \lim_{x\to 0^+}\frac{\cos\,x-x\sin\,x-\cos\,x}{\sin\,x+x\cos\,x} \quad \left[\frac{0}{0}\right] \\
&= \lim_{x\to 0^+}\frac{-\sin\,x-x\cos\,x}{\cos\,x+\cos\,x-x\sin\,x}=\frac{0}{2}=0
\end{aligned}$$

(n) $\lim_{x\to\infty}(x-\sqrt{x^2+1})$

$$\begin{aligned}
\lim_{x\to\infty}(x-\sqrt{x^2+1}) &= [\infty-\infty] \\
&= \lim_{x\to\infty}\left(x-\sqrt{x^2\left(1+\frac{1}{x^2}\right)}\right) \\
&= \lim_{x\to\infty}\left(x-x\sqrt{1+\frac{1}{x^2}}\right) \\
&= \lim_{x\to\infty}x\left(1-\sqrt{1+\frac{1}{x^2}}\right) \quad [\infty\cdot 0]
\end{aligned}$$

By changing the variable of the function to t s.t $t=\frac{1}{x}$ we get

$$\begin{aligned}
\lim_{x\to\infty}x\left(1-\sqrt{1+\frac{1}{x^2}}\right) &= [\infty\cdot 0] \\
&= \lim_{t\to 0^+}\frac{1}{t}(1-\sqrt{1+t^2}) \\
&= \lim_{t\to 0^+}\frac{1-\sqrt{1+t^2}}{t} \quad \left[\frac{0}{0}\right] \\
&= \lim_{t\to 0^+}\frac{-\frac{t}{\sqrt{1+t^2}}}{1}=0
\end{aligned}$$

(o) $\lim_{\theta\to(\pi/2)^-}(\tan\,\theta - \sec\,\theta)$

$$\begin{aligned}
\lim_{\theta\to(\pi/2)^-}(\tan\,\theta - \sec\,\theta) &= [\infty - \infty] \\
&= \lim_{\theta\to(\pi/2)^-}\left(\frac{\sin\,\theta}{\cos\,\theta} - \frac{1}{\cos\,\theta}\right) \\
&= \lim_{\theta\to(\pi/2)^-}\left(\frac{\sin\,\theta - 1}{\cos\,\theta}\right) \quad \left[\frac{0}{0}\right] \\
&= \lim_{\theta\to(\pi/2)^-}\left(\frac{\cos\,\theta}{-\sin\,\theta}\right) = \frac{0}{-1} = 0
\end{aligned}$$

(p) $\lim_{x\to 0^+} x^{2x}$

Note that :

$$\begin{aligned}
\lim_{x\to 0^+} x^{2x} &= \left[0^0\right] \\
&= \lim_{x\to 0^+} e^{\ln\,x^{2x}} \\
&= exp\left\{\lim_{x\to 0^+}\ln\,x^{2x}\right\} = exp\{\,L\}
\end{aligned}$$

Thus let us evaluate:

$$\begin{aligned}
L = \lim_{x\to 0^+}\ln x^{2x} &= \lim_{x\to 0^+} 2x\ln\,x \quad [0\cdot\infty] \\
&= 2\lim_{x\to 0^+} x\ln\,x \\
&= 2\lim_{x\to 0^+}\frac{\ln\,x}{\frac{1}{x}} \quad \left[\frac{\infty}{\infty}\right] \\
&= 2\lim_{x\to 0^+}\frac{\frac{1}{x}}{\frac{-1}{x^2}} \\
&= 2\lim_{x\to 0^+}(-x) = 0
\end{aligned}$$

Therefore

$$\lim_{x\to 0^+} x^{2x} = exp\left\{\lim_{x\to 0^+}\ln x^{2x}\right\} = exp\{L\} = exp\{0\} = 1$$

(q) $\lim\limits_{\theta\to(\pi/2)^-}(\tan\,\theta)^{\cos\,\theta}$

Note that :

$$\begin{aligned}\lim_{\theta\to(\pi/2)^-}(\tan\,\theta)^{\cos\,\theta} &= \left[\infty^0\right]\\ &= \lim_{\theta\to(\pi/2)^-} e^{\ln\,(\tan\,\theta)^{\cos\,\theta}}\\ &= exp\Big\{\lim_{\theta\to(\pi/2)^-}\ln\,(\tan\,\theta)^{\cos\,\theta}\Big\} = exp\{L\}\end{aligned}$$

Thus let us evaluate:

$$\begin{aligned}L=\lim_{\theta\to(\pi/2)^-}\ln\,(\tan\,\theta)^{\cos\,\theta} &= \lim_{\theta\to(\pi/2)^-}\cos\,\theta\ln\,(\tan\,\theta) \quad [0\cdot\infty]\\ &= \lim_{\theta\to(\pi/2)^-}\frac{\ln\,(\tan\,\theta)}{\sec\,\theta} \quad \left[\frac{\infty}{\infty}\right]\\ &= \lim_{\theta\to(\pi/2)^-}\frac{\frac{\sec^2\theta}{\tan\,\theta}}{\sec\,\theta\,\tan\,\theta}\\ &= \lim_{\theta\to(\pi/2)^-}\frac{\sec\,\theta}{\tan^2\theta}\\ &= \lim_{\theta\to(\pi/2)^-}\frac{\cos\,\theta}{\sin^2\theta} = 0\end{aligned}$$

Therefore

$$\lim_{\theta\to(\pi/2)^-}(\tan\,\theta)^{\cos\,\theta} = exp\Big\{\lim_{\theta\to(\pi/2)^-}\ln\,(\tan\,\theta)^{\cos\,\theta}\Big\} = exp\{\,L\} = exp\{\,0\} = 1$$

(r) $\lim\limits_{x\to0^+}(1+x)^{\cot\,x}$

Note that :

$$\begin{aligned}\lim_{x\to0^+}(1+x)^{\cot\,x} &= [1^\infty]\\ &= \lim_{x\to0^+}e^{\ln\,(1+x)^{\cot\,x}}\\ &= exp\Big\{\lim_{x\to0^+}\ln\,(1+x)^{\cot\,x}\Big\} = exp\{L\}\end{aligned}$$

Thus let us evaluate:

$$\begin{aligned}L=\lim_{x\to0^+}\ln\,(1+x)^{\cot\,x} &= \lim_{x\to0^+}\cot\,x\ln\,(1+x) \quad [\infty\cdot0]\\ &= \lim_{x\to0^+}\frac{\ln\,(1+x)}{\tan\,x} \quad \left[\frac{0}{0}\right]\\ &= \lim_{x\to0^+}\frac{\frac{1}{1+x}}{\sec^2 x}\\ &= \lim_{x\to0^+}\frac{\cos^2 x}{1+x} = 1\end{aligned}$$

Therefore

$$\lim_{x\to 0^+}(1+x)^{\cot\, x} = exp\Big\{\lim_{x\to 0^+}\ln{(1+x)^{\cot\, x}}\Big\} = exp\{L\} = exp\{1\} = e$$

(s) $\lim\limits_{x\to\infty}\left(1+\dfrac{1}{x}\right)^{\ln x}$

Note that:

$$\begin{aligned}\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{\ln\, x} &= [1^\infty] \\ &= \lim_{x\to\infty} e^{\ln\left(1+\frac{1}{x}\right)^{\ln\, x}} \\ &= exp\Big\{\lim_{x\to\infty}\ln\left(1+\frac{1}{x}\right)^{\ln x}\Big\} = exp\{L\}\end{aligned}$$

Thus let us evaluate:

$$\begin{aligned}L = \lim_{x\to\infty}\ln\, x\ln\left(1+\frac{1}{x}\right) &= \lim_{x\to\infty}\frac{\ln\left(1+\frac{1}{x}\right)}{\frac{1}{\ln\, x}} \quad \left[\frac{0}{0}\right] \\ &= \lim_{x\to\infty}\frac{\frac{1}{1+\frac{1}{x}}\left(\frac{-1}{x^2}\right)}{\frac{-1}{(\ln\, x)^2}\frac{1}{x}} = \lim_{x\to\infty}\frac{(\ln x)^2}{x+1} = 0\end{aligned}$$

The last limit is 0 since $x+1$ grows faster than $(\ln\, x)^2$ as $x\to\infty$.

Therefore

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{\ln x} = exp\Big\{\lim_{x\to\infty}\ln\left(1+\frac{1}{x}\right)^{\ln x}\Big\} = exp\{L\} = exp\{0\} = 1$$

(t) $\lim\limits_{x\to 0^+}(\tan\, x)^x$

Note that :

$$\begin{aligned}\lim_{x\to 0^+}(\tan\, x)^x &= \left[0^0\right] \\ &= \lim_{x\to 0^+} e^{\ln{(\tan\, x)^x}} \\ &= exp\Big\{\lim_{x\to 0^+}\ln{(\tan\, x)^x}\Big\} = exp\{L\}\end{aligned}$$

Thus let us evaluate:

$$\begin{aligned} L = \lim_{x\to 0^+} \ln(\tan\ x)^x &= \lim_{x\to 0^+} x\ln(\tan\ x) \quad [0\cdot\infty] \\ &= \lim_{x\to 0^+} \frac{\ln\ \tan\ x}{\frac{1}{x}} \quad \left[\frac{\infty}{\infty}\right] \\ &= \lim_{x\to 0^+} \frac{\frac{\sec^2 x}{\tan\ x}}{-\frac{1}{x^2}} \\ &= -\lim_{x\to 0^+} \frac{x^2}{\sin\ x\ \cos\ x} \\ &= -\lim_{x\to 0^+} \frac{x}{\sin\ x} \cdot \lim_{x\to 0^+} \frac{x}{\cos\ x} = -1\cdot 0 = 0 \end{aligned}$$

where the first limit is evaluated by L'Hopital's rule. Therefore

$$\lim_{x\to 0^+} (\tan\ x)^x = exp\Big\{\lim_{x\to 0^+} \ln(\tan\ x)^x\Big\} = exp\{L\} = exp\{0\} = 1$$

Questions related to SECTION 4.8

1. Determine the following indefinite integrals.

(a) $\int (3x^5 - 5x^9)\, dx$

$$\int (3x^5 - 5x^9)\, dx = \frac{3x^6}{6} - \frac{5x^{10}}{10} + C = \frac{x^6}{2} - \frac{x^{10}}{2} + C$$

(b) $\int (3u^{-2} - 4u^2 + 1)\, du$

$$\int (3u^{-2} - 4u^2 + 1)\, du = \frac{3u^{-1}}{-1} - \frac{4u^3}{3} + u + C = \frac{-3}{u} - \frac{4u^3}{3} + u + C$$

(c) $\int 6\sqrt[3]{x}\, dx$

$$\int 6\sqrt[3]{x}\, dx = \int 6x^{1/3}\, dx = \frac{6x^{\frac{4}{3}}}{\frac{4}{3}} + C = 6\frac{3}{4}x^{\frac{4}{3}} + C = \frac{9}{2}x^{\frac{4}{3}} + C$$

(d) $\int (\sin 2y + \cos 3y)\, dy$

$$\int (\sin 2y + \cos 3y)\, dy = -\frac{\cos 2y}{2} + \frac{\sin 3y}{3} + C$$

(e) $\int \left(\sec^2(\theta) + \sec\theta\tan\theta\right)\, d\theta$

$$\int (\sec^2(\theta) + \sec\theta\tan\theta)\, d\theta = \tan\theta + \sec\theta + C$$

(f) $\int \frac{6}{\sqrt{25 - x^2}}\, dx$

$$\int \frac{6}{\sqrt{25 - x^2}}\, dx = 6\sin^{-1}\left(\frac{x}{5}\right) + C$$

see Briggs Calculus p. 297, table 4.6, rule 9

(g) $\int \frac{2}{16z^2 + 25}\, dz$

$$\int \frac{2}{16z^2 + 25}\, dz = 2\int \frac{1}{16\left(z^2 + \frac{25}{16}\right)}\, dz = \frac{1}{8}\int \frac{1}{z^2 + \left(\frac{5}{4}\right)^2}\, dz =$$

$$= \frac{1}{8} \cdot \frac{4}{5}\tan^{-1}\left(\frac{4}{5}z\right) + C = \frac{1}{10}\tan^{-1}\left(\frac{4}{5}z\right) + C$$

see Briggs Calculus, p. 297, table 4.6, rule 10