• **Strict and Wide Sense Stationarity**

• **Autocorrelation Function of a Stationary Process**

• **Power Spectral Density**

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**Stationary Random Processes**

- Stationarity refers to *time invariance* of some, or all, the statistics of a random process, *e.g.*, mean, autocorrelation, nth order distribution, *etc*

- We define two types of stationarity, *strict sense* (SSS) and *wide sense* (WSS)

- A random process $X(t)$ (or $X_n$) is said to be SSS if *all* its finite order distributions are time invariant, *i.e.*, the joint cdf (pdf, or pmf) of $X(t_1), X(t_2), \ldots, X(t_k)$ is the same as for $X(t_1 + \alpha), X(t_2 + \alpha), \ldots, X(t_k + \alpha)$, for all $k$, all $t_1, t_2, \ldots, t_k$, and all time shifts $\alpha$

- So for a SSS process, the first order distribution is independent of $t$, and the second order distribution, *i.e.*, the distribution of any two samples $X(t_1)$ and $X(t_2)$, depends only on $\tau = t_2 - t_1$

To see this, note that from the definition of stationarity, for any $t$, the joint distribution of $X(t_1)$ and $X(t_2)$ is the same as the joint distribution of $X(t_1 + (t - t_1))$ and $X(t_2 + (t - t_1)) = X(t + (t_2 - t_1))$
A random process $X(t)$ is said to be WSS if its mean and autocorrelation functions are time invariant, i.e., $E(X(t)) = \mu$, independent of $t$ and $R_X(t_1, t_2)$ is only a function of $(t_2 - t_1)$.

Since $R_X(t_1, t_2) = R_X(t_2, t_1)$, if $X(t)$ is WSS, $R_X(t_1, t_2)$ is only a function of $|t_2 - t_1|$. Clearly SSS $\Rightarrow$ WSS, the converse, however, is not necessarily true.

For GRP, WSS $\Rightarrow$ SSS, since the process is completely specified by its mean and autocorrelation functions.

Random walk is not WSS, since $R_X(n_1, n_2) = \min\{n_1, n_2\}$ is not time invariant -- in fact no independent increment process can be WSS.
Autocorrelation Function of WSS Processes

- Let $X(t)$ be a WSS process and relabel $R_X(t_1, t_2)$ as $R_X(\tau)$, where $\tau = t_2 - t_1$

1. $R_X(\tau)$ is real and even, i.e., $R_X(\tau) = R_X(-\tau)$ for all $\tau$

2. $|R_X(\tau)| \leq R_X(0) = E(X^2(t))$, the “average power” of $X(t)$

   This can be shown using the Schwartz inequality

   For any $t$

   $$
   (R_X(\tau))^2 = (E(X(t)X(t+\tau)))^2 \\
   \leq E(X^2(t))E(X^2(t+\tau)) = (R_X(0))^2
   $$

3. If $R_X(T) = R_X(0)$ for some $T$, then $R_X(\tau)$ is periodic with period $T$ and so is $X(t)$ (with probability 1)
Which Functions can be an $R_X(\tau)$?
Interpretation of Autocorrelation Function

• If $R_X(\tau)$ drops quickly with $\tau$, this means that samples become uncorrelated quickly as we increase $\tau$, conversely, if $R_X(\tau)$ drops slowly with $\tau$, samples are highly correlated.

• So $R_X(\tau)$ is a measure of the rate of change of $X(t)$ with time $t$, i.e., “the frequency response of $X(t)$”

• It turned out that this is not just an interpretation – the Fourier Transform of $R_X(\tau)$ (the power spectral density) is in fact the average power density over frequency.
Power Spectral Density

- The power spectral density (psd) of a WSS random process $X(t)$, is the Fourier Transform of $R_X(\tau)$,

$$S_X(f) = \mathcal{F}[R_X(\tau)] = \int_{-\infty}^{\infty} R_X(\tau)e^{-j2\pi f \tau}d\tau$$

- For a discrete time process $X_n$, the power spectral density is the discrete Fourier Transform (DFT) of the sequence $R_X(n)$,

$$S_X(f) = \sum_{n=-\infty}^{\infty} R_X(n)e^{-j2\pi nf}, \text{ for } |f| < \frac{1}{2}$$

- $R_X(\tau)$ (or $R_X(n)$) can be recovered from $S_X(f)$ by taking the inverse Fourier Transform, i.e.,

$$R_X(\tau) = \int_{-\infty}^{\infty} S_X(f)e^{j2\pi f \tau}df, \text{ and inverse DFT,}$$

$$R_X(n) = \int_{-\frac{1}{2}}^{\frac{1}{2}} S_X(f)e^{j2\pi nf}df$$

Properties of the Power Spectral Density

1. $S_X(f)$ is real and even, since the Fourier Transform of a real and even function is real and even ($R_X(\tau)$ is real and even)

2. $S_X(f)$ is the power density, i.e., the average power of $X(t)$ in frequency band $[f_1, f_2]$ is $2 \int_{f_1}^{f_2} S_X(f)df$

- From (2) it follows that

$$S_X(f) \geq 0, \text{ and } E(X^2(t)) = \int_{-\infty}^{\infty} S_X(f)df,$$

i.e., the average power of $X(t)$ is the area under $S_X(f)$

- In general a function $S(f)$ is a psd iff it is real, even, nonnegative, and

$$\int_{-\infty}^{\infty} S(f)df < \infty$$
Examples

1. \( R_X(\tau) = e^{-2\alpha|\tau|} \)
   \( S_X(f) = \frac{\alpha}{\alpha^2 + (\pi f)^2} \)

2. \( R_X(\tau) = \frac{a^2}{2} \cos \omega \tau \)
   \( S_X(f) = \frac{a^2}{4} \)

3. \( R_X(n) = 2^{-|n|} \)
   \( S_X(f) = \frac{3}{5-4\cos 2\pi f} \)

4. Discrete time white noise process: \( X_n \) such that \( X_1, X_2, \ldots \) are zero mean, uncorrelated r.v.s with the same variance \( N \)

\[ R_X(n) = \begin{cases} N & n = 0 \\ 0 & \text{otherwise} \end{cases} \]

\[ S_X(f) \]

If \( X_n \) is also a GRP, then we get a discrete time WGN process
5. Bandlimited white noise process: WSS zero mean process \( X(t) \) with

\[
R_X(\tau) = NB \text{sinc} 2B\tau
\]

\[
S_X(f) = \frac{N}{2}
\]

For any \( t \), the samples \( X(t \pm \frac{n\tau}{2B}) \), for \( n = 0, 1, 2, \ldots \), are uncorrelated

6. White noise process: Now if we let \( B \rightarrow \infty \) in the previous example, we get a **white noise process**, which has

\[
S_X(f) = \frac{N}{2}, \text{ for all } f, \text{ and}
\]

\[
R_X(\tau) = \frac{N}{2} \delta(\tau)
\]

If, in addition, \( X(t) \) is a GRP, then we get the famous white gaussian noise (WGN) process

• Remarks on white noise:
  – For a white noise process all samples are uncorrelated
  – The process is not physically realizable, since it has infinite power
  – However, it plays a similar role in random processes to the role of a point mass in physics and delta function in EE
  – Thermal and shot noise are well modelled as white gaussian noise, since they have very flat psd over very wide band (GHzs)