VECTOR ANALYSIS

In this course, vectors like electric and magnetic fields will be discussed both in free space and in material media.

In order to do this, some numerical and mathematical tools; like addition, multiplication, differentiation and integration of scalars and vectors will be used.

SCALARS:

A scalar is s quantity that has only magnitude. i.e. mass, temperature, electric potential, charge density.

VECTORS:

A vector is a quantity that has both magnitude and direction. i.e. force, velocity, field.

A field is a function that specifies a particular quantity everywhere in a region.

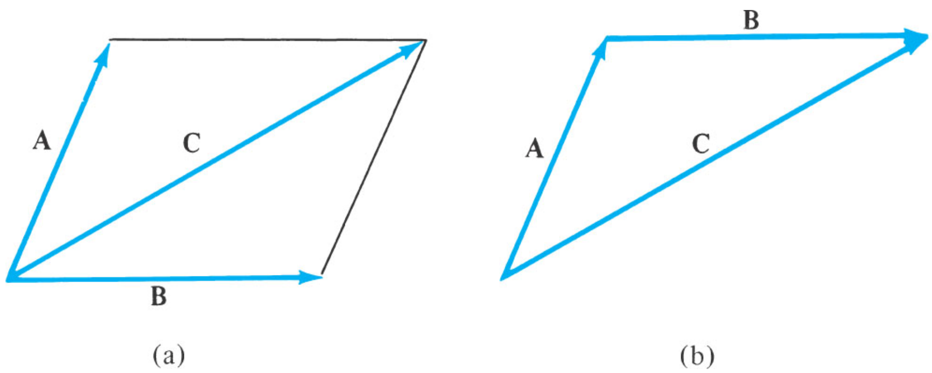
In electromagnetics, common fields that will be used are the electric field intensity $\overline{E}$, electric flux density $\overline{D}$ (displacement vector), magnetic field intensity vector $\overline{H}$ and the magnetic flux density vector $\overline{B}$.

VECTOR ALGEBRA

Vector algebra governs the laws of addition, subtraction and multiplication of vectors in any given coordinate system.

Vector Addition: Consider two vectors $\overline{A}$ and $\overline{B}$.

Their sum is: $\overline{C} = \overline{A} + \overline{B}$
We can add these vectors graphically to form vector $\overline{C}$.



Figs.1-1. **a** and **b** shows Parallelogram and Head to Tail Rules for adding vectors.

Properties of addition:

i) Addition is commutative, i.e.

$$\overline{A}_1 + \overline{A}_2 = \overline{A}_2 + \overline{A}_1$$

ii) Addition is associative i.e.

$$\left(\overline{A}_1 + \overline{A}_2\right) + \overline{A}_3 = \overline{A}_1 + \left(\overline{A}_2 + \overline{A}_3\right) = \overline{A}_1 + \overline{A}_2 + \overline{A}_3$$

iii) On the side two sides of vector equality we can add the same vector. i.e.

If $\overline{A} = \overline{B}$, then

$$\overline{A} + \overline{C} = \overline{B} + \overline{C}$$

iv) The magnitude of the sum of N vectors is less than or equal to the sum of their magnitudes. i.e.

$$\left|\overline{A}_1 + \overline{A}_2 + \ldots + \overline{A}_N\right| \leq \left|\overline{A}_1\right| + \left|\overline{A}_2\right| + \ldots + \left|\overline{A}_N\right|$$

If the N vectors are all in the same direction and have the same senses, then the equality holds.

Multiplication of a Vector by a Scalar

Consider a vector $\bar{A}$ and a scalar λ. We define the multiplication of a vector $\bar{A}$ by a scalar λ as the vector $\bar{B}$ such that,

a) if $\bar{A}$ or λ is zero (or both), then $\bar{B} = 0$,

b) if $\bar{A} \neq 0$ and $\lambda \neq 0$, then

i) $\bar{B}$ has the same direction of $\bar{A}$

ii) If $\lambda > 0$, $\bar{B}$ and $\bar{A}$ have the same sense

iii) If $\lambda < 0$, $\bar{B}$ and $\bar{A}$ have opposite senses.

iv) Magnitude of $\bar{B}$ is, $|\bar{B}| = |\lambda||\bar{A}|$

We denote the multiplication as:

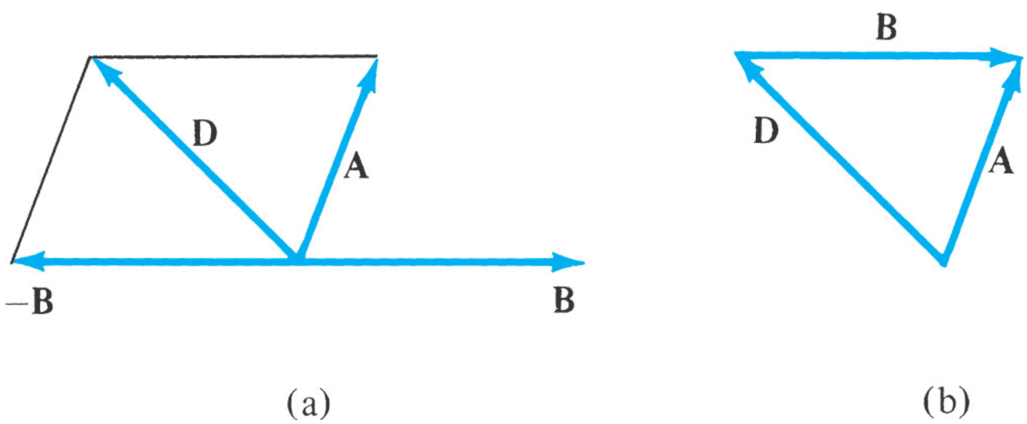
$$\bar{B} = \lambda \bar{A}$$

Consider two non-zero vectors. If they have the same direction, then they are said to be parallel.

Then, all vectors of the form $\lambda \bar{A}$ are parallel to $\bar{A}$. Conversely, every vector which is parallel to $\bar{A}$ has the form of $\lambda \bar{A}$.

VECTOR SUBTRUCTION

Vector subtraction is $\bar{D} = \bar{A} - \bar{B}$.



Figs.1-2. **a** and **b** shows Parallelogram and Head to Tail Rules for subtracting vectors.

UNIT VECTOR:

A unit vector is a vector of unity magnitude. A unit vector in the same direction (with the same sense) of a vector $\bar{A}$ is:

$$\frac{\bar{A}}{\left|\bar{A}\right|}$$

If we denote this unit vector as $\hat{a}_A$, then,

$$\hat{a}_A = \frac{\bar{A}}{\left|\bar{A}\right|}$$

We can write $\bar{A}$ by multiplying each side by $\left|\bar{A}\right|$,

$$\bar{A} = \hat{a}_A \left|\bar{A}\right|$$

Cartesian Components of a Vector

Consider the Cartesian coordinate system.

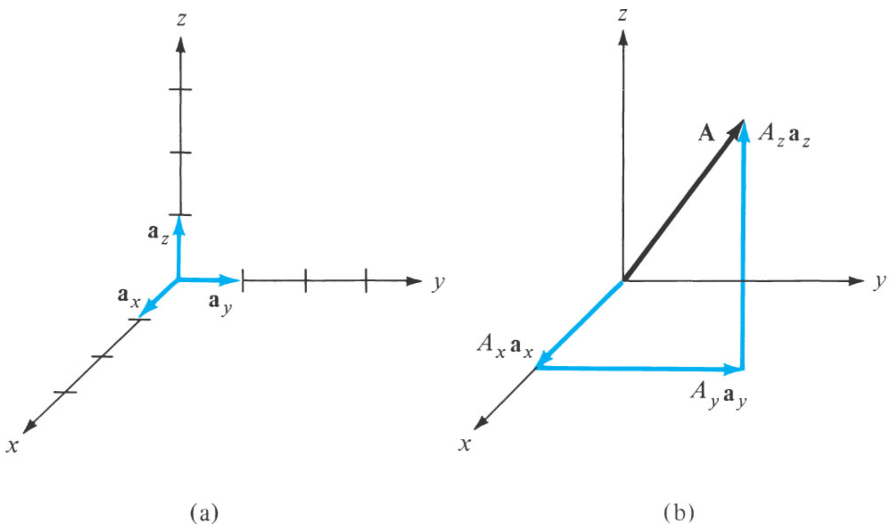


Fig. 1-3

Denote the unit vectors along the x , y and z axes as $\hat{a}_x$, $\hat{a}_y$ and $\hat{a}_z$. Then, any vector along the $x-axis$ is:

$$\overline{A}_x = A_x \hat{a}_x$$

Similarly $\overline{A}_y = A_y \hat{a}_y$ and $\overline{A}_z = A_z \hat{a}_z$.

Then any vector $\overline{A}$ can be written as:

$$\overline{A} = A_x \hat{a}_x + A_y \hat{a}_y + A_z \hat{a}_z$$

The magnitude of $\overline{A}$ is:

$$\left|\overline{A}\right| = \sqrt{A_x^{\,2} + A_y^{\,2} + A_z^{\,2}}$$

VECTOR MULTIPLICATION

I) SCALAR (DOT) MULTIPLICATION

Let $\overline{A}$ and $\overline{B}$ be two non-zero vectors. Let the angle between $\overline{A}$ and $\overline{B}$ be θ_{AB}. By the scalar product of these two vectors, we mean the following scalar:

$$\left|\overline{A}\right|\left|\overline{B}\right|\cos\theta_{AB}$$

The scalar product of $\overline{A}$ and $\overline{B}$ is denoted as $\overline{A}.\overline{B}$. Then,

$$\overline{A}.\overline{B} = \left|\overline{A}\right|\left|\overline{B}\right|\cos\theta_{AB}$$

$\overline{A}.\overline{B}$ is a signed scalar.

Let $\overline{A} \neq 0$ and $\overline{B} \neq 0$, if $0 \leq \theta_{AB} < \frac{\pi}{2}$ then $\overline{A}.\overline{B} > 0$

if $\frac{\pi}{2} < \theta_{AB} \leq \pi$ then $\overline{A}.\overline{B} < 0$

if $\theta_{AB} = \frac{\pi}{2}$, $\cos\theta_{AB} = 0$ and $\overline{A}.\overline{B} = 0$

Scalar Product of Two Vectors in Terms of Their Cartesian Components

Let $\overline{A} = A_x\hat{a}_x + A_y\hat{a}_y + A_z\hat{a}_z$ and

$\overline{B} = B_x\hat{a}_x + B_y\hat{a}_y + B_z\hat{a}_z$

Then, $\overline{A}.\overline{B} = \left(A_x\hat{a}_x + A_y\hat{a}_y + A_z\hat{a}_z\right).\left(B_x\hat{a}_x + B_y\hat{a}_y + B_z\hat{a}_z\right)$

$$\overline{A}.\overline{B} = A_x B_x + A_y B_y + A_z B_z$$

Since,

$$\hat{a}_x.\hat{a}_x = \hat{a}_y.\hat{a}_y = \hat{a}_z.\hat{a}_z = 1$$

$$\hat{a}_x . \hat{a}_y = \hat{a}_x . \hat{a}_z = \hat{a}_y . \hat{a}_z = 0$$

II) VECTOR (CROSS) PRODUCT

Consider two arbitrary vectors $\overline{A}$ and $\overline{B}$. We define the vector or cross product of $\overline{A}$ and $\overline{B}$ as a new vector $\overline{C}$ with the following properties:

i) If either $\overline{A}$ and $\overline{B}$ (or both) is zero then $\overline{C} = 0$. Now consider, $\overline{A} \neq 0$, $\overline{B} \neq 0$

ii) The direction of $\overline{C}$ is perpendicular to the plane which is formed by $\overline{A}$ and $\overline{B}$, hence $\overline{C}$ is perpendicular to both $\overline{A}$ and $\overline{B}$.

iii) Let the angle between $\overline{A}$ and $\overline{B}$ be θ. The magnitude of $\overline{C}$ is: $|\overline{C}| = |\overline{A}||\overline{B}|\sin\theta$ (since $0 \leq \theta \leq \pi$, $\sin\theta \geq 0$), in other words, $|\overline{C}|$ is equal to the area of the parallelogram formed by $\overline{A}$ and $\overline{B}$:

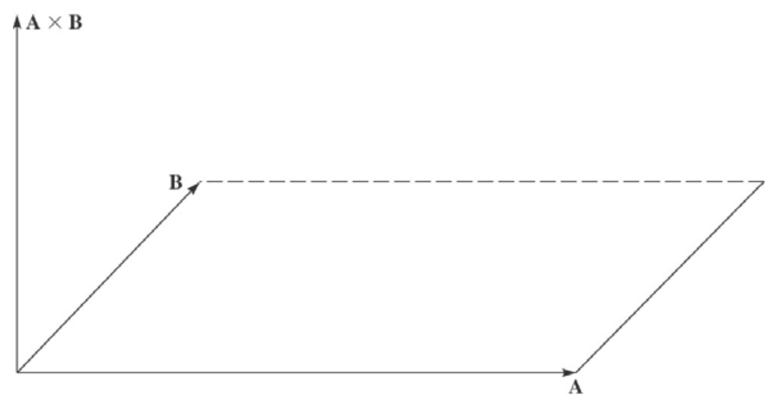


Fig. 1-4 The cross product of $\overline{A}$ and $\overline{B}$ is a vector with magnitude equal to the area of the parallelogram and direction as indicated.

iv) The sense of $|\overline{C}|$ is determined by the right hand rule.

We denote the vector product of two vectors as: $\overline{A} X \overline{B} = |\overline{A}||\overline{B}|\sin\theta \, \hat{a}_n$ where $\hat{a}_n$ is the unit vector perpendicular to both $\overline{A}$ and $\overline{B}$.

v) $\overline{A} X \overline{B} = 0$ if $\overline{A}$ and $\overline{B}$ are parallel.

vi) $\overline{A} X \overline{B} = -\overline{B} X \overline{A}$

vii) $\left|\bar{A}X\bar{B}\right| = \left|\bar{B}X\bar{A}\right|$

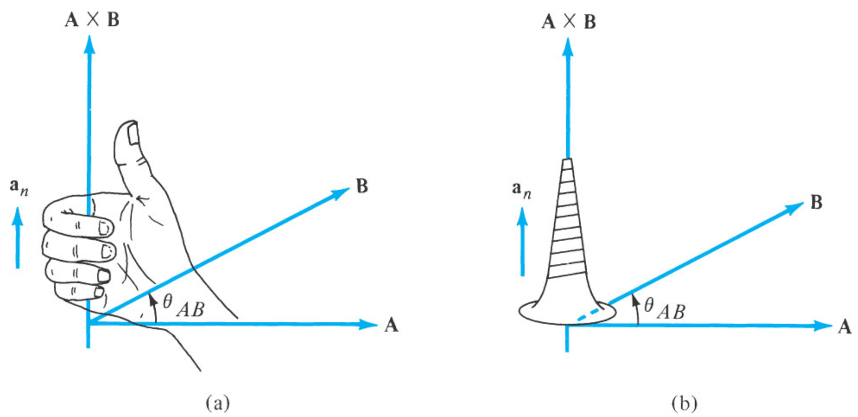


Fig. 1-5 Direction of $\bar{A}X\bar{B}$ and $\hat{a}_n$ using a) right-hand rule b) right-handed screw rule.

Representation of $\bar{A}X\bar{B}$ in Terms of the Cartesian Components

From the definition of the cross-product, we can write:

$$\hat{a}_x X\ \hat{a}_y = \hat{a}_z$$
$$\hat{a}_y X\ \hat{a}_z = \hat{a}_x$$
$$\hat{a}_z X\ \hat{a}_x = \hat{a}_y$$

$$\bar{A}X\bar{B} = \left(A_x\hat{a}_x + A_y\hat{a}_y + A_z\hat{a}_z\right) X \left(B_x\hat{a}_x + B_y\hat{a}_y + B_z\hat{a}_z\right)$$

We have a compact form for the $\bar{A}X\bar{B}$ in terms of a determinant:

$$\bar{A}X\bar{B} = \begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

III) Scalar Triple Product

Given three vectors $\overline{A}$, $\overline{B}$ and $\overline{C}$, we define the scalar triple product as:

$$\overline{A}.\left(\overline{B}X\overline{C}\right)=\overline{B}.\left(\overline{C}X\overline{A}\right)=\overline{C}.\left(\overline{A}X\overline{B}\right)$$

$\left|\overline{A}.\left(\overline{B}X\overline{C}\right)\right|$ is the volume of the parallelepiped having $\overline{A}$, $\overline{B}$ and $\overline{C}$ as edges and is easily obtained by finding the determined:

$$\overline{A}.\left(\overline{B}X\overline{C}\right)=\begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix}$$

$\left|\overline{A}.\left(\overline{B}X\overline{C}\right)\right|=0$ if $\overline{A}$, $\overline{B}$ and $\overline{C}$ are coplanar.