Maxwell’s Equations

We have considered Maxwell’s curl equations for electrostatic fields and modified for time-varying situations to satisfy Faraday’s Law. i.e \( \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \).

We shall now consider Maxwell’s curl equation for magnetic fields (Ampere’s Law) for time-varying conditions.

For static fields EM fields, we have

\[ \nabla \times \vec{H} = \vec{J} \]

\( \vec{J} \), includes conduction and convection currents.

\[ \vec{J} = \sigma \vec{E} + \rho \vec{v} \]

\( \sigma \vec{E} \) = Conduction current, due to the presence of the electric filed in the conducting medium,

\( \rho \vec{v} \) = Convection current, due to the motion of a free charge distribution.
The equations are:

\[
\nabla \bar{X} \bar{E} = -\frac{\partial \bar{B}}{\partial t} \quad \nabla \bar{X} \bar{H} = \bar{J} \\
\n\nabla \cdot \bar{D} = \rho \quad \nabla \cdot \bar{B} = 0
\]

Although the four Maxwell’s equations are consistent, they are not independent. The two divergence equations can be derived from the two curl equations by making use of the Continuity Equation \( \nabla \cdot \bar{J} = -\frac{\partial \rho}{\partial t} \) and Continuity Equation can be derived from four Maxwell’s Equations.

Consider \( \nabla \bar{X} \bar{H} = \bar{J} \) and \( \nabla \nabla \bar{X} \bar{H} = \nabla \bar{J} = 0 \), but the continuity equation is,

\[
\nabla \cdot \bar{J} = -\frac{\partial \rho}{\partial t}
\]

for time-varying fields. Ampere’s Law equation needs modification. Add a term to the Ampere’s Law equation:

\[
\nabla \bar{X} \bar{H} = \bar{J} + \bar{J}_d
\]

and operate \( \nabla \cdot \) on both sides:

\[
\nabla \cdot (\nabla \bar{X} \bar{H}) = \nabla \cdot \bar{J} + \nabla \cdot \bar{J}_d
\]
\[ 0 = -\frac{\partial \rho}{\partial t} + \nabla \cdot \bar{J}_d \]

Since,

\[ \nabla \cdot \bar{D} = \rho \]

\[ \nabla \left( \bar{J}_d - \frac{\partial \bar{D}}{\partial t} \right) = 0 \]

Let, \( \bar{J}_d = \frac{\partial \bar{D}}{\partial t} \), Displacement current density, A/m². The Ampere’s Law equation for time-varying fields takes the form:

\[ \nabla \times \bar{H} = \bar{J} + \frac{\partial \bar{D}}{\partial t} \]

Then four consistent equations,

\[ \nabla \times \bar{E} = -\frac{\partial \bar{B}}{\partial t} \quad \text{Faraday Law} \]

\[ \nabla \times \bar{H} = \bar{J} + \frac{\partial \bar{D}}{\partial t} \quad \text{Ampere’s Law} \]

\[ \nabla \cdot \bar{D} = \rho \quad \text{Gauss Law} \]

\[ \nabla \cdot \bar{B} = 0 \quad \text{Magnetic Flux Law} \]

are known as Maxwell’s equations.
Maxwell’s equations, together with

\[ \mathbf{\bar{F}} = q \left( \mathbf{\bar{E}} + \mathbf{\bar{u}} \times \mathbf{\bar{B}} \right) \]  
Lorentz Force Equation

\[ \nabla \cdot \mathbf{\bar{J}} + \frac{\partial \rho}{\partial t} = 0 \]  
Continuity Equation

from the foundation of the electromagnetic theory.

**INTEGRAL FORMS OF MAXWELL’S EQUATIONS**

Consider

\[ \nabla \times \mathbf{\bar{E}} = -\frac{\partial \mathbf{\bar{B}}}{\partial t} \]

and integrate over an open surface \( S \) with a contour \( C \) and apply Stoke’s Theorem:

\[ \int_{s} \nabla \times \mathbf{\bar{E}} \cdot \mathbf{n} ds = -\int_{s} \frac{\partial \mathbf{\bar{B}}}{\partial t} \cdot \mathbf{n} ds \]

Now consider,

\[ \nabla \times \mathbf{H} = \mathbf{\bar{J}} + \frac{\partial \mathbf{\bar{D}}}{\partial t} \]

And integrate over a surface \( S \)

\[ \int_{s} \nabla \times \mathbf{\bar{H}} \cdot \mathbf{n} ds = \int_{s} \left( \mathbf{\bar{J}} + \frac{\partial \mathbf{\bar{D}}}{\partial t} \right) \cdot \mathbf{n} ds \]
\[ \oint_{C} \vec{H} \cdot d\vec{l} = \int_{S} \left( \vec{J} + \frac{\partial \vec{D}}{\partial t} \right) \hat{n} ds \]

\[ \oint_{C} \vec{H} \cdot d\vec{l} = I + \int_{S} \left( \frac{\partial \vec{D}}{\partial t} \right) \hat{n} ds \]

For the divergence equations:

Consider \( \nabla \cdot \vec{B} = 0 \),

Take the volume integral of both sides of the divergence equation over a volume \( V \) and apply the divergence theorem:

\[ \int_{V} (\nabla \cdot \vec{B}) dv = 0 \]

results,

\[ \oint_{S} \vec{B} \cdot \hat{n} ds = 0 \]

And similarly, \( \nabla \cdot \vec{D} = \rho \)

\[ \int_{V} (\nabla \cdot \vec{D}) dv = \int_{V} \rho dv \]

\[ \oint_{S} \vec{D} \cdot d\vec{s} = Q \]
Consequently, the boundary conditions remain valid for the time-varying fields, where $\hat{a}_n$ is the unit vector normal to the boundary.

\[
E_{2r} = E_{1l} \quad \text{or} \quad (\vec{E}_1 - \vec{E}_2) \times \hat{a}_n = 0
\]
\[
H_{1t} - H_{2t} = J \quad \text{or} \quad (\vec{H}_1 - \vec{H}_2) \times \hat{a}_n = \vec{J}
\]
\[
D_{1n} - D_{2n} = \rho_s \quad \text{or} \quad (D_1 - D_2) \cdot \hat{a}_n = \rho_s
\]
\[
B_{1n} - B_{2n} = 0 \quad \text{or} \quad (B_1 - B_2) \cdot \hat{a}_n = 0
\]