In the previous chapters the electric field intensity has been determined by using the Coulomb’s and Gauss’s Laws when the charge distribution was known or by using $\bar{E} = -\nabla V$ when the potential $V$ is known throughout the region.

Now, we will consider electrostatic problems where only electrostatic conditions (charge and potential) at some boundaries are known and it is desired to find the electric field and the electrostatic potential. Such problems are usually solved by using Laplace’s and Poisson’s equations which are in general referred as *Boundary Value Problems*.

**Laplace’s and Poisson’s Equation’s**

Poisson’s and Laplace’s equations are derived by using the Gauss’s Law (in a linear medium) as:

$$\nabla \cdot \bar{D} = \nabla \cdot \varepsilon \bar{E} = \rho_v$$

and,

Substituting, $\bar{E} = -\nabla V$ into this equation we get,
\[ \nabla^2 V = -\frac{\rho_v}{\varepsilon} \]

which is known as **Poisson’s equation**. In the source free region a special case occurs:

\[ \nabla^2 V = 0 \]

This equation is known as the Laplace’s Equation. \( \nabla^2 \), is the Laplacian operator.

In Cartesian Coordinates the Laplace equation is:

\[ \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0 \]

In Cylindrical coordinates:

\[ \frac{1}{\rho} \frac{\partial}{\partial \rho} \left( \rho \frac{\partial V}{\partial \rho} \right) + \frac{1}{\rho^2} \frac{\partial V^2}{\partial \phi^2} + \frac{\partial V^2}{\partial \phi^2} = 0 \]

In Spherical Coordinates:

\[ \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial V}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial V}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 V}{\partial \phi^2} = 0 \]
General Procedure for Solving Poisson’s or Laplace’s Equation

1) Solve the Laplace or the Poisson equation subject to the Boundary Conditions to find $V$.
2) Calculate $E$ from $E = -\nabla V$ and $D$ from $D = \varepsilon E$.
3) If it is desired, find the charge $Q$ from $Q = \int_s \rho_s ds$ where $\rho_s = D_n$ and $D_n$ is the component of $D$ normal to the conductor.