Obtaining Spectral Information from an FDTD Simulation

- It is known that electromagnetic fields are expressed in terms of harmonic, or frequency-domain, signals.
- The temporal dependence of e^{jwt} is understood and therefore one only has to consider the spatial variation.
- In the frequency domain, the fields and quantities such as the propagation constant and the characteristic impedance are represented by complex numbers.
- These complex numbers give the magnitude and phase of the value and will be function of frequency.
- Given the frequency domain representation of the field at a point, the temporal signal is recovered by multiplying with e^{jwt} and taking the real part. Thus we may write a field in any of these equivalent forms as:

$$E_x(z,t) = \mathrm{Re}\left[E_x^+(z) e^{jwt} \right] = \mathrm{Re}\left[E_o^+ e^{-\gamma z} e^{jwt} \right] = \mathrm{Re}\left[E_o^+ e^{-(\alpha + j\beta) z} e^{jwt} \right]$$

- Where $E_x^+(z)$ is the frequency-domain representation of a wave traveling in the positive z -direction, γ is the propagation constant, with α attenuation constant and β is the phase constant, and E_o^+ is a complex number which gives the amplitude of the wave.
- In FDTD simulations, the time-domain form of the signal is obtained directly. However, we are interested in the behavior of the fields as a function of frequency.

Mapping Frequencies to Discrete Fourier Transforms

- Assume that the field was recorded during an FDTD simulation and then the recorded field was transformed to the frequency domain via a discrete Fourier transform.
- The discrete transform will yield a set of complex numbers that represent the amplitude of discrete spectral components.

- The question naturally arises: what is the correspondence between the indices of the transformed set and the actual frequency?
- In any simulation, the highest frequency $f_{\max}$ that can exists, is the inverse of the shortest possible period.
- In a discrete simulation one must have at least two samples per period. Since the time samples are Δt apart, the shortest possible period is $2\Delta t$. Therefore,

$$f_{\max} = \frac{1}{2\Delta t}$$

- The change in frequency from one discrete frequency to the next is the spectral resolution Δf . N_T is the total number of samples which correspond to the number of time steps, and

$$\Delta f = \frac{f_{\max}}{N_T / 2}$$

Combining two equations:

$$\Delta f = \frac{1}{N_T \Delta t}$$

- Here we will work through a couple of simple examples to illustrate how a broad range of spectral information can be obtained from FDTD simulations.

- Consider the planar interface at z=0, and a wave is incident on the interface from the left. When the impedances of the two media are not matched, a reflected wave must exist in order to satisfy the boundary conditions.

Transmission through a Planar Interface

- Consider the planar interface at z=0. Medium 1(z<0) is free space and medium 2 (z>0) is a dielectric.
- The electric field is normally incidence from the first medium.
- In the frequency domain, the incident, reflected, and transmitted fields are given by

$$E_x^i(z) = E_{o1}^+ e^{-\gamma_1 z}$$

$$E_x^r(z) = E_{o1}^- e^{+\gamma_1 z} = \Gamma E_o^+ e^{+\gamma_1 z}$$

$$E_x^t(z) = E_{o2}^+ e^{-\gamma_2 z} = T E_o^+ e^{-\gamma_2 z}$$

Γ is the reflection coefficient,

$$\Gamma = \frac{E_x^r(z)}{E_x^i(z)} \quad at \;\; z = 0$$

$$\Gamma = \frac{E_{01}^-}{E_{01}^+}$$

and T is the transmission coefficient.

$$T = \frac{E_x^t(z)}{E_x^i(z)} \quad at \;\; z = 0$$

$$T = \frac{E_{02}^+}{E_{01}^+}$$

- Where the magnetic field is related to the electric field by:

$$H_y^i(z) = \frac{E_{o1}^+}{\eta_1} e^{-\gamma z}$$

$$H_y^r(z) = -\frac{\Gamma E_o^+}{\eta_1} e^{+\gamma_1 z}$$

$$H_y^t(z) = \frac{T E_o^+}{\eta_2} e^{-\gamma_2 z}$$

- For the normal incidence case:

$$\Gamma = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1}$$

$$T = \frac{2\eta_2}{\eta_2 + \eta_1}$$

Obtaining the Transmission Coefficient Using FDTD

- Now consider two FDTD simulations. In the first, the second medium is removed and the field is recorded at some observation point (any k=const. point say kc).
- Since nothing is present to interfere with the incident field, the recorded field will be simply the incident field at this location, i.e. $E_x^i(kc,t)$.
- Next, replace the second medium such that the observation point is inside the second medium.
- Performing an FDTD simulation in this case yields the transmitted field $E_x^t(kc,t)$.
- The goal now is to construct the transmission coefficient using the temporal recordings of the field obtained from these two simulations.

- First, it should be noted that one cannot use

$$\frac{E_z^t(kc,t)}{E_z^i(kc,t)}$$

to obtain the transmission coefficient.

- The transmission coefficient is a frequency-domain concept and we currently have time domain-signals.
- The division of these temporal signals is essentially meaningless (the result is undefined when the incident signal is zero).
- The incident and transmitted fields must be converted to the frequency domain using a Fourier transform. Thus,

$$E_x^i(kc) = \Im(E_x^i(kc,t))$$
$$E_x^t(kc) = \Im(E_x^t(kc,t))$$

where, $\Im$ indicates the Fourier transform.

- The division of these two functions *is* meaningful; at least the incident field is non-zero for all frequencies.
- To demonstrate how the transmission coefficient can be reconstructed from the FDTD simulations let us consider an example where the first medium is free space and the second one has a relative permittivity of 9.

- The transmission coefficient can be calculated easily by using:

$$T = \frac{2\eta_2}{\eta_1 + \eta_2}$$

and this provides a reference solution, where

$$\eta_1 = \eta_o$$
$$\eta_2 = \eta_o / 3$$

and

$$T = \frac{2\frac{\eta_o}{3}}{\frac{\eta_o}{3} + \eta_o} = \frac{1}{2}$$

- Ideally the FDTD simulation would yield this same value for all frequencies.