Chapter 8: Convection in External Turbulent Flow

8.1. Introduction

- Turbulent flow is disordered, with random and unsteady velocity fluctuations; hence, exact predictions cannot be determined.
- Turbulence affects local velocity distribution, drag force, and heat transfer in both natural and industrial processes.
- Understanding of turbulent flow leads to the ability to make design improvements to either reduce or enhance turbulent effects
- Our understanding still relies on empirical data and rudimentary conceptual drawings, and, more recently, computer simulations.
- Exact solutions are not possible.
- Chapter Focus: Wall-bounded shear flows

8.1.1. Examples of Turbulent Flows

- **(a) Mixing Processes**
  - Combustion processes – proper mix of fuel and air is one of the requirements for combustion efficiency
  - Chemical processing, such as the production of polymers
  - Laminar mixing occurs when liquids are viscous and/or slowly mixed, and can be problematic
- **(b) Free Shear Flow**
  - Jet Flows (refer to fig. 8.1a): jet energy is dissipated to the surrounding fluid
  - Turbulent wake (refer to fig. 8.1b): transfers energy between an object and the ambient flow, contributes to an object's drag
  - Smoke stack exhaust is dispersed by turbulence (refer to figure 8.c)
- **(c) Wall-Bounded Flows**
  - Flow of air over a flat plate or airfoil
  - Flow of fluid in a pipe
  - Irregular or random motions cause the shape of the velocity profile and the boundary layer edge location to change with time
  - Instantaneous velocity profiles are time-averaged for simplicity: $\overline{u}$
  - Turbulent velocity can be decomposed into steady (mean, $\overline{u}$) and unsteady (fluctuating, u') components
  - Note: Refer to fig. 8.2 for details on the velocity profiles and unsteady components

- The mixing of velocity fluctuations in turbulent flow creates a steeper profile than that of a laminar flow, with a larger boundary layer and higher wall shear stress.
- Turbulent fluctuations enhance momentum transfer between the surface and the flowing fluid, resulting in higher skin friction; this suggests that surface fluctuations similarly enhance heat transfer.
- Note: See figure 8.3 for a comparison of laminar and turbulent velocity profiles.
- By understanding turbulence, we can alter designs to take advantage of turbulent effects, such as increased heat transfer and reduced drag.
- Note: See the golf ball example in fig. 8.4 for an example of drag reduction via turbulence.

8.1.2. The Reynolds Number and the Onset of Turbulence

- The Reynolds number is the ratio of inertial to viscous forces, and indicates the onset of turbulence.
- Reynolds equation:

$$Re_D = \frac{\overline{u}D}{\nu} \tag{8.1}$$

- The onset of turbulence for flow through tubes is approximately $Re_t = \overline{u}D/\nu \approx 2300$.
- For uniform flow over a semi-infinite flat plate, the onset of turbulence is approximately $Re_t = V_\infty x_t/\nu \approx 500{,}000$.
- Viscous forces dominate at low flow velocity.
- At high velocity, inertial forces acting on individual particles dominate; the flow amplifies these disturbances, creating a more chaotic flow.
- Turbulence initiates near the wall in wall-bounded flows.
- Note: See fig. 8.5 for the development of turbulent flow over a semi-infinite flat plate.

8.1.3. Eddies and Vorticity

- Eddies are regions of intermittent, swirling patches of fluid
- An eddy is a particle of vorticity, ω:

$$\vec{\omega} = \nabla \times \vec{V} \tag{8.2}$$

- Eddies form in regions of velocity gradient.
    - Example: for 2D flow over a flat plate:

$$\omega_z = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$$

- High shear stress results in high vorticity.

- The formation and behavior of eddies within the boundary layer is not fully understood.
- Vortex stretching increases the kinetic energy of the vortex, and is thought to be a major mechanism for the main flow to transfer energy to the turbulence.
- Note: Refer to fig. 8.6 for details of eddy formation, vortex formation and vortex stretching in a wall-bounded flow.
- Eddies provide bulk motion (advection) and mixing within the boundary layer.
- Advection differentiates turbulent flow from laminar flow; laminar flow has no bulk motion and relies solely on viscous diffusion to transfer momentum.

8.1.4. Scales of Turbulence

- Viscous effects play a vital role in turbulence.
  - Most turbulence originates from shear flow.
  - Rotation of a fluid element occurs under the action of viscous shear.
- Lewis Richardson proposed the concept of energy cascade in 1922:
  - Turbulence is comprised of a range of different eddy sizes.
  - Treating the eddy as a distinct fluid structure with:
    - Characteristic size: ℓ
    - Characteristic velocity: u
    - Eddy turnover time: $t = \ell / u$
    - Reynolds number for the eddy: $Re = u\ell / \nu$
  - Inertial forces in a turbulent flow cause large, unstable eddies with high kinetic energy and high Reynolds numbers (on the scale of the main flow) to break up into smaller and smaller eddies.
  - Energy is transferred as each eddy breaks into smaller ones.
  - The breaking up process continues until the Reynolds numbers of the eddies approach unity; at this time, viscous forces dissipate the energy of the small eddies into heat.
  - Note: Fig. 8.7 illustrates the cascade process.
  - Eddies decrease in size much faster than in velocity; therefore, the smaller eddies experience very high velocity gradients and have high vorticity.
- The largest eddies contain the bulk of the kinetic energy in a turbulent flow; the smallest eddies contain the bulk of vorticity and therefore the mechanism of dissipation.
- Andrey Kolmogorov proposed a model in 1942 based on the ideas above:
  - The largest eddies contain the bulk of the kinetic energy
  - The smallest scales reach a Reynolds number of unity prior to dissipating into heat

- Kolmogorov's relations:

$$\eta / \ell \sim Re^{-3/4} \tag{8.3a}$$

$$\upsilon / u \sim Re^{-1/4} \tag{8.3b}$$

$$\tau / t \sim Re^{-1/2} \tag{8.3c}$$

  - η / ℓ is the ratio of the length scales of the largest and smallest eddies
  - υ / u is the ratio of their velocities
  - τ / t is the ratio of their time scales
  - The Reynolds number is that of the largest eddy: $Re = u\ell / \nu$
- The variables η, υ and τ are called the *Kolmogorov microscales*
- The variables ℓ, u and t are called the *integral scales*

- Important points from Kolmogorov's model:
  - There is a vast range of eddy sizes, velocities and time scales in turbulent flow, making modeling difficult
  - The smallest eddies are not infinitely small, since they are dissipated into heat by viscous forces.
  - The scale of the smallest eddies is determined by the scale of the largest eddies through the Reynolds number.
  - Turbulent flow responds to an increase in velocity by producing smaller eddies, hence, viscous dissipation is increased.
  - Faster turbulent flows have finer turbulent structure; this is observed in the real world.

8.1.5. Characteristics of Turbulence

- Turbulence is comprised of irregular, chaotic, three-dimensional fluid motion, but containing coherent structures.
- Turbulence occurs at high Reynolds numbers, where instabilities give way to chaotic motion.
- Turbulence is comprised of many scales of eddies, which dissipate energy and momentum through a series of scale ranges.
  - The largest eddies contain the bulk of the kinetic energy and break up by inertial forces.
  - The smallest eddies contain the bulk of the vorticity and dissipate into heat.
- Turbulent flows are not only dissipative, but also dispersive through the advection mechanism.

8.1.6. Analytical Approaches

- The continuum hypothesis still applies to the governing equations of fluid mechanics for turbulent flow.
  - This is based on the example in section 8.1.4, where the microscale for air flowing over a 10 cm diameter cylinder at 10 m/s was approximately 20 microns.
  - The mean free path of air at standard conditions is 10^{-8} m; this is three orders of magnitude smaller.
- The computing power required to process a direct numerical simulation (DNS) using Computational Fluid Dynamics (CFD) is enormous and exceeds practical limitations, due to the vast range of scales of turbulence between the largest and smallest eddies.
- An analytical approach is used to predict macroscale properties, such as velocity, drag force and heat transfer.
- Microscopic flow structure is ignored; turbulent fluctuations are analyzed statistically.
- Important idealizations are used to simplify statistical analysis:
  - *Homogeneous turbulence*: turbulence whose microscale motion does not, on average, change from location to location or from time to time
  - *Isotropic turbulence*: turbulence whose microscale motion does not, on average, change as the coordinate axes are rotated
  - These idealizations are not necessarily realistic, but they can be approximated in the laboratory, so experimental data can be compared to the simplified statistical analytic flow models.

8.2. Conservation Equations for Turbulent Flow

8.2.1. Reynolds Decomposition

- Reynolds proposed that turbulent flow can be considered as the superposition of a time-averaged and a fluctuating component.
- Reynolds Decomposition Approach:
  - Each fluctuating property in the governing equations is decomposed into a time averaged and a fluctuating component.
  - The entire equation is time averaged.

For an arbitrary property *g*:

$$g = \bar{g} + g' \tag{8.4}$$

Where $\bar{g}$, the time-averaged component, is determined by:

$$\bar{g} = \frac{1}{\tau}\int_0^{\tau} g(t)dt \tag{8.5}$$

By definition, the time average of the fluctuating property, g', is 0:

$$\overline{g'} = \frac{1}{\tau}\int_0^\tau g'(t)dt = 0 \tag{8.6}$$

- Useful averaging identities:

$$\overline{\overline{a}} = \overline{a} \tag{8.7a}$$
$$\overline{\overline{a}\,\overline{b}} = \overline{a}\,\overline{b} \tag{8.7b}$$
$$\overline{(\overline{a})^2} = (\overline{a})^2 \tag{8.7c}$$
$$\overline{\overline{a}a'} = 0 \tag{8.7d}$$
$$\overline{ab} = \overline{a}\,\overline{b} + \overline{a'b'} \tag{8.7e}$$
$$\overline{a^2} = (\overline{a})^2 + \overline{(a')^2} \tag{8.7f}$$
$$\overline{a+b} = \overline{a} + \overline{b} \tag{8.7g}$$
$$\overline{\frac{\partial a}{\partial x}} = \frac{\partial \overline{a}}{\partial x} \tag{8.7h}$$
$$\frac{\partial \overline{a}}{\partial t} = 0 \tag{8.7i}$$
$$\overline{\frac{\partial a}{\partial t}} = 0 \tag{8.7j}$$

- An example proving identities (8.7a) and (8.7g) is given.

8.2.2. Conservation of Mass

- Reynolds decomposition is applied to the Cartesian conservation of mass equation:

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z} = 0 \tag{2.2a}$$

  - Analysis is limited to incompressible, two-dimensional flow.

Substituting the Reynolds decomposed velocities, $u = \overline{u} + u'$, and $v = \overline{v} + v'$:

$$\frac{\partial(\overline{u} + u')}{\partial x} + \frac{\partial(\overline{v} + v')}{\partial y} = 0 \tag{a}$$

This is expanded:

$$\frac{\partial \overline{u}}{\partial x} + \frac{\partial u'}{\partial x} + \frac{\partial \overline{v}}{\partial y} + \frac{\partial v'}{\partial y} = 0 \tag{b}$$

The equation is time averaged:

$$\overline{\frac{\partial \overline{u}}{\partial x}} + \overline{\frac{\partial u'}{\partial x}} + \overline{\frac{\partial \overline{v}}{\partial y}} + \overline{\frac{\partial v'}{\partial y}} = 0 \tag{c}$$

Invoking identity (8.7h), (c) becomes:

$$\frac{\partial \overline{\overline{u}}}{\partial x} + \frac{\partial \overline{u'}}{\partial x} + \frac{\partial \overline{\overline{v}}}{\partial y} + \frac{\partial \overline{v'}}{\partial y} = 0 \tag{d}$$

Using identity (8.7a), $\overline{\overline{u}} = \overline{u}$ and $\overline{\overline{v}} = \overline{v}$, and by (8.6), $\overline{u'} = \overline{v'} = 0$, so (d) reduces to the time averaged turbulent flow continuity equation, identical in form to (2.2):

$$\frac{\partial \overline{u}}{\partial x} + \frac{\partial \overline{v}}{\partial y} = 0 \tag{8.8}$$

Subtracting (8.8) from (b) demonstrates that the divergence of the fluctuating term is 0:

$$\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} = 0 \tag{8.9}$$

8.2.3. Momentum Equations

- Reynolds decomposition is applied to the Cartesian momentum equations:

$$\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right) = \rho g_x - \frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) \tag{2.10x}$$

$$\rho\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}\right) = \rho g_y - \frac{\partial p}{\partial y} + \mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right) \tag{2.10y}$$

- **Assumptions**
  - (1) Two-dimensional flow
  - (2) Incompressible flow
  - (3) Constant properties
  - (4) Body forces are neglected
  - (5) Flow is, on average, steady-state, so $\overline{u}$ and $\overline{v}$ are constant
- **Formulation**

Equations (2.10x) and (2.10y) become:

$$\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) = -\frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \tag{8.10x}$$

$$\rho\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}\right) = -\frac{\partial p}{\partial y} + \mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) \tag{8.10y}$$

From the product rule of differentiation, $u(\partial u/\partial x)$ and $v(\partial u/\partial y)$ in the x-momentum equation become:

$$u\frac{\partial u}{\partial x} = \frac{\partial u^2}{\partial x} - u\frac{\partial u}{\partial x} \tag{a}$$

$$v\frac{\partial u}{\partial y} = \frac{\partial (uv)}{\partial y} - u\frac{\partial v}{\partial y} \tag{b}$$

Substituting (a) into the x-momentum equation (8.10x):

$$\rho\left(\frac{\partial u}{\partial t}+\frac{\partial u^2}{\partial x}-\underset{(A)}{u\frac{\partial u}{\partial x}}+\frac{\partial(uv)}{\partial y}-\underset{(B)}{u\frac{\partial v}{\partial y}}\right)=-\frac{\partial p}{\partial x}+\mu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right) \quad \text{(c)}$$

The terms (A) and (B) in (c) above are combined, and by conservation of mass:

$$-u\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)=0 \quad \text{(d)}$$

The *x*-momentum equation reduces to:

$$\rho\left(\frac{\partial u}{\partial t}+\frac{\partial u^2}{\partial x}+\frac{\partial(uv)}{\partial y}\right)=-\frac{\partial p}{\partial x}+\mu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right) \quad (8.11)$$

The *y*-momentum equation reduces in a similar fashion

- Reynolds decomposition is now performed on the simplified *x*- and *y*-momentum equations, resulting in the turbulent *x*- and *y*-momentum equations for turbulent flow:

$$\rho\left(\bar{u}\frac{\partial \bar{u}}{\partial x}+\bar{v}\frac{\partial \bar{u}}{\partial y}\right)=-\frac{\partial \bar{p}}{\partial x}+\mu\left(\frac{\partial^2 \bar{u}}{\partial x^2}+\frac{\partial^2 \bar{u}}{\partial y^2}\right)-\rho\frac{\partial\overline{(u')^2}}{\partial x}-\rho\frac{\partial\overline{u'v'}}{\partial y} \quad (8.12\text{x})$$

$$\rho\left(\bar{u}\frac{\partial \bar{v}}{\partial x}+\bar{v}\frac{\partial \bar{v}}{\partial y}\right)=-\frac{\partial \bar{p}}{\partial y}+\mu\left(\frac{\partial^2 \bar{v}}{\partial x^2}+\frac{\partial^2 \bar{v}}{\partial y^2}\right)-\rho\frac{\partial\overline{u'v'}}{\partial x}-\rho\frac{\partial\overline{(v')^2}}{\partial y} \quad (8.12\text{y})$$

- Equations (8.12) are identical to equations (8.10) with two notable exceptions:
  - The transient terms $\partial u/\partial t$ and $\partial v/\partial t$ disappear.
  - More importantly, new terms indicating fluctuating velocity are introduced.

8.2.4. Energy Equation

- Reynolds decomposition is applied to the Energy equation for incompressible flow:

$$\rho c_p\left(\frac{\partial T}{\partial t}+u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}+w\frac{\partial T}{\partial z}\right)=k\left(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}+\frac{\partial^2 T}{\partial z^2}\right)+\mu \quad (2.19\text{b})$$

- **Assumptions**
  - (1) Heat generation is neglected
  - (2) Thermal properties are considered constant
  - (3) Steady-on-average flow
  - (4) Two dimensional flow
  - (5) The dissipation function $\mu\Phi$ is neglected; this is appropriate as long as the flow is not highly viscous or compressible
- **Formulation**

The Energy equation reduces to:

$$\rho c_p\left(\frac{\partial T}{\partial t}+u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}\right)=k\left(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}\right) \tag{8.13}$$

After Reynolds decomposition and time averaging is applied (left as an exercise, Problem 8.7), equation (8.13) becomes:

$$\rho c_p\left(\overline{u}\frac{\partial \overline{T}}{\partial x}+\overline{v}\frac{\partial \overline{T}}{\partial y}\right)=k\left(\frac{\partial^2 \overline{T}}{\partial x^2}+\frac{\partial^2 \overline{T}}{\partial y^2}\right)-\rho c_p\frac{\partial\left(\overline{u'T'}\right)}{\partial x}-\rho c_p\frac{\partial\left(\overline{v'T'}\right)}{\partial y} \tag{8.14}$$

- Equation (8.14) is almost identical to the steady-state Energy equation (8.10) with two notable exceptions:
  - The transient term $\rho c_p \partial T/\partial t$ disappears
  - Two terms indicating fluctuating velocities and temperature are introduced, and come out of the convective terms on the left side of the equation.

8.2.5. Summary of Governing Equations for Turbulent Flow

- Continuity:

$$\frac{\partial \overline{u}}{\partial x}+\frac{\partial \overline{v}}{\partial y}=0 \tag{8.8}$$

- *x*-momentum:

$$\rho\left(\overline{u}\frac{\partial \overline{u}}{\partial x}+\overline{v}\frac{\partial \overline{u}}{\partial y}\right)=-\frac{\partial \overline{p}}{\partial x}+\mu\left(\frac{\partial^2 \overline{u}}{\partial x^2}+\frac{\partial^2 \overline{u}}{\partial y^2}\right)-\rho\frac{\partial\overline{(u')^2}}{\partial x}-\rho\frac{\partial\overline{u'v'}}{\partial y} \tag{8.12x}$$

- *y*-momentum:

$$\rho\left(\overline{u}\frac{\partial \overline{v}}{\partial x}+\overline{v}\frac{\partial \overline{v}}{\partial y}\right)=-\frac{\partial \overline{p}}{\partial y}+\mu\left(\frac{\partial^2 \overline{v}}{\partial x^2}+\frac{\partial^2 \overline{v}}{\partial y^2}\right)-\rho\frac{\partial\overline{u'v'}}{\partial x}-\rho\frac{\partial\overline{(v')^2}}{\partial y} \tag{8.12y}$$

- Energy:

$$\rho c_p\left(\overline{u}\frac{\partial \overline{T}}{\partial x}+\overline{v}\frac{\partial \overline{T}}{\partial y}\right)=k\left(\frac{\partial^2 \overline{T}}{\partial x^2}+\frac{\partial^2 \overline{T}}{\partial y^2}\right)-\rho c_p\frac{\partial\left(\overline{u'T'}\right)}{\partial x}-\rho c_p\frac{\partial\left(\overline{v'T'}\right)}{\partial y} \tag{8.14}$$

8.3. Analysis of External Turbulent Flow

- Goal: Solve the governing equations for turbulent flow for forces of interest, such as drag force and heat transfer.
- Focus: Flow along a surface
- Boundary layer concept is invoked
- Similar development of boundary layer equations to those of laminar flow (Chapter 4), with two important differences:

- Turbulent flow governing equations contain time-averaged quantities (i.e. $\overline{u}$, $\overline{T}$), which are handled exactly like u and T previously
- Turbulent equations contain fluctuating terms (i.e. u' and T') that require additional consideration

8.3.1. Turbulent Boundary Layer Equations

(i) Turbulent Momentum Boundary Layer Equation

- Turbulent flow over a flat plate is considered (refer to Fig. 8.8)
- Assumptions
  - (1) Steady-state
  - (2) Incompressible flow
  - (3) Constant properties
  - (4) The boundary layer is thin, $\delta << L$, so:

$$\frac{\delta}{L} << 1 \tag{8.15}$$

- Formulation

  Following the arguments used for the laminar boundary layer, the following scalar arguments are made:

$$\overline{u} \sim V_\infty \tag{8.16a}$$

$$x \sim L \tag{8.16b}$$

$$y \sim \delta \tag{8.16c}$$

  Following an analysis similar to that in Section 4.2, the viscous dissipation terms in (8.12x) compare as follows:

$$\frac{\partial^2 \overline{u}}{\partial x^2} << \frac{\partial^2 \overline{u}}{\partial y^2} \tag{8.17}$$

  The pressure gradient in the y-direction is negligible:

$$\frac{\partial \overline{p}}{\partial y} \approx 0 \tag{8.18}$$

  The pressure gradient in the x-direction is expressed as:

$$\frac{\partial \overline{p}}{\partial x} = \frac{d\overline{p}}{dx} = \frac{dp_\infty}{dx} \tag{8.19}$$

  - The fluctuation terms $\dfrac{\partial \overline{(u')^2}}{\partial x}$ and $\dfrac{\partial \overline{u'v'}}{\partial y}$ require additional scaling arguments.

There is no preferred direction to the fluctuations; this is equivalent to assuming the turbulence is isotropic, therefore

$$u' \sim v' \tag{8.20}$$

$$\overline{(u')^2} \sim \overline{u'v'} \tag{8.21}$$

Comparing the fluctuating terms using scale analysis:

$$\frac{\partial \overline{(u')^2}}{\partial x} \sim \frac{\overline{(u')^2}}{L} \tag{a}$$

$$\frac{\partial \overline{u'v'}}{\partial y} \sim \frac{\overline{u'v'}}{\delta} \sim \frac{\overline{(u')^2}}{\delta} \tag{b}$$

Since $\delta << L$, we can conclude that

$$\frac{\partial \overline{(u')^2}}{\partial x} << \frac{\partial \overline{u'v'}}{\partial y} \tag{8.22}$$

Using the simplifications (8.17) and (8.22), the *x*-momentum equation for the turbulent boundary layer reduces to:

$$\rho\left(\overline{u}\,\frac{\partial \overline{u}}{\partial x} + \overline{v}\,\frac{\partial \overline{u}}{\partial y}\right) = -\frac{d\overline{p}}{dx} + \mu\frac{\partial^2 \overline{u}}{\partial y^2} - \rho\frac{\partial \overline{u'v'}}{\partial y} \tag{8.23}$$

(ii) Turbulent Energy Equation

- The derivation for laminar boundary layer equations is again followed.
- Formulation

Scaling arguments for the thermal boundary layer:

$$x \sim L \tag{8.16b}$$

$$y \sim \delta_t \tag{8.24}$$

$$\Delta T \sim T_s - T_\infty \tag{8.25}$$

- The scale for the velocity sizes depend on δ_t and δ, as in Chapter 4.

The second derivative terms compare as follows

$$\frac{\partial^2 \overline{T}}{\partial x^2} << \frac{\partial^2 \overline{T}}{\partial y^2} \tag{8.26}$$

- The fluctuation terms $\rho c_p \frac{\partial}{\partial x}\left(\overline{u'T'}\right)$ and $\rho c_p \frac{\partial}{\partial y}\left(\overline{v'T'}\right)$ require additional scaling arguments:

  There is no preferred direction for the fluctuations, so:

$$u' \sim v' \tag{8.20}$$

$$\overline{u'T'} \sim \overline{v'T'} \tag{8.27}$$

The fluctuating terms compare as follows (it is left to the reader to derive this):

$$\frac{\partial\left(\overline{u'T'}\right)}{\partial x} << \frac{\partial\left(\overline{v'T'}\right)}{\partial y} \tag{8.28}$$

- Applying simplifications (8.26) and (8.28), the energy equation for the turbulent boundary layer reduces to:

$$\rho c_p\left(\overline{u}\frac{\partial \overline{T}}{\partial x} + \overline{v}\frac{\partial \overline{T}}{\partial y}\right) = k\frac{\partial^2 \overline{T}}{\partial y^2} - \rho c_p \frac{\partial\left(\overline{v'T'}\right)}{\partial y} \tag{8.29}$$

8.3.2. Reynolds Stress and Heat Flux

- Equations (8.23) and (8.29), the x-momentum and energy equations for the turbulent boundary layer, are written as follows to provide physical insight and a way to model the time-averaged fluctuation terms, $\overline{u'v'}$ and $\overline{v'T'}$:

$$\rho\left(\overline{u}\frac{\partial \overline{u}}{\partial x} + \overline{v}\frac{\partial \overline{u}}{\partial y}\right) = -\frac{d\overline{p}}{dx} + \frac{\partial}{\partial y}\left(\mu\frac{\partial \overline{u}}{\partial y} - \rho\overline{u'v'}\right) \tag{8.30}$$

$$\rho c_p\left(\overline{u}\frac{\partial \overline{T}}{\partial x} + \overline{v}\frac{\partial \overline{T}}{\partial y}\right) = \frac{\partial}{\partial y}\left(k\frac{\partial \overline{T}}{\partial y} - \rho c_p\overline{v'T'}\right) \tag{8.31}$$

- The terms in parenthesis on the right side of equation (8.30) represent shear stress
  - The first term represents the molecular shear stress from the time-averaged velocity
  - Boussinesq suggested (1877) that the second term can be viewed as shear imposed by the time-averaging velocity fluctuations.
  - Note: Refer to Fig. 8.9 and to the related discussion in the text for a visualization of Boussinesq's idea.
  - The time average fluctuation term $\overline{u'v'}$ is proportional to the velocity gradient, just like for the viscous shear stress, suggesting $\overline{u'v'}$ behaves like shear in the flow.

$$-\overline{u'v'} \propto \frac{\partial \overline{u}}{\partial y}$$

- The *apparent shear stress* experienced by the flow is made up of two parts
  - Molecular shear imposed by the time-averaged velocity component
  - *Turbulent shear stress* $\rho\overline{u'v'}$, or *Reynolds stress*, imposed by the time-averaged velocity fluctuations
- The *apparent energy flux* in the energy equation contains a turbulence-induced heat flux, $\rho c_p\overline{v'T'}$, called the *turbulence heat flux* or *Reynolds heat flux*.

8.3.3. The Closure Problem of Turbulence

Summary of turbulence boundary equations:

- Continuity

$$\frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{v}}{\partial y} = 0 \tag{8.8}$$

- x-momentum

$$\rho\left(\bar{u}\frac{\partial \bar{u}}{\partial x} + \bar{v}\frac{\partial \bar{u}}{\partial y}\right) = -\frac{d\bar{p}}{dx} + \frac{\partial}{\partial y}\left(\mu\frac{\partial \bar{u}}{\partial y} - \rho\overline{u'v'}\right) \tag{8.30}$$

- Energy

$$\rho c_p\left(\bar{u}\frac{\partial \bar{T}}{\partial x} + \bar{v}\frac{\partial \bar{T}}{\partial y}\right) = \frac{\partial}{\partial y}\left(k\frac{\partial \bar{T}}{\partial y} - \rho c_p\overline{v'T'}\right) \tag{8.31}$$

- The boundary conditions for these equations are:

$$\bar{u}(x,0) = 0 \tag{8.31a}$$

$$\bar{v}(x,0) = 0 \tag{8.31b}$$

$$\bar{u}(x,\infty) = V_\infty \tag{8.31c}$$

$$\bar{u}(0,y) = V_\infty \tag{8.31d}$$

$$\bar{T}(x,0) = T_s \tag{8.31e}$$

$$\bar{T}(x,\infty) = T_\infty \tag{8.31f}$$

$$\bar{T}(0,y) = T_\infty \tag{8.31g}$$

- If we know the velocity field outside of the boundary layer, the pressure can be determined:
  - The pressure gradient is expressed as

$$\frac{d\bar{p}}{dx} = \frac{dp_\infty}{dx} \tag{8.32}$$

  - From inviscid flow theory, outside the boundary layer:

$$V_\infty\frac{dV_\infty}{dx} = -\frac{1}{\rho}\frac{dp_\infty}{dx} \tag{8.33}$$

- We are left with three equations, (8.8), (8.30) and (8.31), and five unknowns, $\bar{u}$, $\bar{v}$, $\bar{T}$, $\overline{u'v'}$, and $\overline{v'T'}$; this is *the closure problem of turbulence*.
- The two terms $\overline{u'v'}$ and $\overline{v'T'}$ are nonlinear terms
  - There is no exact solution to the turbulence boundary layer equations
  - Modeling will facilitate numerical and approximate solutions

8.3.4. Eddy Diffusivity

- Reynolds Stress, based on Boussinesq's hypothesis, is modeled as

$$-\rho\overline{u'v'} = \rho\varepsilon_M \frac{\partial \overline{u}}{\partial y} \tag{8.34}$$

  - ε_M is the *momentum eddy diffusivity*
  - $\rho\varepsilon_M$ is referred to as the *eddy viscosity*
- The Reynolds heat flux is modeled as

$$-\rho c_p \overline{v'T'} = \rho c_p \varepsilon_H \frac{\partial \overline{T}}{\partial y} \tag{8.35}$$

  - ε_H is the *thermal eddy diffusivity*
  - $\rho c_p \varepsilon_H$ is referred to as the *eddy conductivity*
- The boundary layer momentum and energy equations are written as:

$$\rho\left(\overline{u}\frac{\partial \overline{u}}{\partial x} + \overline{v}\frac{\partial \overline{u}}{\partial y}\right) = -\frac{d\overline{p}}{dx} + \frac{\partial}{\partial y}\left[\left(\mu + \rho\varepsilon_M\right)\frac{\partial \overline{u}}{\partial y}\right] \tag{8.36}$$

$$\rho c_p\left(\overline{u}\frac{\partial \overline{T}}{\partial x} + \overline{v}\frac{\partial \overline{T}}{\partial y}\right) = \frac{\partial}{\partial y}\left[\left(k + \rho c_p \varepsilon_H\right)\frac{\partial \overline{T}}{\partial y}\right] \tag{8.37}$$

- These are simplified by dividing (8.36) through by ρ and (8.37) by ρc_p:

$$\overline{u}\frac{\partial \overline{u}}{\partial x} + \overline{v}\frac{\partial \overline{u}}{\partial y} = \frac{\partial}{\partial y}\left[\left(\nu + \varepsilon_M\right)\frac{\partial \overline{u}}{\partial y}\right] \tag{8.38}$$

$$\overline{u}\frac{\partial \overline{T}}{\partial x} + \overline{v}\frac{\partial \overline{T}}{\partial y} = \frac{\partial}{\partial y}\left[\left(\alpha + \varepsilon_H\right)\frac{\partial \overline{T}}{\partial y}\right] \tag{8.39}$$

- The terms in brackets in (8.38) represent the *apparent shear stress*:

$$\frac{\tau_{app}}{\rho} = \left(\nu + \varepsilon_M\right)\frac{\partial \overline{u}}{\partial y} \tag{8.40}$$

- The terms in brackets in (8.39) represent the *apparent heat flux*:

$$-\frac{q''_{app}}{\rho c_p} = \left(\alpha + \varepsilon_H\right)\frac{\partial \overline{T}}{\partial y} \tag{8.41}$$

  - The negative sign in (8.41) assigns the correct direction to heat transfer.
- ε_M and ε_H are properties of the flow and are dependent on the velocity and temperature fields, respectively

- The number of unknowns has not been reduced; the velocity fluctuation terms have been replaced with expressions containing different unknowns
- Finding ways to evaluate ε_M and ε_H is one of the main goals of turbulence research

8.4. Momentum Transfer in External Turbulent Flow

Section Focus: Finding ways to evaluate ε_M and ε_H

- Energy and momentum equations are decoupled
  - Energy equation solution requires knowledge of both velocity and temperature fields
  - Momentum equation requires knowledge of velocity field, so solving for momentum transfer requires solutions of continuity and momentum equations only
    - This still leaves two equations and three unknowns, but the situation is simplified; now only ε_M requires modeling

8.4.1. Modeling Eddy Diffusivity: Prandtl's Mixing Length Theory

- Boussinesq postulated that ε_M is constant
  - This does not allow $\overline{u'v'}$ to approach zero at the wall, which is unrealistic, since turbulent fluctuations are expected to dampen out near the wall
- Prandtl's model reasons that fluid particle behavior is analogous to that of molecules in the kinetic theory of gases.
  - Note: Refer to Fig. 8.10 for visual details, (i.e. directions) of variables discussed below.
  - In a two-dimensional flow, Prandtl defines:
  - v' as a velocity fluctuation forcing a particle towards the wall
  - ℓ, the mixing length, as the distance the particle travels as a result of the velocity fluctuation
  - The resulting velocity fluctuation u' is approximated using Taylor series expansion:

$$u_{final} \approx u_{initial} + \frac{\partial \overline{u}}{\partial y} dy \tag{a}$$

  - So, with $u' = u_{final} - u_{initial}$,

$$u' \sim \ell \frac{\partial \overline{u}}{\partial y} \tag{b}$$

  - Again assuming that the velocity fluctuations have no preferred direction, $u' \sim v'$, we have

$$v' \sim \ell \frac{\partial \overline{u}}{\partial y} \tag{c}$$

- By the above scales, one argues that the scale of the turbulent stress term $-\overline{u'v'}$ is:

$$-\overline{u'v'} \sim (u')(v') \sim \ell^2 \left(\frac{\partial \overline{u}}{\partial y}\right)^2 \tag{d}$$

- Therefore, solving equation (8.34) for eddy viscosity gives:

$$\varepsilon_M = \frac{-\overline{u'v'}}{\partial \overline{u} / \partial y} \sim \ell^2 \left|\frac{\partial \overline{u}}{\partial y}\right| \tag{8.42}$$

- The absolute value on the derivative in equation (8.42) ensures the eddy diffusion remains positive.
- Note that the mixing length itself must still be modeled, and is dependent on the type of flow.

- Prandtl's model for mixing length for flow over a flat plat is:

$$\ell = \kappa y \tag{8.43}$$

  - where κ is some constant
  - This implies that the mixing length approaches zero as y approaches zero.
  - Thus equation (8.42) becomes *Prandtl's mixing-length model*:

$$\varepsilon_M = \kappa^2 y^2 \left|\frac{\partial \overline{u}}{\partial y}\right| \tag{8.44}$$

  - However:
    - κ is unknown
    - No single value for κ is effective throughout the boundary layer.
- Before developing a suitable model for κ, boundary layer behavior must be further understood. This is presented next.

8.4.2. Universal Turbulent Velocity Flow Profile

- In Chapter 5, an approximate solution for the integral form of the boundary layer equations for laminar flow was found using a *universal velocity profile* $u(y)$; this approach is used here.
- This approach also provides physical insight that can be applied to other solution techniques.
- Section Goal: Find a universal velocity profile function

(i) Large-Scale Velocity Distribution: "Velocity Defect Law"

- Variables making up the velocity distribution are normalized:
  - y/δ becomes one axis

- $\bar{u}/V_\infty$ becomes the other axis, where V_∞ is the velocity outside of the boundary layer
- Figure 8.11 shows velocity curves at different values of wall friction; if the plot were truly universal, all curves would collapse into one.
- Data is normalized by the wall friction factor, $C_f = \dfrac{\tau_o}{(1/2)\rho V_\infty^2}$, where τ_0 is the shear stress at the wall.
- Further manipulations to the data are performed:
  - For convenience, we cast the data relative to the velocity outside the boundary layer:

$$(\bar{u} - V_\infty) \tag{8.45}$$

  - A *friction velocity* is defined:

$$u^* \equiv \sqrt{\tau_o / \rho} \tag{8.46}$$

  - Friction velocity can also be written as:

$$u^* = V_\infty \sqrt{C_f / 2} \tag{8.47}$$

  - Note: u^* has the same dimension as velocity.
  - The velocity difference is normalized by u^* to produce the *velocity defect*:

$$\frac{(\bar{u} - V_\infty)}{u^*} \tag{8.48}$$

- The velocity defect is plotted against y/δ in Figure 8.12, which also shows the range of experimental data. Note that the curves from Fig. 8.11 collapse into a single curve

- Note: The defect plot does not show enough detail near the wall.
- An improved plot will probably be logarithmic, and will require an entirely new set of coordinates.

(ii) Wall Coordinates

- The following coordinates will collapse the boundary layer velocity data into a single curve reasonably well:

$$u^+ \equiv \frac{\bar{u}}{u^*} \tag{8.49}$$

$$y^+ \equiv \frac{y u^*}{\nu} \tag{8.50}$$

- Similarly, $v^+ = \bar{v}/u^*$, and $x^+ = x u^*/\nu$.

- All of the above variables are dimensionless, and are called *wall coordinates*.
- Note: Refer to Figure 8.13 to see a plot of data produced by Clauser using these coordinates.
    - The boundary layer extends out to approximately $\delta \approx 2000$ to 5000
    - The plot includes data obtained from flow in a pipe as well as flow along a flat plate, so the profile appears to be universal.

(iii) Near-Wall Profile: Couette Flow Assumption

- The turbulent boundary layer momentum equation, (8.38), is invoked to develop a model of the velocity profile near the wall.
- **Assumption**: Flow is over a flat plate, so $d\overline{p}/dx = 0$.
- The momentum equation is simplified by recognizing that the flow is nearly parallel to the wall, so $\overline{v} \sim 0$.
- Conservation of mass implies that $\partial\overline{u}/\partial x \sim 0$; therefore the $\overline{u}$ component of velocity does not change significantly along the wall.
- This scaling argument suggests that the convective terms in the momentum equation, $\overline{u}\,\partial\overline{u}/\partial x$ and $\overline{v}\,\partial\overline{u}/\partial y$, are each approximately zero, therefore, in (8.38):

$$\frac{\partial}{\partial y}\left[(\nu + \varepsilon_M)\frac{\partial\overline{u}}{\partial y}\right] \sim 0 \text{ near the wall.}$$

- The bracketed term is the apparent shear stress, τ_{app}/ρ (Eqn. 8.40)
- The above relationship implies that the apparent stress is approximately constant (with respect to y), resulting in the *Couette Flow Assumption*:

$$\frac{\tau_{app}}{\rho} = (\nu + \varepsilon_M)\frac{\partial\overline{u}}{\partial y} \sim \text{constant} \tag{8.51}$$

- Recall from Chapter 3: this result is similar to Couette Flow.
- Since the local shear is constant in (8.51), we can replace τ_{app} with its value at the wall, τ_o.
- The presence of the eddy diffusivity makes (8.51) different from Couette flow, implying that the shear stress is constant, but not necessarily linear.
- The Couette Flow Assumption is used to develop a velocity profile at the wall:
    - Substituting the definitions of u^+ and y^+ into (8.51), it can be shown that:

$$\left(1 + \frac{\varepsilon_M}{\nu}\right)\frac{\partial u^+}{\partial y^+} = 1 \tag{8.52}$$

    - After rearranging and integrating:

$$u^+ = \int_0^{y^+} \frac{dy^+}{(1+\varepsilon_M/\nu)} \tag{8.53}$$

- This is a general expression for the universal velocity profile in wall coordinates

- This integral can be evaluated more simply by dividing the boundary layer into two near-wall regions:
  - (1) a region very close to the wall where viscous forces dominate
  - (2) a region where turbulent fluctuations dominate

(iv) Viscous Sublayer

- The wall tends to damp out or prevent turbulent fluctuations, so viscous forces dominate very close to the wall: $\nu >> \varepsilon_M$.

- The Couette Flow Assumption reduces to $\frac{\partial u^+}{\partial y^+} = 1$.

- Integrating, with boundary condition $u^+ = 0$ at $y^+ = 0$ yields:

$$u^+ = y^+, \left(0 \le y^+ \le 7\right) \tag{8.54}$$

- This relation compares well to experimental data from $y^+ \approx 0$ to 7, which is called the *viscous sublayer*.

- Equation (8.54) is illustrated in Figure 8.13. Note that the curvature in the plot results from the semi-logarithmic coordinates.

(v) Fully Turbulent Region: "Law of the Wall"

- Further away from the wall, turbulent fluctuations (i.e., Reynolds stresses) dominate, so $\varepsilon_M >> \nu$.

- The Couette Flow Assumption (8.52) therefore becomes:

$$\frac{\varepsilon_M}{\nu}\frac{\partial u^+}{\partial y^+} = 1 \tag{8.55}$$

- As before, τ is constant; this is the same value as in the viscous sublayer (which is the value at the wall), so discontinuity between regions is avoided.

- Substituting wall coordinates into Prandtl's mixing length theory (8.44) yields:

$$\varepsilon_M = \kappa^2 (y^+)^2 \nu \frac{\partial u^+}{\partial y^+} \tag{8.56}$$

- Equation (8.56) is substituted into (8.55):

$$\kappa^2 (y^+)^2 \left(\frac{\partial u^+}{\partial y^+} \right)^2 = 1$$

- Solving the above for the velocity gradient:

$$\frac{\partial u^+}{\partial y^+} = \frac{1}{\kappa y^+} \tag{8.57}$$

- Integrating the above results in the *Law of the Wall*:

$$u^+ = \frac{1}{\kappa} \ln y^+ + B \tag{8.58}$$

- κ is called *von Kármán's constant*, and experiments show that $\kappa \approx 0.41$.
- B is a constant of integration, and is found as follows:
  - The viscous sublayer and the Law of the Wall region appear to intersect at roughly $y^+ = u^+ \sim 10.8$.
  - Using the above as a boundary condition, $B \approx 5.0$.
- An approximation for the Law of the Wall region is:

$$u^+ = 2.44 \ln y^+ + 5.0, \quad \left(50 \le y^+ \le 1500 \right) \tag{8.59}$$

(vi) Other Models

- The velocity profile presented in the previous sections is a *two-layer model*, and is called the Prandtl-Taylor model.
- *Three-layer models*, like that of von Kármán, also exist.
- van Driest's *continuous law of the wall* is a single equation model that illustrates how single equation models work:
  - The eddy diffusion must diminish as y approaches zero, so van Driest proposed a mixing length model of this form:

$$\ell = \kappa y \left(1 - e^{-y/A} \right) \tag{8.60}$$

  - The term in parentheses is damping factor that makes (8.60) approach zero at the wall.
  - Using (8.60) with (8.42) in (8.51) yields:

$$\frac{\tau_{app}}{\rho} = \left[\nu + \kappa^2 y^2 \left(1 - e^{-y/A} \right)^2 \right] \frac{\partial \overline{u}}{\partial y} \tag{8.61}$$

  - As y approaches zero, the eddy diffusivity approaches zero, leaving pure viscous shear.
  - Transforming (8.61) into wall coordinates, and solving for $\partial u^+ / \partial y^+$, yields:

$$\frac{\partial u^+}{\partial y^+} = \frac{2}{1+\sqrt{1+4\kappa^2 y^{+2}\left(1-e^{-y^+/A^+}\right)^2}} \quad (8.62)$$

- For flow over a smooth, flat plate, van Driest used $\kappa = 0.4$ and $A^+ = 26$.
- When integrated numerically, Equation (8.62) produces the curve shown in Fig. 8.13.

- D.B. Spalding's model is commonly used for both flat plates and pipe flow:

$$y^+ = u^+ + e^{-\kappa B}\left[e^{\kappa u^+} - 1 - \kappa u^+ - \frac{(\kappa u^+)^2}{2} - \frac{(\kappa u^+)^3}{6}\right] \quad (8.63)$$

  - where Spalding used $\kappa = 0.40$ and $B = 5.5$
  - Note that that u^+ is implicit in (8.63), so there is no closed-form solution and one must numerically solve for u^+

- Reichardt developed a profile frequently applied to pipe flow:

$$u^+ = \frac{1}{\kappa}\ln\left(1+\kappa y^+\right) + C\left[1 - e^{-y^+/X} - \frac{y^+}{X}e^{-0.33y^+}\right] \quad (8.64)$$

  - where $\kappa = 0.40$, $C = 7.8$, and $X = 11$.

(vii) Effect of Pressure Gradient

- The velocity profiles modeled up to this point assume the pressure gradient is zero.
- Figure 8.14 depicts how the velocity profile is affected by a pressure gradient.
  - The plot, for flow over a flat plate, shows that in the presence of an adverse pressure gradient, the velocity profile beyond $y^+ \approx 350$ deviates from the Law of the Wall model.
  - The deviation is referred to as a "wake."
  - The region $y^+ > 350$ is commonly referred to as the *wake region*.
  - The region where the data continue to adhere to the Wall Law is called the *overlap region*.
  - The wake increases with adverse pressure gradient, until separation, where the velocity profile deviates even from the overlap region.
  - A slight wake exists for zero or even a strong favorable pressure gradient, although the difference between the two sets of data is negligible.
- Wake models are not addressed in this text.
- The Law of the Wall-type models developed earlier model flat plate flow reasonably well in the presence of zero pressure gradient.
- A favorable pressure gradient is approximately what we encounter in pipe flow; this is one reason why the models developed here apply as well to pipe flow.

8.4.3. Approximate Solution for Momentum Transfer: Momentum Integral Method

- To obtain drag force on the surface of a body, the momentum integral equation is invoked, as in Chapter 5.
- In independent works, both Prandtl and von Kármán used this approach to estimate the friction factor on a flat plate.

(i) Prandtl-von Kármán Model

- Considering a flat, impermeable plate exposed to incompressible, zero-pressure-gradient flow, the integral momentum equation reduces to equation (5.5):

$$\nu \frac{\partial \bar{u}(x,0)}{\partial y} = V_\infty \frac{d}{dx} \int_0^{\delta(x)} \bar{u}\, dy - \frac{d}{dx} \int_0^{\delta(x)} \bar{u}^2 dy \tag{5.5}$$

  - This equation applies to turbulent flows if the behavior of the flow on average is considered, and the flow properties are interpreted as time-averaged values
- Recall that the integral method requires an estimate for the velocity profile in the boundary layer.
- Prandtl and von Kármán used Blasius' model for the shear at the wall of a circular pipe, based on dimensional analysis and experimental data:

$$C_f \approx 0.07910\ Re_D^{-1/4}, \quad \left(4000 < Re_D < 10^5\right) \tag{a}$$

  - where $C_f = 2\tau_o / \rho u_m^2$, and u_m is the mean velocity over the pipe cross-section.
- Based on the above, the velocity profile in the pipe could be modeled as

$$\frac{\bar{u}}{u_{CL}} = \left(\frac{y}{r_o}\right)^{1/7} \tag{b}$$

  - where y is the distance from the wall, and u_{CL} is the centerline velocity
  - This is the well-known *1/7th Law velocity profile*, further discussed in Chapter 9.
- In an external flow a mean velocity is not defined, nor is r_o,, so the following adjustments are made:
  - r_o is approximated by the edge of the boundary layer δ
  - u_{CL} is approximated as representing the free-stream velocity V_∞
  - The velocity profile in the boundary layer is then modeled as:

$$\frac{\bar{u}}{V_\infty} = \left(\frac{y}{\delta}\right)^{1/7} \tag{8.65}$$

- The fundamental problem with using this model in the integral equation is that the velocity profile gradient goes to infinity as *y* goes to zero.
- Equation (8.65) cannot be used directly to estimate the wall shear in equation (5.5).
- Prandtl and von Kármán modeled the wall shear differently:
  - Since the characteristics of the flow near the surface of the plate are similar to that of pipe flow, the Blasius correlation was adapted to find an expression for the wall shear on a flat plate:
  - Recasting (a) in terms of the wall shear and the tube radius:

$$\tau_o = 0.03326 \rho u_m^2 \left(\frac{\nu}{r_o u_m} \right)^{1/4}$$

  - It can be shown that for the 1/7th velocity profile, $u_m = 0.8167 u_{CL}$.
  - Recall, u_{CL} is modeled as V_∞
  - The expression for τ_0 is written as:

$$\frac{\tau_o}{\rho} = \nu \frac{\partial \overline{u}(x,0)}{\partial y} = 0.02333 V_\infty^2 \left(\frac{\nu}{V_\infty \delta} \right)^{1/4} \qquad (8.66)$$

  - In terms the friction factor, this is:

$$\frac{C_f}{2} = \frac{\tau_o}{\rho V_\infty^2} = 0.02333 \left(\frac{V_\infty \delta}{\nu} \right)^{-1/4} \qquad (8.67)$$

  - This can now be used in the momentum equation.
- An example (Example 8.2) is presented in which the expressions for the boundary layer thickness and the friction factor for flow over a flat plate are developed using the 1/7th power law for the velocity profile (8.65) and the expression for friction factor (8.67).
  - The following equations are found within the example:

$$\frac{\tau_o}{\rho V_\infty^2} = \frac{7}{72} \frac{d\delta}{dx} \qquad (8.68)$$

$$\frac{4}{5} \delta^{5/4} = 0.02333 \left(\frac{72}{7} \right) \left(\frac{V_\infty}{\nu} \right)^{-1/4} x + C \qquad (8.69)$$

  - The expression for the boundary layer thickness is:

$$\frac{\delta}{x} = \frac{0.3816}{Re_x^{1/5}} \qquad (8.70)$$

  - The expression for the friction factor is:

$$\frac{C_f}{2} = \frac{0.02968}{Re_x^{1/5}} \tag{8.71}$$

- Note: Refer to Fig. 8.15 for additional considerations for the boundary layer over a flat plate.
- Note that, according to this model, the turbulent boundary layer δ / x varies as $Re_x^{-1/5}$, as does the friction factor C_f; this is contrast to laminar flow, in which δ / x and C_f vary as $Re_x^{-1/2}$.

(ii) Newer Models

- A limitation of the Prandtl-von Kármán model is that the approximation for the wall shear, Eqn. (8.66), is based on limited experimental data, and is of limited applicability even for pipe flow
- White presents a method that makes use of the Law of the Wall velocity profile (8.59):
  - Because the wall coordinates u^+ and y^+ can be expressed as functions of $\sqrt{C_f / 2}$, an expression for the friction factor can be developed from the Law of the Wall (a technique which is also seen in analysis of pipe flow – see Section 9.5).
  - Substituting the definitions of u^+ and y^+, as well as u^*, into the Law of the Wall expression (8.59):

$$\frac{\bar{u}}{V_\infty}\sqrt{\frac{2}{C_f}} = 2.44 \ln\left(\frac{yV_\infty}{\nu}\sqrt{\frac{C_f}{2}}\right) + 5.0$$

  - Any y value within the wall law layer would satisfy this expression, but a useful value to choose is the edge of the boundary layer, where $\bar{u}(y = \delta) = V_\infty$, so:

$$\frac{1}{\sqrt{C_f / 2}} = 2.44 \ln\left(Re_\delta \sqrt{\frac{C_f}{2}}\right) + 5.0 \tag{8.72}$$

    - where $Re_\delta = V_\infty \delta / \nu$

- Equation (8.72) relates the skin friction to the boundary layer thickness, and can be used in the integral momentum equation, though it's cumbersome.
- By curve-fitting values obtained from (8.72) over a range of values from $Re_\delta \approx 10^4$ to 10^7, yielding the approximate relation:

$$C_f \approx 0.02 Re_\delta^{-1/6} \tag{8.73}$$

- The above is used to estimate wall shear using the integral method, along with the 1/7th power law for the velocity profile, resulting in the following solutions to the integral momentum equation:

$$\frac{\delta}{x} = \frac{0.16}{Re_x^{1/7}} \tag{8.74}$$

$$\frac{C_f}{2} = \frac{0.0135}{Re_x^{1/7}} \tag{8.75}$$

- The above equations replace the less accurate Prandtl-von Kármán correlations; White recommends these for general use.
- Kestin and Persen used Spalding's law of the wall for the velocity profile to develop a more-accurate, but more cumbersome correlation; White simplified their model to:

$$C_f = \frac{0.455}{\ln^2(0.06Re_x)} \tag{8.76}$$

- According to White, the above relation is accurate to Kestin and Persen's model to within 1%.

(iii) Total Drag

- Total drag is found by integrating the wall shear along the entire plate.
- **Assumptions**
  - Laminar flow exists along the initial portion of the plate.
  - The plate has width *w*.
- **Formulation**
  - The total drag over the entire plate is:

$$F_D = \int_0^{x_{crit}} (\tau_o)_{lam}\, w dx + \int_{x_{crit}}^{L} (\tau_o)_{turb}\, w dx \tag{8.77}$$

  - Dividing by $\frac{1}{2}\rho V_\infty^2 A = \frac{1}{2}\rho V_\infty^2 wL$, the *drag coefficient* C_D is:

$$C_D = \frac{1}{L}\left[\int_0^{x_{crit}} C_{f,lam} dx + \int_{x_{crit}}^{L} C_{f,turb} dx\right] \tag{8.78}$$

  - Substituting Equation (4.48) for laminar flow and using White's model (8.75) for turbulent flow, we obtain with some manipulation:

$$C_D = \frac{0.0315}{Re_L^{1/7}} - \frac{1477}{Re_L} \tag{8.79}$$

  - The above assumes $x_{crit} = 5\times10^5$.

8.4.4. Effect of Surface Roughness on Friction Factor

- The previous models have all assumed flow over smooth walls.
- The interaction between the turbulent flow and the complex, random geometric features of a rough wall is the subject of advanced study and numerical modeling.
- Crude modeling and experimental study give some physical insight.
- k is defined as the average height of roughness elements on the wall
- k is transformed into wall coordinates: $k^+ = ku^* / \nu$
- For small values of k^+, ($k^+ \leq 5$), experiments show that the velocity profile and friction factor are unaffected by roughness.
  - Roughness is contained within the viscous sublayer; disturbances are likely dampened out by the viscosity-dominated flow.
- For $k^+ > 10$, the roughness extends beyond the viscous sublayer and the viscous sublayer begins to disappear, likely due to enhanced mixing the roughness provides.
- For $k^+ > 70$, viscous effects are virtually eliminated and the flow is *fully rough*.
  - The shape of the velocity profile changes little after this roughness value, so it is expected that increasing the roughness will not change the friction factor.
- Refer to Figure 8.16 for an illustration of how the near-wall velocity profile is affected by roughness.
  - Roughness tends to shift the Law of the Wall down and to the right
  - The shift in velocity profile means that the velocity gradient at the wall is greater, and therefore the friction factor increases
  - The slope of the wall law curve is not affected by roughness
- Correlations for friction factor on rough plates are highly dependent on roughness geometry; there are few models.
- For fully rough flow over sand-rough plates, White [14] suggests the following correlation, based on a wall law velocity profile developed from experimental data

$$C_f = \left[1.4 + 3.7\log_{10}\left(\frac{x}{k}\right)\right]^{-2}, \quad \frac{x}{k} > \frac{Re_x}{1000} \qquad (8.80)$$

  - Note that this correlation does not include the Reynolds number

8.5. Energy Transfer in External Turbulent Flow

- From Chapter 2, the heat transfer for flow over a geometrically similar body like a flat plate can be correlated through dimensionless analysis by

$$Nu_x = f(x^*, Re, Pr) \qquad (2.52)$$

  - where x^* is the dimensionless location along the body

- This neglects buoyancy and viscous dissipation

- The momentum and thermal eddy diffusivities introduced in equations (8.38) and (8.39), ε_M and ε_H, are used to create the *turbulent Prandtl number*, a new dimensionless parameter:

$$Pr_t = \frac{\varepsilon_M}{\varepsilon_H} \tag{8.81}$$

- There are several options to develop suitable models for turbulent heat transfer:
  - Find an analogy between heat and mass transfer
  - Develop a universal temperature profile and then attempt to obtain an approximate solution for heat transfer using the integral method
  - The universal temperature profile may also lend itself to a simple algebraic method for evaluating the heat transfer
  - More advanced methods, including numerical solutions to the boundary layer flow, are not covered in this text

8.5.1. Momentum and Heat Transfer Analogies

- Reynolds theorized that heat transfer and the frictional resistance in a pipe are proportional, based on his study of turbulent flow in steam boilers
- This implies that if the friction along a pipe wall is measured or predicted, heat transfer can be determined by using a multiplying factor, allowing one to solve directly for heat transfer.

(i) Reynolds Analogy

- The Reynolds analogy for external flow is developed:
- **Assumptions**
  - Flow is parallel and is over a flat plate
  - The pressure gradient dp/dx is zero
- **Formulation**
  - The boundary layer momentum and energy equations reduce to:

$$\bar{u}\frac{\partial \bar{u}}{\partial x} + \bar{v}\frac{\partial \bar{u}}{\partial y} = \frac{\partial}{\partial y}\left[(\nu + \varepsilon_M)\frac{\partial \bar{u}}{\partial y}\right] \tag{8.82a}$$

$$\bar{u}\frac{\partial \bar{T}}{\partial x} + \bar{v}\frac{\partial \bar{T}}{\partial y} = \frac{\partial}{\partial y}\left[(\alpha + \varepsilon_H)\frac{\partial \bar{T}}{\partial y}\right] \tag{8.82b}$$

  - The boundary conditions are

$$\bar{u}(y=0) = 0, \quad \bar{T}(y=0) = T_s, \tag{8.83a}$$

$$\bar{u}(y \to \infty) = V_\infty, \qquad \bar{T}(y \to \infty) = T_\infty \tag{8.83b}$$

- The variables are normalized as follows:

$$U = \frac{\bar{u}}{V_\infty}, \; V = \frac{\bar{v}}{V_\infty}, \; \theta = \frac{\bar{T} - T_s}{T_\infty - T_s}, \; X = \frac{x}{L}, \text{ and } Y = \frac{y}{L}$$

- Equations (8.82) and boundary conditions (8.83) become:

$$U\frac{\partial U}{\partial X} + V\frac{\partial U}{\partial Y} = \frac{1}{V_\infty L}\frac{\partial}{\partial Y}\left[(\nu + \varepsilon_M)\frac{\partial U}{\partial Y}\right] \tag{8.84a}$$

$$U\frac{\partial \theta}{\partial x} + V\frac{\partial \theta}{\partial y} = \frac{1}{V_\infty L}\frac{\partial}{\partial Y}\left[(\alpha + \varepsilon_H)\frac{\partial \theta}{\partial Y}\right] \tag{8.84b}$$

$$U(Y = 0) = 0, \qquad \theta(Y = 0) = 0 \tag{8.85a}$$

$$U(Y \to \infty) = 1, \qquad \theta(Y \to \infty) = 1 \tag{8.85b}$$

  - Note that normalizing the boundary conditions has made them identical.
- Equations (8.84) can then be made identical if $(\nu + \varepsilon_M) = (\alpha + \varepsilon_H)$, which is possible under two conditions:
  - The kinematic viscosity and thermal diffusivity are equal:

$$\nu = \alpha \tag{8.86}$$

    - This condition limits the analogy to fluids with $Pr = 1$
    - This also suggests that the velocity and thermal boundary layers are approximately the same thickness, $\delta \approx \delta_t$
  - The eddy diffusivities are equal

$$\varepsilon_M = \varepsilon_H \tag{8.87}$$

    - This is justified by arguing that the same turbulent mechanism—the motion and interaction of fluid particles—is responsible for both momentum and heat transfer; Reynolds essentially made this argument, so equation (8.87) is sometimes referred to as the Reynolds Analogy.
    - This also means that the turbulent Prandtl number Pr_t is equal to 1
- The analogy is now complete; the normalized velocity and temperature profiles, $U(X,Y)$ and $\theta(X,Y)$ are equal.
- Refer to Figure 8.17 for an illustration.

* Derivation of a relationship between the shear stress and heat flux at the wall:
  - The ratio of the apparent heat flux and shear stress (equations. 8.40 and 8.41) is

$$\frac{q''_{app} / \rho c_p}{\tau_{app} / \rho} = -\frac{(\alpha + \varepsilon_H)\partial \overline{T} / \partial y}{(\nu + \varepsilon_M)\partial \overline{u} / \partial y} \tag{8.88}$$

- By imposing conditions (8.86) and (8.87), the terms in parenthesis cancel and, after substituting the dimensionless variables into (8.88):

$$\frac{q''_{app}}{\tau_{app}} = \frac{c_p (T_s - T_\infty)\partial \theta / \partial Y}{V_\infty \partial U / \partial Y} \tag{8.89}$$

- Since the dimensionless velocity and temperature profiles are identical, their derivates cancel
- The ratio q''_{app} / τ_{app} is constant throughout the boundary layer, so this ratio can be represented by the same ratio at the wall , and equation (8.89) becomes:

$$\frac{q''_o}{\tau_o} = \frac{c_p (T_s - T_\infty)}{V_\infty}$$

- This can be recast into a more convenient form by substituting $q''_o = h(T_s - T_\infty)$ and $\tau_o = \frac{1}{2} C_f \rho V_\infty^2$ into the above, and rearranging:

$$\frac{h}{\rho V_\infty c_p} = \frac{C_f}{2}$$

- The terms on the left side can also be written in terms of the Reynolds, Nusselt, and Prandtl numbers

$$St_x \equiv \frac{Nu_x}{Re_x Pr} = \frac{C_f}{2} \tag{8.90}$$

  - where St_x is called the *Stanton number*

- Equation (8.90) is commonly referred to as the *Reynolds Analogy*; it can also be derived for laminar flow over a flat plate for $Pr = 1$.

- The Reynolds Analogy is limited to $Pr = 1$ fluids; it is appropriate for gases, but not for most liquids.

(ii) Prandtl-Taylor Analogy

- The Reynolds analogy doesn't account for the varying intensity of molecular and turbulent diffusion in the boundary layer
- Very close to the wall, molecular forces are expected to dominate:

$$\nu >> \varepsilon_M \text{, and } \alpha >> \varepsilon_H \tag{8.91}$$

- Further away from the wall, turbulent effects dominate:

$$\varepsilon_M >> \nu \text{, and } \varepsilon_H >> \alpha \tag{8.92}$$

- Neither condition above restricts us to $Pr = 1$ fluids.

- Prandtl and Taylor independently divided the boundary layer into two regions:
  - A viscous sublayer where molecular effects (8.91) dominate
  - A turbulent outer layer, where (8.92) is assumed to hold
- In order for an analogy to exist, the momentum and boundary layer equations, and their boundary conditions, must be identical in both regions, so:
  - The viscous sublayer is defined as the portion of the boundary layer beneath $y = y_1$, where y_1 is some threshold value, with boundary conditions:

$$\bar{u}(0) = 0, \quad \bar{T}(0) = T_s,$$

$$\bar{u}(y_1) = \bar{u}_1, \quad \bar{T}(y_1) = \bar{T}_1$$

  - The following normalized variables make the boundary conditions and equation (8.82) identical:

$$U = \frac{\bar{u}}{\bar{u}_1}, \quad V = \frac{\bar{v}}{\bar{u}_1}, \quad \theta = \frac{\bar{T} - T_s}{\bar{T}_1 - T_s}, \quad X = \frac{x}{y_1} \text{ and } Y = \frac{y}{y_1}.$$

  - For the viscous sublayer, the ratio of the apparent heat flux and apparent shear stress (Eqn. 8.89) leads to the following:

$$T_s - \bar{T}_1 = \frac{q''_o}{\tau_o c_p} Pr\, \bar{u}_1 \tag{8.93}$$

    - where $q''_{app} / \tau_{app} = q''_o / \tau_o$ = constant
  - The outer layer closely resembles the Reynolds Analogy, with $\varepsilon_M = \varepsilon_H$ (or $Pr_t = 1$), but this time we assume that the turbulent effects outweigh the molecular effects; this region has boundary conditions:

$$\bar{u}(y_1) = \bar{u}_1, \quad \bar{T}(y_1) = \bar{T}_1,$$

$$\bar{u}(y \to \infty) = V_\infty, \quad \bar{T}(y \to \infty) = T_\infty$$

  - The following normalizing variables make the analogy valid in this region:

$$U = \frac{\bar{u} - \bar{u}_1}{V_\infty - \bar{u}_1}, \quad V = \frac{\bar{v} - \bar{u}_1}{V_\infty - \bar{u}_1}, \quad \theta = \frac{\bar{T} - \bar{T}_1}{\bar{T}_\infty - \bar{T}_1}, \quad X = \frac{x}{L}, \text{ and } Y = \frac{y}{L}$$

  - For the outer region, the ratio of the apparent heat flux and apparent shear stress (equation 8.89) leads to:

$$\bar{T}_1 - T_\infty = \frac{q''_o}{\tau_o c_p} (V_\infty - \bar{u}_1) \tag{8.94}$$

  - The ratio q''_{app} / τ_{app} is constant, so we have chosen the value at y=y_1, which can be represented by q''_o / τ_o.
  - Adding (8.93) and (8.94) yields:

$$T_s - T_\infty = \frac{q_o''}{\tau_o c_p} V_\infty \left[\frac{\bar{u}_1}{V_\infty} (Pr - 1) + 1 \right]$$

- Substituting $\tau_o = \frac{1}{2} C_f \rho V_\infty^2$ into the above yields

$$St = \frac{q_o''}{\rho V_\infty c_p (T_s - T_\infty)} = \frac{C_f / 2}{\left[\frac{\bar{u}_1}{V_\infty} (Pr - 1) + 1 \right]}$$

- The velocity at the edge of the viscous sublayer, $\bar{u}_1$, is estimated using the universal velocity profile (Fig. 8.13); a value of $u^+ = y^+ \approx 5$ is chosen to approximate the edge of the viscous sublayer.
- From the definition of u^+:

$$u^+ = 5 = \frac{\bar{u}_1}{V_\infty} \sqrt{\frac{2}{C_f}} \text{ , or}$$

$$\frac{\bar{u}_1}{V_\infty} = 5 \sqrt{\frac{C_f}{2}} \qquad (8.95)$$

- Thus the *Prandtl-Taylor analogy* is:

$$St_x \equiv \frac{Nu_x}{Re_x Pr} = \frac{C_f / 2}{\left[5 \sqrt{\frac{C_f}{2}} (Pr - 1) + 1 \right]} \qquad (8.96)$$

(iii) von Kármán Analogy

- Theodore von Kármán extended the Reynolds analogy even further to include a third layer – a buffer layer – between the viscous sublayer and outer layer:

$$St_x \equiv \frac{Nu_x}{Re_x Pr} = \frac{C_f / 2}{1 + 5 \sqrt{\frac{C_f}{2}} \left\{ (Pr - 1) + \ln \left[\frac{5 Pr + 1}{6} \right] \right\}} \qquad (8.97)$$

- Note: Refer to Appendix D for development.

(iv) Colburn Analogy

- Colburn proposed a purely empirical modification to the Reynolds analogy that accounts for fluids with varying Prandtl number, using an empirical fit of available experimental data:

$$St_x Pr^{2/3} = \frac{C_f}{2} \qquad (8.98)$$

- The exponent (2/3) is entirely empirical

- The *Colburn Analogy* yields acceptable results for $Re_x < 10^7$ (including the laminar flow regime) and Prandtl number ranging from about 0.5 to 60.
- An example, Example 8.3, is presented in which the average Nusselt number for heat transfer along a flat plate of length L with constant surface temperature is determined, using White's model for turbulent friction factor and assuming the flow over the plate has an initial laminar region.
- Equations developed within the example:
  - The average heat transfer coefficient from Equation (2.50) is split into laminar and turbulent regions:

$$\bar{h}_L = \frac{1}{L}\left[\int_0^{x_c} h_{lam}(x)dx + \int_{x_c}^{L} h_{turb}(x)dx\right] \tag{8.99}$$

  - After several manipulations, substitutions and assumptions, the equation for the average Nusselt number for heat transfer along a flat plate of length L with constant surface temperature and initial laminar region is:

$$\overline{Nu}_L = \left(0.0158 Re_L^{6/7} - 739\right) Pr^{1/3} \tag{8.100}$$

  - If the laminar region had been neglected, the above would be:

$$\overline{Nu}_L = 0.0158 Re_L^{6/7} Pr^{1/3} \tag{8.101}$$

8.5.2. Validity of Analogies

- Momentum-heat transfer analogies are frequently used to develop heat transfer models for many types of flows and geometries.
- Although derived for a flat plate, these analogies are considered generally valid for slender bodies, where pressure gradient does not vary greatly from zero.
- They are approximately valid for internal flows in circular pipes, although other analogies have been developed specifically for internal flow (See Chapter 9).
- Although they are derived assuming constant wall temperature, the above correlations work reasonably well even for constant heat flux.
- The large temperature variation near the wall means that the assumption of uniform properties, particularly for Pr, becomes a weakness; this is overcome by evaluating properties at the film temperature:

$$T_f = \frac{T_s + T_\infty}{2} \tag{8.102}$$

- The analogies were derived assuming that the turbulent Prandtl number is equal to unity.
  - Experimentally-measured values of Pr_t are as high as 3 very near the wall, though outside the viscous sublayer the values range from around 1 to 0.7.

- The turbulent Prandtl number seems to be affected slightly by pressure gradient, though largely unaffected by surface roughness or the presence of boundary layer suction or blowing.
- A value of $Pr_t \approx 0.85$ is considered reasonable for most flows, so the analogies should be approximately valid for real flows.

- Though only an empirical correlation, the Colburn analogy was shown to represent experimental data well over a variety of fluids.
- However, it has been demonstrated that the Colburn analogy under-predicts the Nusselt number by 30-40% for fluids with Prandtl numbers greater than 7.

8.5.3. Universal Turbulent Temperature Profile

- Development of a universal temperature profile in turbulent flow provides physical insight.
- An approximate temperature profile can be used in an integral approach to solve for the heat transfer.

(i) Near Wall Profile

- Beginning with the turbulent energy equation, assume that, near the wall, the velocity component $\overline{v} \sim 0$, as is the temperature gradient $\partial\overline{T}/\partial x$.
  - The left-hand side of (8.39) approaches 0:

$$\frac{\partial}{\partial y}\left[(\alpha+\varepsilon_H)\frac{\partial\overline{T}}{\partial y}\right]\sim 0\text{, near wall}$$

  - This implies that the apparent heat flux is approximately constant with respect to y:

$$\frac{q''_{app}}{\rho c_p}=-(\alpha+\varepsilon_H)\frac{\partial\overline{T}}{\partial y}\sim\text{constant} \qquad (8.103)$$

- Solve the above relation for the temperature profile:
  - Since $q''_{app}/\rho c_p$ is constant throughout this region, replace q''_{app} with q''_o and substitute wall coordinates into (8.103) and rearrange so both sides are dimensionless:

$$-\frac{\partial\overline{T}}{\partial y^+}\frac{\rho c_p u^*}{q''_o}=\frac{\nu}{(\alpha+\varepsilon_H)} \qquad (8.104)$$

  - (8.104) suggests a definition for a temperature wall coordinate:

$$T^+\equiv(T_s-\overline{T})\frac{\rho c_p u^*}{q''_o} \qquad (8.105)$$

  - The above is cast into a simpler form:

$$\frac{\partial T^+}{\partial y^+} = \frac{\nu}{(\alpha + \varepsilon_H)} \quad (8.106)$$

- Though the above can now be integrated, ε_H is unknown, but substituting ε_M into (8.106) by using the definition of the turbulent Prandtl number (8.81) yields:

$$T^+ = \int_0^{y^+} \frac{\nu\, dy^+}{\alpha + \varepsilon_H} \quad (8.107)$$

- This is a general expression for the temperature profile in wall coordinates.
- The above is evaluated by dividing it into two regions, as was done with the universal velocity profile.

(ii) Conduction Sublayer

- Very close to the wall, molecular effects are expected to dominate the heat transfer, so $\alpha >> \varepsilon_H$.
- By evoking this, (8.107) reduces to:

$$T^+ = \int Pr\, dy^+ = Pr\, y^+ + C$$

- The constant of integration, C, is found by applying the boundary condition that $T^+(y^+ = 0) = 0$, resulting in $C = 0$ and a temperature profile in the conduction sublayer of:

$$T^+ = Pr\, y^+, \; (y^+ < y_1^+) \quad (8.108)$$

- y_1^+ is the dividing point between the conduction and outer layers

(iii) Fully Turbulent Region

- Outside the conduction-dominated region close to the wall, turbulent effects dominate, so $\varepsilon_H >> \alpha$
- The temperature profile (8.107) becomes:

$$T^+ = T_1^+ + \int_{y_1^+}^{y^+} \frac{\nu}{\varepsilon_H} dy^+ \quad (8.109)$$

- ε_H must be evaluated to evaluate (8.109); this is done by relating it to the momentum eddy diffusivity:

$$\varepsilon_H = \frac{\varepsilon_M}{Pr_t}$$

- Recall (8.44), the model for the eddy diffusivity from Prandtl's mixing length theory:

$$\varepsilon_M = \kappa^2 y^2 \left| \frac{\partial \overline{u}}{\partial y} \right| \quad (8.44)$$

- This is written in terms of wall coordinates:

$$\varepsilon_M = \kappa^2 (y^+)^2 \nu \frac{\partial u^+}{\partial y^+} \quad (8.110)$$

- The partial derivative $\partial u^+ / \partial y^+$ can be found from the Law of the Wall
- Substituting the above and (8.58) into (8.109) yields:

$$T^+ = \int_{y_1^+}^{y^+} \frac{Pr_t}{\kappa y^+} dy^+ \quad (8.111)$$

- Assume Pr_t and κ are constants, so (8.111) becomes:

$$T^+ = \frac{\mathrm{Pr}_t}{\kappa} \ln\left(\frac{y^+}{y_1^+} \right), \ \left(y^+ > y_1^+ \right) \quad (8.112)$$

- The temperature profile defined by Equations (8.108) and (8.112) depends on the fluid (Pr), as well as the parameters Pr_t and κ.
- Kays et al. assumed $Pr_t = 0.85$ and $\kappa = 0.41$, but found that the thickness of the conduction sublayer (y_1^+) varies by fluid
- White reports a correlation that can be used for any fluid with $Pr \geq 0.7$:

$$T^+ = \frac{Pr_t}{\kappa} \ln y^+ + 13\,\mathrm{Pr}^{2/3} - 7 \quad (8.113)$$

  - In this model, Pr_t is assumed to be approximately 0.9 or 1.0
  - Figure 8.18 is a plot of this model, along with viscous sublayer, for various values of Pr
  - Note that the temperature profile increases with increasing Prandtl number.

(iv) A 1/7th Law for Temperature

- A simpler 1/7th power law relation is sometimes used for the temperature profile:

$$\frac{\overline{T} - T_s}{T_\infty - T_s} = \left(\frac{y}{\delta_t} \right)^{1/7} \quad (8.114)$$

8.5.4. Algebraic Method for Heat Transfer Coefficient

- The existence of a universal temperature and velocity profile makes for a fairly simple method to estimate the heat transfer:
  - The definition of the Nusselt number, expressed using Newton's law of cooling is:

$$Nu_x \equiv \frac{hx}{k} = \frac{q''_o x}{(T_s - T_\infty)k} \tag{8.115}$$

- The universal temperature profile T^+ is to be invoked, so using the definition of T^+, equation (8.105), the free stream temperature is defined:

$$T_\infty^+ = (T_s - T_\infty)\frac{\rho c_p u^*}{q''_o} = (T_s - T_\infty)\frac{\rho c_p V_\infty \sqrt{C_f/2}}{q''_o} \tag{8.116}$$

  - where (8.47) was substituted for the friction velocity u^*
- Substituting (8.116) into (8.115) for $(T_s - T_\infty)$ and rearranging yields:

$$Nu_x = \frac{\rho c_p V_\infty \sqrt{C_f/2}\, x}{T_\infty^+ \, k}$$

- Then, multiplying the numerator and denominator by ν yields:

$$Nu_x = \frac{Re_x Pr \sqrt{C_f/2}}{T_\infty^+} \tag{8.117}$$

- The universal temperature profile, Equation (8.113),is used to evaluate T_∞^+:

$$T_\infty^+ = \frac{Pr_t}{\kappa}\ln y_\infty^+ + 13Pr^{2/3} - 7 \tag{8.118}$$

- A precise value for y_∞^+ is not easy to determine, but a clever substitution is made by using the Law of the Wall velocity profile.
- Equation (8.58) is evaluated in the free stream as:

$$u_\infty^+ = \frac{1}{\kappa}\ln y_\infty^+ + B \tag{8.119}$$

- Substituting (8.119) into (8.118) for $\ln y_\infty^+$, the Nusselt number relation then becomes

$$Nu_x = \frac{Re_x Pr \sqrt{C_f/2}}{Pr_t(u_\infty^+ - B) + 13Pr^{2/3} - 7}$$

- The above is simplified using the definition of Stanton number, $St = Nu_x/(Re_x Pr)$, selecting $B = 5.0$ and $Pr_t = 0.9$, and noting that the definition of u^+ leads to $u_\infty^+ = \sqrt{2/C_f}$

$$St_x = \frac{C_f/2}{0.9 + 13\left(Pr^{2/3} - 0.88\right)\sqrt{C_f/2}} \tag{8.120}$$

- Note how similar this result is to the more advanced momentum-heat transfer analogies, particularly those by Prandtl and Taylor (8.96) and von Kármán (8.97).

8.5.5. Integral Methods for Heat Transfer Coefficient

- The universal temperature profile allows us to model heat transfer using the integral energy equation.
- A case of turbulent flow over a flat plate where a portion of the leading surface is unheated is examined, illustrated in Figure 8.19.
  - The simplest solution is to assume the $1/7^{th}$ power law for both the velocity and temperature profiles
  - This is mathematically cumbersome, and is developed in Appendix E.
  - The result of the analysis is:

$$St_x \equiv \frac{Nu_x}{Re_x Pr} = \frac{C_f}{2}\left[1 - \left(\frac{x_o}{x}\right)^{9/10}\right]^{-1/9} \quad (8.121)$$

    - where x_o is the unheated starting length
- Note that Equation (8.121) reduced to the Reynolds Analogy when $x_o = 0$, (the Prandtl number in that derivation was assumed to be 1).
- The model has been used to approximate heat transfer for other fluids:
  - Equation (8.121) is expressed as:

$$Nu_x = \frac{Nu_{x_o=0}}{\left[1 - (x_o / x)^{9/10}\right]^{1/9}} \quad (8.122)$$

  - where $Nu_{x_o=0}$ represents the heat transfer in the limit of zero insulated starting length
  - In this form, other models for heat transfer, like von Kármán's analogy, could be used to approximate $Nu_{x_o=0}$ for $Pr \neq 1$ fluids.

8.5.6. Effect of Surface Roughness on Heat Transfer

- Surface roughness is important in such applications as turbomachinery, where rough surfaces can be used to enhance heat transfer in the cooling of turbine blades.
- Figure 8.16 shows that the viscous sublayer diminishes and disappears as roughness increases, implying that the turbulent fluid elements exchange momentum with the surface directly (i.e. *profile* or *pressure drag*) and the role of molecular diffusion (i.e. *skin friction*) is diminished.
- Heat transfer relies on molecular conduction at the surface, no matter how rough the surface, or how turbulent the flow.
- Fluid in the spaces between roughness elements is largely stagnant, and transfers heat entirely by molecular conduction.

- The conduction sublayer can be viewed as the average height of the roughness elements.
- The major resistance to heat transfer is formed by the stagnant regions between roughness elements

- Roughness cannot improve heat transfer as much as it increases friction.
  - This means that we cannot predict the heat transfer by using a friction factor for rough plates along with one of the momentum-heat transfer analogies
- In developing a model for heat transfer on a rough surface, the following are expected:
  - Roughness size has no influence until it extends beyond the viscous and conduction sublayers.
  - The influence of roughness reaches a maximum beyond some roughness size (the *fully rough* limit).
  - The Prandtl number should be present in any model, and fluids with higher Prandtl number (lower conductivity) would be affected more by roughness
    - In these fluids, lower-conductivity fluid trapped between the roughness elements will have a higher resistance to heat transfer.
    - The conduction sublayer is shorter for these fluids, so roughness elements penetrate relatively further into the thermal boundary layer
- A correlation for a rough plate was developed by Kays, et al. [29]:

$$St = \frac{C_f}{2}\left[Pr_t + C(k_s^+)^{0.2} Pr^{0.44} \sqrt{C_f/2}\right]^{-1} \tag{8.123}$$

  - where $k_s^+ = k_s u^* / \nu$ is based on the equivalent sand-grain roughness, k_s, and C is a constant that depends on roughness geometry
- This model displays the expected behavior.
  - For high-Prandtl-number fluids the second term in the parentheses dominates
  - For sufficiently low-Pr fluids the second term diminishes, in spite of roughness size
  - Bogard, et al. showed a 50% increase in heat transfer on rough turbine blades compared to heat transfer in smooth turbine blades, and that increasing roughness beyond some value showed little increase in the heat transfer, which is consistent with the above expectations.