

Chapter 3: Turbulence and its modeling

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Is the Flow Turbulent?

External Flows

$$Re_x \geq 5 \times 10^5 \quad \text{along a surface}$$

$$Re_D \geq 20,000 \quad \text{around an obstacle}$$

Internal Flows

$$Re_{D_h} \geq 2,300$$

Natural Convection

$$Ra \geq 10^9 \text{ Pr}$$

where $Re_L \equiv \frac{\rho UL}{\mu}$

$$L = x, D, D_h, \text{ etc.}$$

Other factors such as free-stream turbulence, surface conditions, and disturbances may cause earlier transition to turbulent flow.

where $Ra = \frac{g\beta\Delta TL^3}{\alpha\nu}$

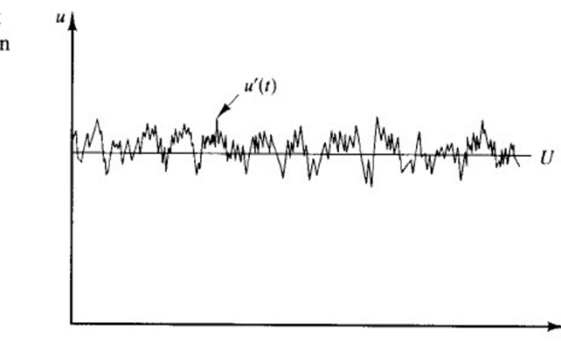
is the Rayleigh number

Introduction

- For $Re > Re_{critical} \Rightarrow$ Flow becomes **Turbulent**.
- Chaotic and random state of motion develops.
- Velocity and pressure change continuously with time
- The velocity fluctuations give rise to additional stresses on the fluid
 \Rightarrow these are called **Reynolds stresses**
- We will try to model these extra stress terms

3.1 What is turbulence?

Fig. 3.1 Typical point velocity measurement in turbulent flow



$$u(t) = U + u'(t)$$

Instantaneous Velocity

Mean Value

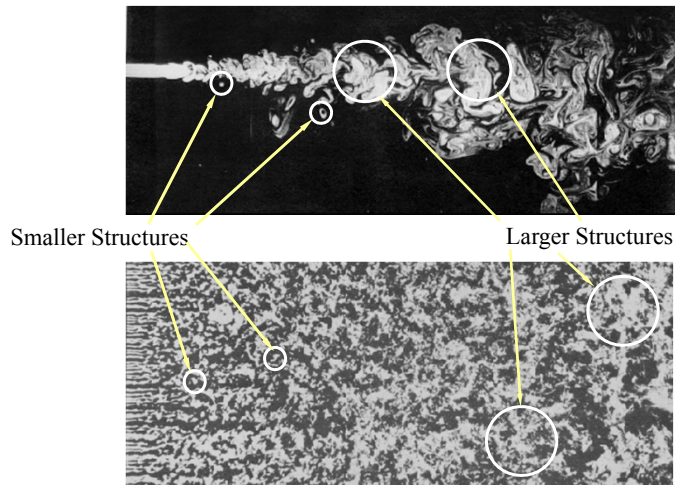
Fluctuating Component

1D,2D or 3D

Always 3D

Two Examples of Turbulent Flow

In turbulent flows there are rotational flow structures called turbulent eddies, which have a wide range of length scales.



■ In turbulent flow:

A streak of dye which is introduced at a point will rapidly break up and dispersed → effective mixing



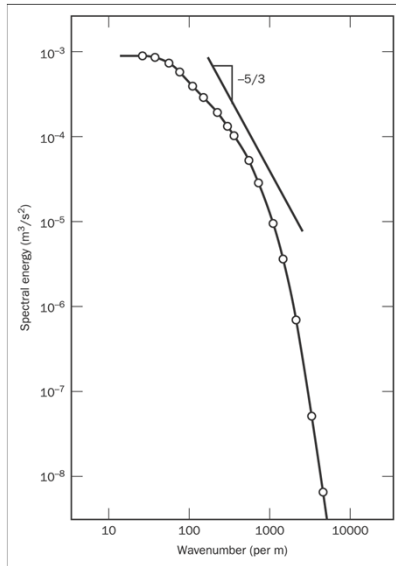
Give rise to high values of diffusion coefficient for;

- Mass
- Momentum
- and heat

All fluctuating components contain energy across a wide range of frequencies or wave numbers

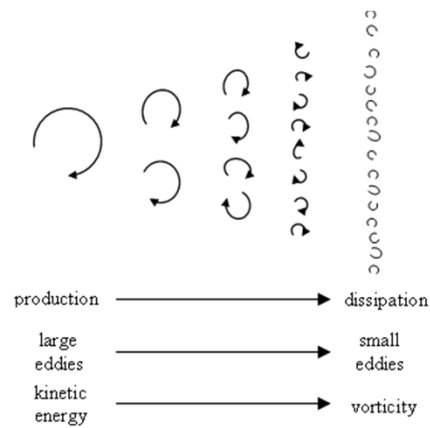
$$\text{wave number} = \frac{2\pi f}{U} \quad f = \text{frequency}$$

Fig.3.3 Energy Spectrum of turbulence behind a grid



Scales of turbulence

- Largest eddies break up due to inertial forces
- Smallest eddies dissipate due to viscous forces
- Richardson Energy Cascade (1922)



■ **Large Eddies:**

- have large eddy Reynolds number, vl/ν
- are dominated by inertia effects
- viscous effects are negligible
- are effectively inviscid

■ **Small Eddies:**

- motion is dictated by viscosity
- $Re \approx 1$
- length scales : 0.1 – 0.01 mm
- frequencies : ≈ 10 kHz

■ **Energy associated with eddy motions is dissipated and converted into thermal internal energy**

- increases energy losses.
- Largest eddies → anisotropic
- Smallest eddies → isotropic

3.2 Transition from laminar to turbulent flow

■ Can be explained by considering the stability of laminar flows to small disturbances

→ Hydrodynamic instability

■ **Hydrodynamic stability of laminar flows;**

a) **Inviscid instability:** flows with velocity profile having a point of inflection.

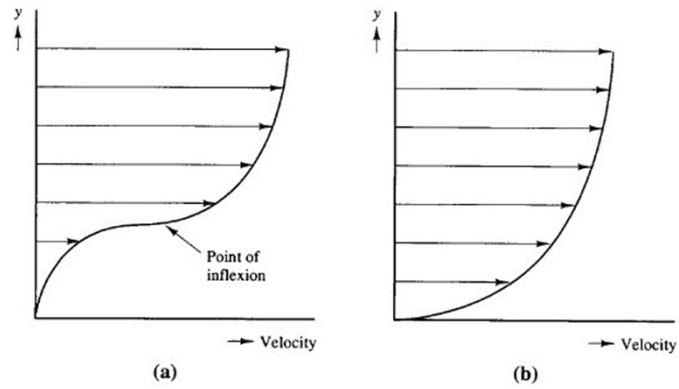
1. jet flows
2. mixing layers and wakes
3. boundary layers with adverse pressure gradients

b) **Viscous instability:** flows with laminar profile having no point of inflection

- occurs near solid walls

Fig. 3.4 Velocity profiles susceptible to

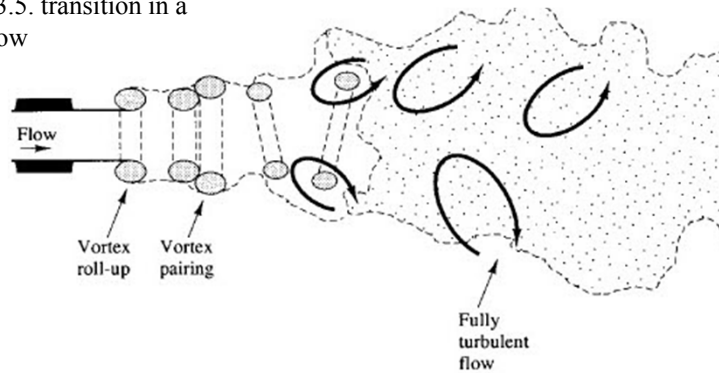
- a) Inviscid instability and
- b) Viscous instability



■ Transition to Turbulence

Jet flow: (example of a flow with point of inflection in velocity profile)

Fig. 3.5. transition in a jet flow



■ **Boundary layers on a flat plate** (Example with no inflection point in the velocity profile)

The unstable two-dimensional disturbances are called Tolmien-Schlichting (T-S) waves

Fig. 3.6 plan view sketch of transition processes in boundary layer flow over a flat plate

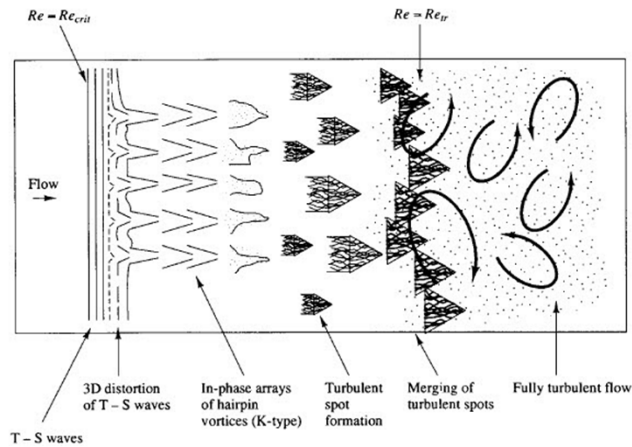
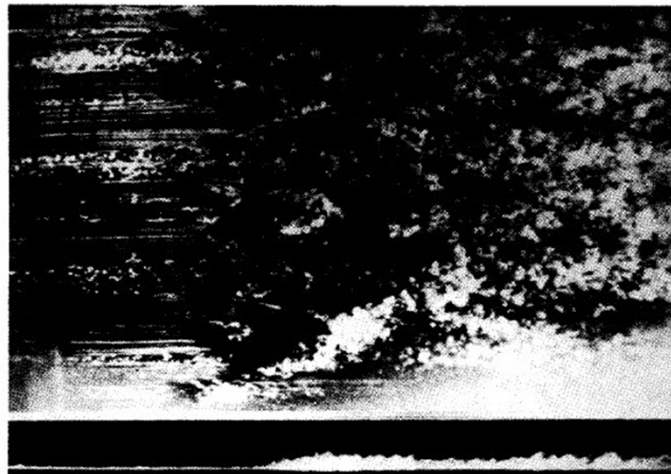


Fig. 3.7 merging of turbulent spots and transition to turbulence in a natural flat plate boundary layer



■ **Common features in the transition process:**

- (i) The amplification of initially small disturbances
- (ii) The development of areas with concentrated rotational structures
- (iii) The formation of intense small scale motions
- (iv) The growth and merging of these areas of small scale motions into fully turbulent flows

Transition to turbulence is strongly affected by:

- Pressure gradient
- Disturbance levels
- Wall roughness
- Heat transfer

The transition region often comprises only a very small fraction of the flow domain

→ Commercial CFD packages often ignore transition entirely (classify the flow as only laminar or turbulent)

3.3 Effect of turbulence on time-averaged Navier-Stokes eqns

- In turbulent flow there are eddying motions of a wide range of length scales
- A domain of 0.1×0.1 m contains smallest eddies of 10-100 μm size
- We need $10^9 - 10^{12}$ mesh points
- The frequency of fastest events ≈ 10 kHz $\rightarrow \Delta t \approx 100 \mu\text{s}$ needed
- DNS of turbulent pipe flow of $\text{Re} = 10^5$ requires a computer which is 10 million times faster than CRAY supercomputer
- Engineers need only time-averaged properties of the flow
- Let's see how turbulent fluctuations effect the mean flow properties

Descriptors of Turbulent Flow

Time average or mean

First we define the mean Φ of a flow property φ as follows:

$$\Phi = \frac{1}{\Delta t} \int_0^{\Delta t} \varphi(t) dt \quad (3.2)$$

The property φ can be thought of as the sum of a steady mean component Φ and fluctuation component φ'

$$\varphi(t) = \Phi + \varphi'(t)$$

The time average of the fluctuations φ' is, by definition, zero:

$$\overline{\varphi'} = \frac{1}{\Delta t} \int_0^{\Delta t} \varphi'(t) dt \equiv 0 \quad (3.3)$$

Variance, r.m.s. and turbulence kinetic energy

The spread of the fluctuations φ' about the mean Φ are measured by

variance
$$\overline{(\varphi')^2} = \frac{1}{\Delta t} \int_0^{\Delta t} (\varphi')^2 dt \quad (3.4a)$$

and root mean square (r.m.s.)
$$\varphi_{rms} = \sqrt{\overline{(\varphi')^2}} = \left[\frac{1}{\Delta t} \int_0^{\Delta t} (\varphi')^2 dt \right]^{1/2} \quad (3.4b)$$

The rms values of velocity components can be measured (e.g. by hot-wire anemometer)

The kinetic energy k (per unit mass) associated with the turbulence is defined as

$$k = \frac{1}{2} (\overline{u'^2} + \overline{v'^2} + \overline{w'^2}) \quad (3.5)$$

The turbulence intensity T_i is linked to the kinetic energy and a reference mean flow velocity U_{ref} as follows:

$$T_i = \frac{(2/3k)^{1/2}}{U_{ref}} \quad (3.6)$$

Moments of different fluctuating variables

The variance is also called the second moment of the fluctuations.

If $\varphi = \Phi + \varphi'$ and $\psi = \Psi + \psi'$ with $\overline{\varphi'} = \overline{\psi'} = 0$

Their **second moment** is defined as

$$\overline{\varphi'\psi'} = \frac{1}{\Delta t} \int_0^{\Delta t} \varphi'\psi' dt \quad (3.7)$$

If velocity fluctuations in different directions were independent random fluctuations, then $\overline{u'v'}$, $\overline{u'w'}$ and $\overline{v'w'}$ would be equal to zero.

However, although u' , v' and w' are chaotic, they are not independent. As a result the second moments $\overline{u'v'}$, $\overline{u'w'}$ and $\overline{v'w'}$ are non-zero.

Higher order moments

Third moment:

$$\overline{(\varphi')^3} = \frac{1}{\Delta t} \int_0^{\Delta t} (\varphi')^3 dt \quad (3.8)$$

Third moment is related to skewness (asymmetry) of the distribution of the fluctuations:

Fourth moment:

$$\overline{(\varphi')^4} = \frac{1}{\Delta t} \int_0^{\Delta t} (\varphi')^4 dt \quad (3.9)$$

Fourth moment is related to kurtosis (peakedness) of the distribution of the fluctuations:

Correlation functions – time and space

The **autocorrelation** function $R_{\varphi'\varphi'}(\tau)$ based on two measurements shifted by time τ is defined as

$$R_{\varphi'\varphi'}(\tau) = \overline{\varphi'(t)\varphi'(t+\tau)} = \frac{1}{\Delta t} \int_0^{\Delta t} \varphi'(t)\varphi'(t+\tau)dt \quad (3.10)$$

Similarly, the **autocorrelation** function $R_{\varphi'\varphi'}(\xi)$ based on two measurements shifted by a certain distance in function space is defined as

$$R_{\varphi'\varphi'}(\xi) = \overline{\varphi'(\mathbf{x},t)\varphi'(\mathbf{x}+\xi,t)} = \frac{1}{\Delta t} \int_t^{t+\Delta t} \varphi'(\mathbf{x},t')\varphi'(\mathbf{x}+\xi,t')dt' \quad (3.11)$$

When $\tau = 0$, or $\xi = 0$ $R_{\varphi'\varphi'}(0) = \overline{\varphi'^2}$ and it is maximum because the two correlations are perfectly correlated.

Since φ' is chaotic we expect that the fluctuations become increasingly decorrelated as τ or $\xi \rightarrow \infty$ so, $R_{\varphi'\varphi'}(\infty) \rightarrow 0$.

The eddies at the root of turbulence cause a certain degree of local structure in the flow, so there will be a correlation between the values of φ' at time t and a short time later or at a given location \mathbf{x} and a small distance away.

The decorrelation process will take place gradually over the lifetime (or size scale) of a typical eddy.

The integral time and length scale represent concrete measures of the average period or size of a turbulent eddy. They can be computed from integrals of the autocorrelation functions $R_{\varphi'\varphi'}(\tau)$ or $R_{\varphi'\varphi'}(\xi)$.

By analogy it is also possible to define cross-correlation functions $R_{\varphi'\psi'}(\tau)$ or $R_{\varphi'\psi'}(\xi)$ between pairs of different fluctuations.

Probability density function

Probability density function $P(\varphi^*)$ is related to the fraction of time that a fluctuating signal spends between φ^* and $\varphi^* + d\varphi^*$:

$$P(\varphi^*)d\varphi^* = \text{Prob}(\varphi^* < \varphi < \varphi^* + d\varphi^*) \quad (3.12)$$

The **average**, **variance** and **higher moments** of the variables and its fluctuations are related to the probability density functions as follows:

$$\bar{\varphi} = \int_{-\infty}^{\infty} \varphi P(\varphi) d\varphi \quad (3.13a)$$

$$\overline{(\varphi')^n} = \int_{-\infty}^{\infty} (\varphi')^n P(\varphi') d\varphi' \quad (3.13b)$$

If $n = 2 \rightarrow$ we get variance, if $n = 3, 4, \dots$ we get higher order moments.

3.4. Characteristics of simple turbulent flows

In turbulent thin shear layers,

$$\frac{\partial}{\partial x} \ll \frac{\partial}{\partial y}$$

$$\frac{\delta}{L} \ll 1$$

x \rightarrow Flow Direction

y \rightarrow cross - Stream

δ \rightarrow Boundary layer thickness

L \rightarrow length scale in x direction

The following 2D incompressible turbulent flows with constant P will be considered:

■ Free turbulent flows

- Mixing layers
- Jet
- Wake

■ Boundary layer near solid walls

- Flat plate boundary layer
- Pipe flow

Data for U , $-\overline{\rho u'^2}$, $-\overline{\rho v'^2}$, $-\overline{\rho w'^2}$ and $-\overline{\rho u'v'}$ will be reviewed.

These can be measured by 1) Hot wire anemometry 2) Laser doppler anemometers

3.4.1 Free turbulent flows

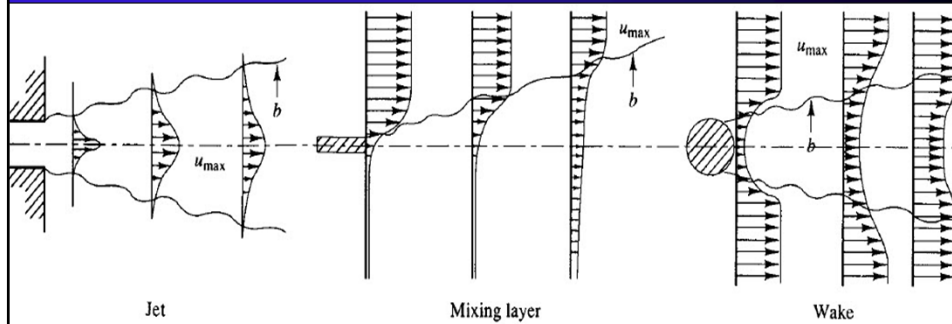


Fig. 3.8 Free turbulent flows

$$\frac{U}{U_{\max}} = g\left(\frac{y}{b}\right)$$

for jets

$$\frac{U - U_{\min}}{U_{\max} - U_{\min}} = f\left(\frac{y}{b}\right)$$

for mixing layers

$$\frac{U_{\max} - U}{U_{\max} - U_{\min}} = h\left(\frac{y}{b}\right)$$

for wakes

b = cross sectional layer width, y = distance in cross-stream direction
 x = distance downstream the source



- Eddies of a wide range of length scales are visible
- Fluid from surroundings is **entrained** into the jet.

- If x is large enough \rightarrow functions f, g and h are independent of x . \rightarrow such flows are called **self-preserving**.

The turbulence structure also reaches a self-preserving state albeit after a greater distance from the flow source than the mean velocity. Then

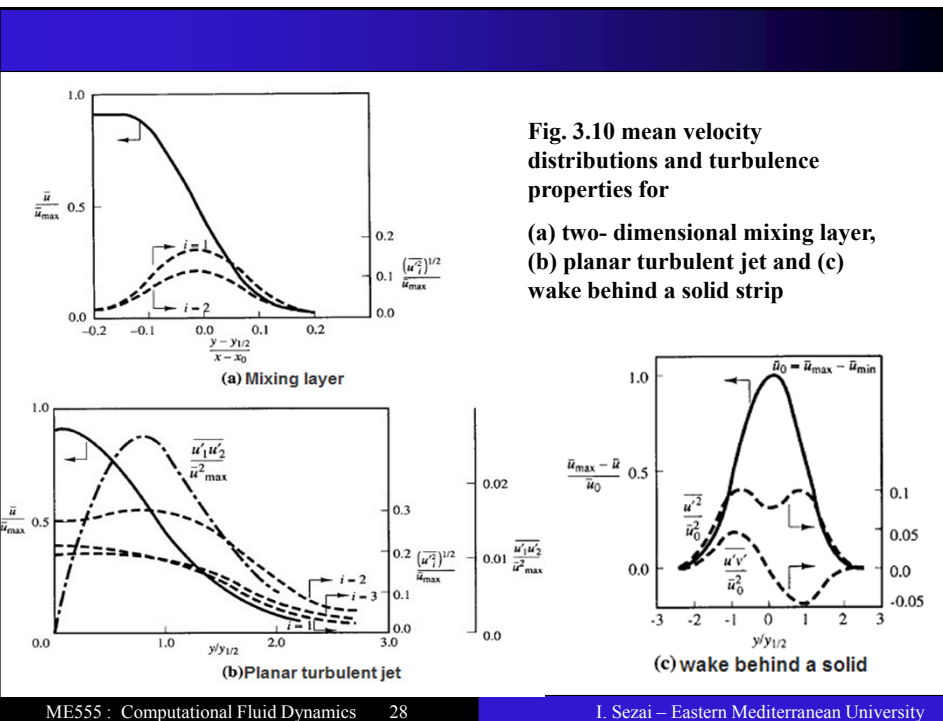
$$\frac{\overline{u'^2}}{U_{ref}^2} = f_1\left(\frac{y}{b}\right) \quad \frac{\overline{v'^2}}{U_{ref}^2} = f_2\left(\frac{y}{b}\right) \quad \frac{\overline{w'^2}}{U_{ref}^2} = f_3\left(\frac{y}{b}\right) \quad \frac{\overline{u'v'}}{U_{ref}^2} = f_4\left(\frac{y}{b}\right)$$

$U_{ref} = U_{max} - U_{min}$ for a **mixing layer** and **wakes**

$U_{ref} = U_{max}$ for jets

- Form of functions f, g, h and f_i varies from flow to flow

\rightarrow See Fig.3.10



- In Fig.3.10:
- $\overline{u'^2}$, $\overline{v'^2}$ and $\overline{w'^2}$ and $-\overline{u'v'}$ are maximum when $\frac{\partial u}{\partial y}$ is maximum
- u' gives the largest of the normal stresses (v' and w').
- $\frac{\overline{u'^2}}{u_{\max}^2} = 0.15 - 0.40$
- Fluctuating velocities are not equal → anisotropic structure of turbulence
- As mean velocity gradients tend to zero → turbulence quantities tend to zero → turbulence can't be sustained in absence of shear
- The mean velocity gradient is also zero at the centerline of **jets** and **wakes**. → No turbulence there.
- The value of $-\overline{u'v'}$ is zero at the centerline of a **jet** and **wake** since shear stress must change sign here.

3.4.2 Flat plate boundary and pipe flow

$$\text{Re} = \frac{\text{inertia forces}}{\text{viscous forces}}$$

$$\text{Re} = \frac{Uy}{\nu} \quad y: \text{distance away from the wall}$$

Near the wall (y small) Re_y is small → viscous forces dominate

Away from the wall (y large) Re_y is large → inertia forces dominate.

Near the wall U only depends on y , ρ , μ and τ (wall shear stress), so

$$U = f(y, \rho, \mu, \tau_w)$$

Dimensional analysis shows that

$$u^+ = \frac{U}{u_\tau} = f\left(\frac{\rho u_\tau y}{\mu}\right) = f(y^+) \quad (3.16)$$

Formula (3.18) is called the **law of the wall**

$$u_\tau = (\tau_w / \rho)^{1/2} = V_\infty (C_f / 2)^{1/2} \quad \text{friction velocity}$$

Far away from the wall:

$$U = g(y, \delta, \rho, \tau_w) \quad \text{independent of } \mu$$

Dimensional analysis yields

$$u^+ = \frac{U}{u_\tau} = g\left(\frac{y}{\delta}\right)$$

The most useful form emerges if we view the wall shear stress as the cause of a velocity deficit $U_{max} - U$ which decreases the closer we get to the edge of the boundary layer or the pipe centerline. Thus

$$\frac{U_{max} - U}{u_\tau} = g\left(\frac{y}{\delta}\right) \quad (3.17)$$

This formula is called the **velocity – defect law**.

Linear sub layer – the fluid layer in contact with smooth wall

- Very near the wall there is no turbulent (Reynolds) shear stresses
→ flow is dominated by viscous shear

For $(y^+ < 5)$ shear stress is approximately constant,

$$\tau(y) = \mu \frac{\partial U}{\partial y} \cong \tau_w$$

Integrating and using $U = 0$ at $y = 0$,

$$U = \frac{\tau_w y}{\mu}$$

After some simple algebra and making use of the definitions of u^+ and y^+ this leads to

$$u^+ = y^+ \quad (3.18)$$

Because of the linear relationship between velocity and distance from the wall the fluid layer adjacent to the wall is often known as the **linear sub-layer**

Log-law layer – the turbulent region close to smooth wall

Outside the viscous sublayer ($30 < y^+ < 500$) a region exists where viscous and turbulent effects are both important.

The shear stress τ varies slowly with distance; it is assumed to be constant and equal to τ_w

Assuming a length scale of turbulence $l_m = \kappa y$ where l_m is mixing length, the following relationship can be derived:

$$u^+ = \frac{1}{\kappa} \ln y^+ + B = \frac{1}{\kappa} \ln(Ey^+) \quad (3.19)$$

$\kappa = 0.4$, $B = 5.5$, $E = 9.8$; (for smooth walls)

B decreases with roughness

κ and B are universal constants valid for all turbulent flows past smooth walls at high Reynolds numbers.

($30 < y^+ < 500$) \rightarrow log – law layer

Outer layer – the inertia dominated region far from the wall

Experimental measurements show that the log-law is valid in the region $0.02 < y/\delta < 0.2$.

For larger values of y the velocity-defect law (3.19) provides the correct form.

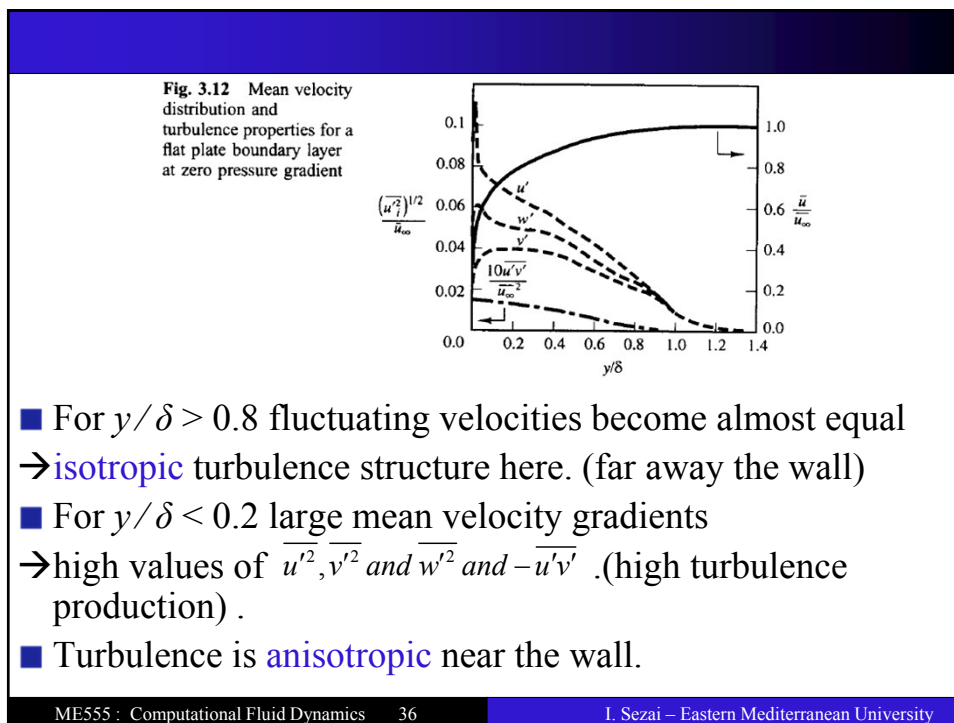
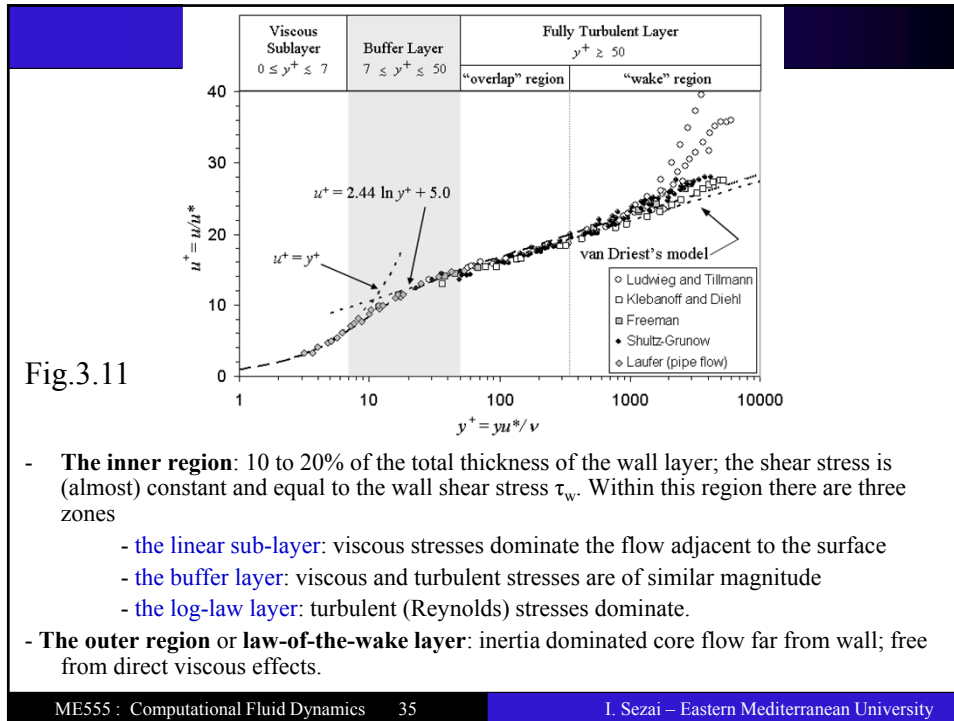
In the overlap region the log-law and velocity-defect law have to become equal.

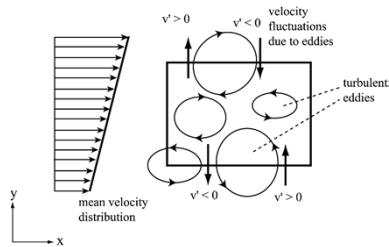
Tennekes and Lumley (1972) show that a matched overlap is obtained by assuming the following logarithmic form:

$$\frac{U_{\max} - U}{u_\tau} = \frac{1}{\kappa} \ln\left(\frac{y}{\delta}\right) + A \quad \text{Law of the Wake} \quad (3.20)$$

where A is constant

See Fig.3.11





Control volume in a 2D turbulent shear flow

Fluid entering from top into the CV will bring in higher momentum fluid and will accelerate the slower moving layer. As a result, the fluid layer will experience additional turbulent shear stresses which are known as the **Reynolds stresses**.

Rules which govern the time averages of fluctuation properties

$$\varphi = \Phi + \varphi' \quad \text{and} \quad \psi = \Psi + \psi' \quad :$$

$$\begin{aligned} \overline{\varphi'} = \overline{\psi'} = 0; & \quad \overline{\Phi} = \Phi; & \quad \frac{\partial \overline{\varphi}}{\partial s} = \frac{\partial \Phi}{\partial s}; & \quad \overline{\int \varphi ds} = \int \Phi ds \\ \overline{\varphi + \psi} = \Phi + \Psi; & \quad \overline{\varphi\psi} = \Phi\Psi + \overline{\varphi'\psi'}; & \quad \overline{\varphi\Psi} = \Phi\Psi; & \quad \overline{\varphi^{\prime n}\Psi} = 0 \end{aligned}$$

Since *div* and *grad* are both differentiations the above rules can be extended to a fluctuating vector quantity $\mathbf{a} = \mathbf{A} + \mathbf{a}'$ and its combinations with a fluctuating scalar $\varphi = \Phi + \varphi'$:

$$\overline{div \mathbf{a}} = div \mathbf{A}; \quad \overline{div(\varphi \mathbf{a})} = div(\overline{\varphi \mathbf{a}}) = div(\Phi \mathbf{A}) + div(\overline{\varphi' \mathbf{a}'}); \quad (3.22)$$

$$\overline{div grad \varphi} = div grad \Phi$$

To illustrate the influence of turbulent fluctuations on mean flow we consider

$$div \mathbf{u} = 0 \quad (3.23)$$

$$\frac{\partial u}{\partial t} + div(\mathbf{u}\mathbf{u}) = -\frac{1}{\rho} \frac{\partial p}{\partial x} + \nu div grad u \quad (3.24a)$$

$$\frac{\partial v}{\partial t} + div(\mathbf{v}\mathbf{u}) = -\frac{1}{\rho} \frac{\partial p}{\partial y} + \nu div grad v \quad (3.24b)$$

$$\frac{\partial w}{\partial t} + div(\mathbf{w}\mathbf{u}) = -\frac{1}{\rho} \frac{\partial p}{\partial z} + \nu div grad w \quad (3.24c)$$

Substitute $\mathbf{u} = \mathbf{U} + \mathbf{u}'$; $u = U + u'$; $v = V + v'$; $w = W + w'$; $p = P + p'$

Consider continuity equation: note that $\overline{div \mathbf{u}} = div \mathbf{U}$ this yields the continuity equation

$$div \mathbf{U} = 0 \quad (3.25)$$

A similar process is now carried out on the x -momentum equation (3.9a). The time averages of the individual terms in this equation can be written as follows:

$$\begin{aligned} \overline{\frac{\partial u}{\partial t}} &= \frac{\partial U}{\partial t}; & \overline{div(\mathbf{u}\mathbf{u})} &= div(\mathbf{U}\mathbf{U}) + div(\overline{u'\mathbf{u}'}) \\ -\frac{1}{\rho} \overline{\frac{\partial p}{\partial x}} &= -\frac{1}{\rho} \frac{\partial P}{\partial x}; & \overline{v div grad u} &= v div grad U \end{aligned}$$

Substitution of these results gives the time-average x -momentum equation

$$\begin{aligned} \overline{\frac{\partial u}{\partial t}} + \overline{div(\mathbf{u}\mathbf{u})} + \overline{div(u'\mathbf{u}')} &= -\frac{1}{\rho} \overline{\frac{\partial p}{\partial x}} + v \overline{div grad u} \\ (I) \quad (II) \quad (III) \quad (IV) \quad (V) & \end{aligned} \quad (3.26a)$$

Repetition of this process on equations (3.9b) and (3.9c) yields the time-average y -momentum and z -momentum equations

$$\begin{aligned} \overline{\frac{\partial V}{\partial t}} + \overline{div(V\mathbf{U})} + \overline{div(v'\mathbf{u}')} &= -\frac{1}{\rho} \overline{\frac{\partial p}{\partial y}} + v \overline{div grad V} \\ (I) \quad (II) \quad (III) \quad (IV) \quad (V) & \end{aligned} \quad (3.26b)$$

$$\begin{aligned} \overline{\frac{\partial W}{\partial t}} + \overline{div(W\mathbf{U})} + \overline{div(w'\mathbf{u}')} &= -\frac{1}{\rho} \overline{\frac{\partial p}{\partial z}} + v \overline{div grad W} \\ (I) \quad (II) \quad (III) \quad (IV) \quad (V) & \end{aligned} \quad (3.26c)$$

- The process of time averaging has introduced new terms (III).
- Their role is additional turbulent stresses on the mean velocity components U , V and W .

■ Placing these terms on the rhs:

$$\frac{\partial U}{\partial t} + \text{div}(U\mathbf{U}) = -\frac{1}{\rho} \frac{\partial P}{\partial x} + \nu \text{div grad } U + \left[-\frac{\overline{\partial u'^2}}{\partial x} - \frac{\overline{\partial u'v'}}{\partial y} - \frac{\overline{\partial u'w'}}{\partial z} \right] \quad (3.27a)$$

Reynolds equations ←

$$\frac{\partial V}{\partial t} + \text{div}(V\mathbf{U}) = -\frac{1}{\rho} \frac{\partial P}{\partial y} + \nu \text{div grad } V + \left[-\frac{\overline{\partial u'v'}}{\partial x} - \frac{\overline{\partial v'^2}}{\partial y} - \frac{\overline{\partial v'w'}}{\partial z} \right] \quad (3.27b)$$

$$\frac{\partial W}{\partial t} + \text{div}(W\mathbf{U}) = -\frac{1}{\rho} \frac{\partial P}{\partial z} + \nu \text{div grad } W + \left[-\frac{\overline{\partial u'w'}}{\partial x} - \frac{\overline{\partial v'w'}}{\partial y} - \frac{\overline{\partial w'^2}}{\partial z} \right] \quad (3.27c)$$

The extra stress terms result from six additional stresses, three normal stresses and three shear stresses:

$$\tau_{xx} = -\overline{\rho u'^2} \quad \tau_{yy} = -\overline{\rho v'^2} \quad \tau_{zz} = -\overline{\rho w'^2} \quad (3.28a)$$

$$\tau_{xy} = \tau_{yx} = -\overline{\rho u'v'} \quad \tau_{xz} = \tau_{zx} = -\overline{\rho u'w'} \quad \tau_{yz} = \tau_{zy} = -\overline{\rho v'w'} \quad (3.28b)$$

-These extra turbulent stresses are termed the **Reynolds stresses**.

-The normal stresses $-\overline{\rho u'^2}$, $-\overline{\rho v'^2}$ and $-\overline{\rho w'^2}$ are always non-zero

-The shear stresses $-\overline{\rho u'v'}$, $-\overline{\rho u'w'}$ and $-\overline{\rho v'w'}$ are also non-zero

If, for example, u' and v' were statistically independent fluctuations, the time average of their product $u'v'$ would be zero

Similar extra turbulent transport terms arise when we derive a transport equation for an arbitrary scalar quantity. The **time average transport equation for scalar ϕ** is

$$\frac{\partial \Phi}{\partial t} + \text{div}(\Phi\mathbf{U}) = \text{div}(\Gamma_{\phi}^* \text{grad } \Phi) + \left[-\frac{\overline{\partial u'\phi'}}{\partial x} - \frac{\overline{\partial v'\phi'}}{\partial y} - \frac{\overline{\partial w'\phi'}}{\partial z} \right] + S_{\phi} \quad (3.29)$$

- In compressible flow density fluctuations are usually negligible
- The density-weighted averaged (or Favre-averaged) form of the compressible turbulent flow are:

Table 3.1 Turbulent flow equations for compressible flows

Continuity

$$\frac{\partial \rho}{\partial t} + \text{div}(\rho \mathbf{U}) = 0 \quad (3.15)$$

Reynolds equations

$$\frac{\partial(\rho U)}{\partial t} + \text{div}(\rho U \mathbf{U}) = -\frac{\partial P}{\partial x} + \text{div}(\mu \text{grad } U) + \left[-\frac{\partial(\overline{\rho u'^2})}{\partial x} - \frac{\partial(\overline{\rho u'v'})}{\partial y} - \frac{\partial(\overline{\rho u'w'})}{\partial z} \right] + S_{Mx} \quad (3.16a)$$

$$\frac{\partial(\rho V)}{\partial t} + \text{div}(\rho V \mathbf{U}) = -\frac{\partial P}{\partial y} + \text{div}(\mu \text{grad } V) + \left[-\frac{\partial(\overline{\rho u'v'})}{\partial x} - \frac{\partial(\overline{\rho v'^2})}{\partial y} - \frac{\partial(\overline{\rho v'w'})}{\partial z} \right] + S_{My} \quad (3.16b)$$

$$\frac{\partial(\rho W)}{\partial t} + \text{div}(\rho W \mathbf{U}) = -\frac{\partial P}{\partial z} + \text{div}(\mu \text{grad } W) + \left[-\frac{\partial(\overline{\rho u'w'})}{\partial x} - \frac{\partial(\overline{\rho v'w'})}{\partial y} - \frac{\partial(\overline{\rho w'^2})}{\partial z} \right] + S_{Mz} \quad (3.16c)$$

Scalar transport equation

$$\frac{\partial(\rho \Phi)}{\partial t} + \text{div}(\rho \Phi \mathbf{U}) = \text{div}(\Gamma_\Phi \text{grad } \Phi) + \left[-\frac{\partial(\overline{\rho u'\phi'})}{\partial x} - \frac{\partial(\overline{\rho v'\phi'})}{\partial y} - \frac{\partial(\overline{\rho w'\phi'})}{\partial z} \right] + S_\Phi \quad (3.17)$$

Closure problem – the need for turbulence modeling

- In the continuity and Navier – Stokes equations (3.8) and (3.9 a-c) there are 6 additional unknowns

→ the **Reynolds stresses**

Similarly in scalar transport eqn (3.14) there are 3 extra unknowns:

$$\overline{u'\phi'}, \overline{v'\phi'} \text{ and } \overline{w'\phi'}$$

- Main task of turbulence modeling:

is to develop computational procedures to predict the 9 extra terms. (Reynolds stresses and scalar transport terms).

3.5 Turbulence models

Table 3.1 Turbulent flow equations for compressible flows

Continuity

$$\frac{\partial \rho}{\partial t} + \text{div}(\rho \mathbf{U}) = 0 \quad (3.15)$$

Reynolds equations

$$\frac{\partial(\rho U)}{\partial t} + \text{div}(\rho U \mathbf{U}) = -\frac{\partial P}{\partial x} + \text{div}(\mu \text{grad } U) + \left[-\frac{\partial(\overline{\rho u'^2})}{\partial x} - \frac{\partial(\overline{\rho u'v'})}{\partial y} - \frac{\partial(\overline{\rho u'w'})}{\partial z} \right] + S_{Mx} \quad (3.16a)$$

$$\frac{\partial(\rho V)}{\partial t} + \text{div}(\rho V \mathbf{U}) = -\frac{\partial P}{\partial y} + \text{div}(\mu \text{grad } V) + \left[-\frac{\partial(\overline{\rho u'v'})}{\partial x} - \frac{\partial(\overline{\rho v'^2})}{\partial y} - \frac{\partial(\overline{\rho v'w'})}{\partial z} \right] + S_{My} \quad (3.16b)$$

$$\frac{\partial(\rho W)}{\partial t} + \text{div}(\rho W \mathbf{U}) = -\frac{\partial P}{\partial z} + \text{div}(\mu \text{grad } W) + \left[-\frac{\partial(\overline{\rho u'w'})}{\partial x} - \frac{\partial(\overline{\rho v'w'})}{\partial y} - \frac{\partial(\overline{\rho w'^2})}{\partial z} \right] + S_{Mz} \quad (3.16c)$$

Scalar transport equation

$$\frac{\partial(\rho \Phi)}{\partial t} + \text{div}(\rho \Phi \mathbf{U}) = \text{div}(\Gamma_{\Phi} \text{grad } \Phi) + \left[-\frac{\partial(\overline{\rho u'\phi'})}{\partial x} - \frac{\partial(\overline{\rho v'\phi'})}{\partial y} - \frac{\partial(\overline{\rho w'\phi'})}{\partial z} \right] + S_{\Phi} \quad (3.17)$$

■ We need expressions for :

- Reynolds stresses: $\overline{u'^2}, \overline{u'v'}, \overline{u'w'}, \overline{v'^2}, \overline{v'w'}, \overline{w'^2}$
- Scalar transport terms : $\overline{u'\phi'}, \overline{v'\phi'}$ and $\overline{w'\phi'}$

Table 3.2 Turbulence models

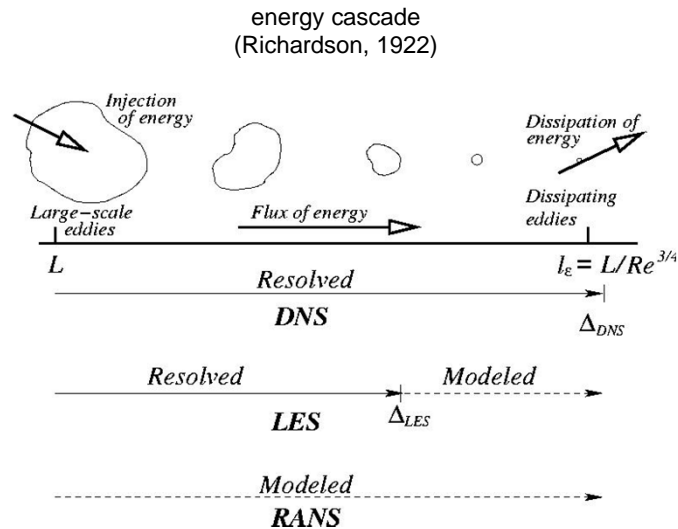
Classical models	Based on (time-averaged) Reynolds equations
	1. zero equation model – mixing length model
	2. two-equation model – $k-\epsilon$ model
	3. Reynolds stress equation model
	4. algebraic stress model
Large eddy simulation	Based on space-filtered equations

■ Classical models: use the Reynolds eqn's.(all commercial CFD codes)

■ Large eddy simulation: the flow eqn's are solved for the

- mean flow and largest eddies
- but the effect of the smaller eddies are modeled
- are at the research state
- calculations are too costly for engineering use.

Turbulence Scales and Prediction Methods



- The **mixing length** and **k- ϵ models** are the most widely used and validated.
- They are based on the presumption that “there exists an analogy between the action of **viscous stresses** and **Reynolds stresses** on the mean flow”
- Viscous stresses are proportional to the rate of deformation. For incompressible flow:

$$\tau_{ij} = 2\mu s_{ij} = \mu \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \quad (2.31)$$

- Notation:

$i = 1$ or $j = 1 \rightarrow$ x-direction
 $i = 2$ or $j = 2 \rightarrow$ y-direction
 $i = 3$ or $j = 3 \rightarrow$ z-direction

For example $\tau_{12} = \tau_{xy} = \mu \left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1} \right) = \mu \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right)$

- Turbulent stresses are found to increase as the mean rate of deformation increases.

- It was proposed by **Boussinesq** in 1877 that

Reynolds stress could be linked to mean rate of deformation

$$\tau_{ij} = -\rho \overline{u'_i u'_j} = \mu_t \left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i} \right) - \frac{2}{3} \rho k \delta_{ij} \quad (3.33)$$

where $k = 0.5(\overline{u'^2} + \overline{v'^2} + \overline{w'^2})$

This is similar to viscous stress equation

$$\tau_{ij} = 2\mu s_{ij} = \mu \left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i} \right)$$

except that μ is replaced by μ_t , where μ_t = turbulent (eddy) viscosity

Similarly; $\nu_t = \mu_t / \rho$ kinematic turbulent or eddy viscosity.

Turbulent momentum transport is assumed to be proportional to mean gradients of velocity

By analogy turbulent transport of a scalar is taken to be proportional to the gradient of the mean value of the transported quantity. In suffix notation we get

$$-\rho \overline{u'_i \phi'} = \Gamma_t \frac{\partial \Phi}{\partial x_i} \quad (3.34)$$

where Γ_t is the turbulent diffusivity.

We introduce a turbulence Prandtl / Schmidt number as

$$\sigma_t = \frac{\mu_t}{\Gamma_t} \quad (3.35)$$

Experiments in many flows have shown that

$$\sigma_t \approx 1$$

Most CFD procedures use $\sigma_t \approx 1$

■ **Mixing length models:**

Attempts to describe the turbulent stresses by means of simple algebraic formulae for μ_t as a function of position

■ **The k - ε models:**

Two transport eqn's (PDE's) are solved:

1. For the turbulent kinetic energy, k
2. For the rate of dissipation of turbulent kinetic energy, ε .

$$\mu_t = C \rho \mathcal{L} \ell = \rho C_\mu k^2 / \varepsilon \quad (C_\mu = \text{constant})$$

Both models assume that μ_t is isotropic (same for u , v , and w equations)

(i.e. the ratio between Reynolds stresses and mean rate of deformation is the same in all directions)

This assumption fails in many categories of flow.

- It is necessary to derive and solve transport eqn's for the 6 **Reynolds stresses** themselves. ($\overline{u'^2}$, $\overline{u'v'}$, $\overline{u'w'}$, $\overline{v'^2}$, $\overline{v'w'}$, $\overline{w'^2}$).

These eqn's contain:

1. Diffusion
2. Pressure strain
3. Dissipation

terms whose individual effects are **unknown** and **cannot be measured**

■ **Reynolds stress equation models:**

Solves the 6 transport equations (PDE's) for the Reynolds stress terms together with k and ε eqn's., by making assumptions about

- diffusion
- pressure strain
- and dissipation terms.

■ Algebraic stress models:

- Approximate the Reynolds stresses in terms of algebraic equations instead of PDE type transport equations.
- Is an economical form of Reynolds stress model.
- Is able to introduce **anisotropic** turbulence effects into CFD simulations

3.5.1 Mixing length model

$$v_t \rightarrow \frac{m^2}{s} = \frac{m}{s} \times m \quad \mathcal{G}: \text{turbulent velocity scale}$$

$$v_t = C \mathcal{G} \ell \quad \ell: \text{length scale of largest eddies} \quad (3.36)$$

where C is a dimensionless constant of proportionality.

Turbulent viscosity is given by $\mu_t = C \rho \mathcal{G} \ell$

This works well in simple 2-D turbulent flows where the only significant

Reynolds stress is $\tau_{xy} = \tau_{yx} = -\rho \overline{u'v'}$ and only significant velocity gradient is $\frac{\partial U}{\partial y}$

$$\text{For such flows } \mathcal{G} = c \ell \left| \frac{\partial U}{\partial y} \right| \quad (3.37)$$

where c is dimensionless constant

Combining (3.26) and (3.27) and absorbing C and c into a new length scale ℓ_m

$$v_t = \ell_m^2 \left| \frac{\partial U}{\partial y} \right| \quad (3.38)$$

This is **Prandtl's mixing length model**.

The **Reynolds stress** is

$$\tau_{xy} = \tau_{yx} = -\overline{\rho u'v'} = \rho \ell_m^2 \left| \frac{\partial U}{\partial y} \right| \frac{\partial U}{\partial y} \quad (3.39)$$

Table 3.3 Mixing lengths for two-dimensional turbulent flows

Flow	Mixing length l_m	L
Mixing layer	0.07L	Layer width
Jet	0.09L	Jet half width
Wake	0.16L	Wake half width
Axisymmetric jet	0.075L	Jet half width
Boundary layer ($\partial p/\partial x = 0$)		
viscous sub-layer and log-law layer ($y/L \leq 0.22$)	$\kappa y [1 - \exp(-y^+/26)]$	Boundary layer thickness
outer layer ($y/L \geq 0.22$)	0.09L	
Pipes and channels (fully developed flow)	$L[0.14 - 0.08(1 - y/L)^2 - 0.06(1 - y/L)^4]$	Pipe radius or channel half width

The transport of scalar quantity ϕ is modeled as

$$-\overline{\rho v' \phi'} = \Gamma_t \frac{\partial \Phi}{\partial y} \quad (3.40)$$

where $\Gamma_t = \mu_t / \sigma_t$ and v_t is found from (3.28).

$\sigma_t = 0.9$ for near wall flows
 $\sigma_t = 0.5$ for jets and mixing layers
 $\sigma_t = 0.7$ in axisymmetric jets
 } → Rodi(1980)

In Table 3.3:

y : distance from the wall

$\kappa = 0.41$ von Karman's constant

Fig. 3.13 Results of calculations using mixing length model for (a) planar jet and (b) wake behind a long slender circular cylinder

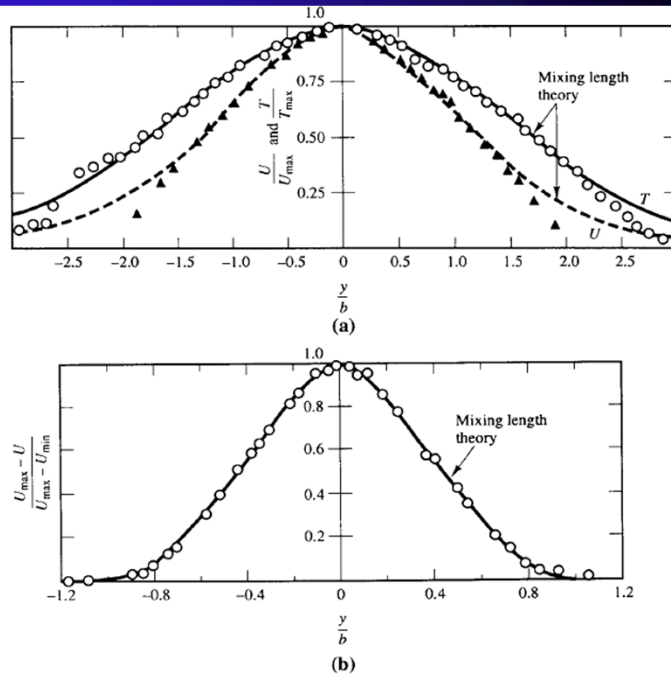


Table 3.4 Mixing length model assessment

Advantages

- easy to implement and cheap in terms of computing resources
- good predictions for thin shear layers: jets, mixing layers, wakes and boundary layers
- well established

Disadvantages

- completely incapable of describing flows with separation and recirculation
- only calculates mean flow properties and turbulent shear stress

3.5.2. The $k - \varepsilon$ model

If convection and diffusion of turbulence properties are not negligible (as in the case of recirculating flows), then the mixing length model is not applicable.

The $k-\varepsilon$ model focuses on the mechanisms that affect the turbulent kinetic energy.

■ Some preliminary definitions:

$$K = \frac{1}{2}(U^2 + V^2 + W^2) \quad \text{mean kinetic energy}$$

$$k = \frac{1}{2}(\overline{u'^2} + \overline{v'^2} + \overline{w'^2}) \quad \text{turbulent kinetic energy}$$

$$k(t) = K + k \quad \text{instantaneous kinetic energy}$$

To facilitate the subsequent calculations it is common to write the components of the rate of deformation s_{ij} and the stresses τ_{ij} in tensor (matrix) form:

$$s_{ij} = \begin{pmatrix} s_{xx} & s_{xy} & s_{xz} \\ s_{yx} & s_{yy} & s_{yz} \\ s_{zx} & s_{zy} & s_{zz} \end{pmatrix} \quad \text{and} \quad \tau_{ij} = \begin{pmatrix} \tau_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \tau_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \tau_{zz} \end{pmatrix}$$

- Using $s_{ij}(t) = S_{ij} + s'_{ij}$ gives

$$s_{xx}(t) = S_{xx} + s'_{xx} = \frac{\partial U}{\partial x} + \frac{\partial u'}{\partial x}; \quad s_{yy}(t) = S_{yy} + s'_{yy} = \frac{\partial V}{\partial y} + \frac{\partial v'}{\partial y};$$

$$s_{zz}(t) = S_{zz} + s'_{zz} = \frac{\partial W}{\partial z} + \frac{\partial w'}{\partial z};$$

$$s_{xy}(t) = S_{xy} + s'_{xy} = s_{yx}(t) = S_{yx} + s'_{yx} = \frac{1}{2} \left[\frac{\partial U}{\partial y} + \frac{\partial V}{\partial x} \right] + \frac{1}{2} \left[\frac{\partial u'}{\partial y} + \frac{\partial v'}{\partial x} \right]$$

$$s_{xz}(t) = S_{xz} + s'_{xz} = s_{zx}(t) = S_{zx} + s'_{zx} = \frac{1}{2} \left[\frac{\partial U}{\partial z} + \frac{\partial W}{\partial x} \right] + \frac{1}{2} \left[\frac{\partial u'}{\partial z} + \frac{\partial w'}{\partial x} \right]$$

$$s_{yz}(t) = S_{yz} + s'_{yz} = s_{zy}(t) = S_{zy} + s'_{zy} = \frac{1}{2} \left[\frac{\partial V}{\partial z} + \frac{\partial W}{\partial y} \right] + \frac{1}{2} \left[\frac{\partial v'}{\partial z} + \frac{\partial w'}{\partial y} \right]$$

- The product of a vector \mathbf{a} and a tensor b_{ij} is a vector \mathbf{c}

$$\mathbf{a}b_{ij} \equiv a_i b_{ij} = (a_1 \quad a_2 \quad a_3) \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix}$$

$$= \begin{pmatrix} a_1 b_{11} + a_2 b_{21} + a_3 b_{31} \\ a_1 b_{12} + a_2 b_{22} + a_3 b_{32} \\ a_1 b_{13} + a_2 b_{23} + a_3 b_{33} \end{pmatrix}^T = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}^T = c_j = \mathbf{c}$$

- The scalar product of two tensors a_{ij} and b_{ij} is evaluated as follows

$$a_{ij} \cdot b_{ij} = a_{11}b_{11} + a_{12}b_{12} + a_{13}b_{13} + a_{21}b_{21} + a_{22}b_{22} + a_{23}b_{23} \\ + a_{31}b_{31} + a_{32}b_{32} + a_{33}b_{33}$$

- Suffix notation:

1 → x – direction

2 → y – direction

3 → z – direction

Governing equation for mean flow kinetic energy K

Reynolds equations: (3.27a), (3.27b), (3.27c)

Multiply x – component Reynolds eqn. (3.27a) by U

Multiply y – component Reynolds eqn. (3.27b) by V

Multiply z – component Reynolds eqn. (3.27c) by W

Then add the result together.

After some algebra the time-average eqn. for mean kinetic energy of the flow is

$$\frac{\partial(\rho K)}{\partial t} + \text{div}(\rho K \mathbf{U}) = \text{div} \left(-P \mathbf{U} + 2\mu \mathbf{U} S_{ij} - \rho \mathbf{U} \overline{u'_i u'_j} \right) - 2\mu S_{ij} \cdot S_{ij} + \rho \overline{u'_i u'_j} \cdot S_{ij}$$

(I)
(II)
(III)
(IV)
(V)
(VI)
(VII)

(3.41)

Or in words, for the mean kinetic energy K , we have

Rate of change of K	+ Transport of K by convection	= Transport of K by pressure	+ Transport of K by viscous stresses	+ Transport of K by Reynolds stress
		- Rate of dissipation of K	+ Turbulence production	

Governing equation for turbulent kinetic energy k

- 1- Multiply x, y and z momentum eqns (3.24a-c) by u' , v' and w' , respectively
- 2- Multiply x, y and z Reynolds eqns (3.27a-c) by u' , v' and w' , respectively
- 3- subtract the two of the resulting equations

Rearrange:

→ yields the equation for turbulent kinetic energy k :

$$\frac{\partial(\rho k)}{\partial t} + \text{div}(\rho k \mathbf{U}) = \text{div} \left(-\overline{p' \mathbf{u}'} + 2\mu \overline{\mathbf{u}' s'_{ij}} - \rho \frac{1}{2} \overline{u'_i \cdot u'_i u'_j} \right) - 2\mu \overline{s'_{ij} \cdot s'_{ij}} + \rho \overline{u'_i u'_j} \cdot S_{ij}$$

(I)
(II)
(III)
(IV)
(V)
(VI)
(VII)

(3.42)

Rate of change of k	+ Transport of k by convection	= Transport of k by pressure	+ Transport of k by viscous stresses	+ Transport of k by Reynolds stress
		- Rate of dissipation of k	+ Turbulence production	

The viscous term (VI)

$$-2\mu \overline{s'_{ij} \cdot s'_{ij}} = -2\mu \left(\overline{s'^2_{11}} + \overline{s'^2_{22}} + \overline{s'^2_{33}} + 2\overline{s'^2_{12}} + 2\overline{s'^2_{13}} + 2\overline{s'^2_{23}} \right)$$

Gives a negative contribution to (3.32) due to the appearance of the sum of squared fluctuating deformation rate s'_{ij}

The dissipation of turbulent kinetic energy is caused by work done by the smallest eddies against viscous stresses.

The rate of dissipation per unit mass (m^2/s^3) is denoted by

$$\varepsilon = 2\nu \overline{s'_{ij} \cdot s'_{ij}} \quad (3.43)$$

■ The $k - \varepsilon$ model equations:

- It is possible to develop similar transport eqns. for all other turbulence quantities
- The exact ε - equation, however, contains many unknown and unmeasurable terms
- The standard $k - \varepsilon$ model (Launder and Spalding, 1974) has two model eqns (a) one for k and (b) one for ε

■ We use k and ε to define velocity scale \mathcal{G} and length scale ℓ representative of the large scale turbulence as follows:

$$\mathcal{G} = k^{1/2} \quad \ell = \frac{k^{3/2}}{\varepsilon}$$

Applying the same approach as in the mixing length model we specify the eddy viscosity as follows

$$\mu_t = C_\mu \rho \mathcal{G} \ell = \rho C_\mu \frac{k^2}{\varepsilon} \quad (3.44)$$

where C_μ is a dimensionless constant

■ The standard model uses the following transport equations used for k and ε

$$\frac{\partial(\rho k)}{\partial t} + \text{div}(\rho k \mathbf{U}) = \text{div} \left(\frac{\mu_t}{\sigma_k} \text{grad } k \right) + \underbrace{2\mu_t S_{ij} \cdot S_{ij}}_{I_k} - \rho \varepsilon \quad (3.45)$$

$$\frac{\partial(\rho \varepsilon)}{\partial t} + \text{div}(\rho \varepsilon \mathbf{U}) = \text{div} \left(\frac{\mu_t}{\sigma_\varepsilon} \text{grad } \varepsilon \right) + C_{1\varepsilon} \frac{\varepsilon}{k} 2\mu_t S_{ij} \cdot S_{ij} - C_{2\varepsilon} \rho \frac{\varepsilon^2}{k} \quad (3.46)$$

In words the equations are

Rate of change of k or ε	Transport + of k or ε by convection	=	Transport of k or ε by diffusion	+	Rate of production of k or ε
				-	Rate of destruction of k or ε

The equations contain five adjustable constants $C_\mu, \sigma_k, \sigma_\varepsilon, C_{1\varepsilon}$ and $C_{2\varepsilon}$. The standard $k - \varepsilon$ model employs values for the constants that are arrived at by comprehensive data fitting for a wide range of turbulent flows:

$$C_\mu = 0.09; \quad \sigma_k = 1.00; \quad \sigma_\varepsilon = 1.30; \quad C_{1\varepsilon} = 1.44; \quad C_{2\varepsilon} = 1.92 \quad (3.47)$$

■ To compute the Reynolds stresses with the $k - \varepsilon$ model (3.44-3.46), Boussinesq relationship is used:

$$-\rho \overline{u'_i u'_j} = \mu_t \left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i} \right) - \frac{2}{3} \rho k \delta_{ij} = 2\mu_t S_{ij} - \frac{2}{3} \rho k \delta_{ij} \quad (3.48)$$

$\delta_{ij} = 1$ if $i = j$, $\delta_{ij} = 0$ if $i \neq j$

Contraction gives $-\rho \overline{u'_i u'_i} = 2\mu_t \frac{\partial U_i}{\partial x_i} - \frac{2}{3} \delta_{ii} \rho k = 2\mu_t \frac{\partial U_i}{\partial x_i} - \frac{2}{3} \rho k$

Using $\sum_3 \frac{\partial U_i}{\partial x_i} = \frac{\partial U_1}{\partial x_1} + \frac{\partial U_2}{\partial x_2} + \frac{\partial U_3}{\partial x_3} = 0$ due to continuity

Then $\sum_{i=1}^3 \overline{u'_i u'_i} = 2k$ which is the definition of k .

$$\left\{ k = \frac{1}{2} (\overline{u'_1 u'_1} + \overline{u'_2 u'_2} + \overline{u'_3 u'_3}) \quad \text{where} \quad \overline{u'_i u'_i} = \overline{u'_i u'_i} \right\}$$

So an equal 1/3 is allocated to each normal stress component in eqn (3.48) to have $-2\rho k$ when summed. This implies an isotropic assumption for the normal Reynolds stresses.

B.C.'s for high Re model k and ε -equations

■ The following boundary conditions are needed

- | | |
|----------------------------|---|
| • inlet: | distributions of k and ε must be given |
| • outlet or symmetry axis: | $\partial k / \partial n = 0$ and $\partial \varepsilon / \partial n = 0$ |
| • free stream: | $k = 0$ and $\varepsilon = 0$ |
| • solid walls: | approach depends on Reynolds number (see below) |

For inlet, if no k and ε is available crude approximations can be obtained from the

- Turbulence intensity, T_i
- Characteristic length, L (equivalent pipe diameter) :

$$k = \frac{2}{3} (U_{ref} T_i)^2; \quad \varepsilon = C_\mu^{3/4} \frac{k^{3/2}}{\ell}; \quad \ell = 0.07L$$

The formulae are closely related to the mixing length formulae in section 3.5.1 and the universal distributions near a solid wall given below.

Wall boundary conditions for high Re models

Near the wall, at high Reynolds numbers, equations (3.44-3.46) of the standard $k - \varepsilon$ model are not integrated right to the wall. Instead, universal behaviour of near wall flows is used.

$$u^+ = \frac{1}{\kappa} \ln E y^+ \quad \text{for} \quad 30 < y^+ < 500 \quad (3.19)$$

In this region measurements of the budgets indicates that:

the rate of turbulence production = rate of dissipation

Using this assumption and $\mu_t = \rho C_\mu k^2 / \varepsilon$, the following wall functions can be derived for a grid point P, which is nearest to the wall, in the region $30 < y^+ < 500$:

$$u_p^+ = \frac{U_p}{u_\tau} = \frac{1}{\kappa} \ln(E y_p^+); \quad k = \frac{u_\tau^2}{\sqrt{C_\mu}}; \quad \varepsilon = \frac{u_\tau^3}{\kappa y} \quad \text{for} \quad 30 < y^+ < 500 \quad (3.49)$$

where $u_\tau = (\tau_w / \rho)^{1/2}$ friction velocity, $y^+ = \frac{\rho u_\tau y}{\mu}$

$\kappa = 0.41$ (Von Karman's constant)

$E = 9.8$ (wall roughness parameter) for smooth walls

Wall b.c.'s for high Re models: Momentum Eqns

Using k from Eqn. (3.49), $k_p = \frac{u_\tau^2}{\sqrt{C_\mu}} \rightarrow u_\tau = k_p^{1/2} C_\mu^{1/4}$

$$y_p^+ = \frac{\rho u_\tau y_p}{\mu} = \frac{\rho k_p^{1/2} C_\mu^{1/4} y_p}{\mu}$$

In the viscous sublayer, near the wall ($y^+ < 5$) the flow is laminar and the velocity is given by

$$u^+ = y^+ \quad (3.18)$$

The position of the interface between the laminar sublayer and the log law layer can be found by equating Eqns. (3.18) and (3.19):

$$y_{\text{int}}^+ = \frac{1}{\kappa} \ln(E y_{\text{int}}^+), \quad \rightarrow y_{\text{int}}^+ = 11.63$$

Then, the velocity at a point P near the wall is found from

$$u_p^+ = \begin{cases} y_p^+ & \text{if } y_p^+ < 11.63 \\ \frac{1}{\kappa} \ln(E y_p^+) & \text{if } y_p^+ \geq 11.63 \end{cases}$$

Wall b.c.'s for high Re models: k and ε -Eqns

k -equation, Eqn. (3.45), is integrated right to the wall. However, the source term of the k -equation given by

$$s_k = 2\mu_t S_{ij} \cdot S_{ij} - \rho\varepsilon = P_k - \rho\varepsilon$$

is calculated at a near wall point P by assuming local equilibrium. Under this assumption,

the rate of turbulence production = rate of dissipation.

Thus, the rate of turbulence production is calculated from

$$P_k \approx \tau_w \frac{\partial U}{\partial y} = \tau_w \frac{\tau_w}{\kappa \rho C_\mu^{1/4} k_p^{1/2} y_p}, \quad \text{where } \tau_w = \rho u_\tau^2, \quad u_\tau = k_p^{1/2} C_\mu^{1/4}$$

and ε is calculated from

$$\varepsilon_p = \frac{C_\mu^{3/4} k_p^{3/2}}{\kappa y_p}$$

Wall b.c.'s for high Re models: k and ε -Eqns

The boundary condition for k imposed at the wall is

$$\partial k / \partial n = 0$$

where n is normal to the wall.

The ε -equation is not solved at the wall-adjacent cells but is computed from

$$\varepsilon_p = \frac{C_\mu^{3/4} k_p^{3/2}}{\kappa y_p}$$

Wall b.c.'s for high Re models: Energy equation

The wall heat flux is given by

$$q_w = -C_p \rho C_\mu^{1/4} k_p^{1/2} (T_p - T_w) / T^+$$

where T^+ is calculated from the universal near wall temperature distribution valid at high Reynolds numbers (Launder and Spalding, 1974):

$$T^+ \equiv -\frac{(T - T_w) C_p \rho u_\tau}{q_w} = \sigma_{T,t} \left[u^+ + P \left(\frac{\sigma_{T,l}}{\sigma_{T,t}} \right) \right] \quad (3.50)$$

with T_p = temperature at near wall point y_p $\sigma_{T,t}$ = turbulent Prandtl number
 T_w = wall temperature $\sigma_{T,l} = \mu C_p / \Gamma_T$ = Prandtl number
 q_w = wall heat flux Γ_T = thermal conductivity
 C_p = fluid specific heat and constant pressure

Finally P is the “pee-function”, a correction function dependent on the ratio of laminar to turbulent Prandtl numbers (Launder and Spalding, 1974)

Low Reynolds number k-ε models

- At **low Reynolds numbers** (which is the case near the wall) the constants C_μ , $C_{1\varepsilon}$ and $C_{2\varepsilon}$ in Eqns. (3.51–3.53) are not valid so these equations cannot be integrated right to the wall. To integrate the governing equations right to the wall, modifications to the standard k - ε model is necessary.
- The equations of the low Reynolds number k - ε model, which replace (3.44-3.46), are given below:

$$\mu_t = \rho C_\mu f_\mu \frac{k^2}{\varepsilon} \quad (3.51)$$

$$\frac{\partial(\rho k)}{\partial t} + \text{div}(\rho k \mathbf{U}) = \text{div} \left(\left(\mu + \frac{\mu_t}{\sigma_k} \right) \text{grad } k \right) + 2\mu_t S_{ij} \cdot S_{ij} - \rho \varepsilon \quad (3.52)$$

$$\begin{aligned} \frac{\partial(\rho \varepsilon)}{\partial t} + \text{div}(\rho \varepsilon \mathbf{U}) = & \text{div} \left(\left(\mu + \frac{\mu_t}{\sigma_\varepsilon} \right) \text{grad } \varepsilon \right) \\ & + C_{1\varepsilon} f_1 \frac{\varepsilon}{k} 2\mu_t S_{ij} \cdot S_{ij} - C_{2\varepsilon} f_2 \rho \frac{\varepsilon^2}{k} \end{aligned} \quad (3.53)$$

■ Modifications:

- A viscous contribution is included
- C_μ , $C_{1\varepsilon}$ and $C_{2\varepsilon}$ are multiplied by wall damping functions f_μ, f_1, f_2 which are functions of turbulence Reynolds number or similar functions

- As an example we quote the Lam and Bremhorst (1981) wall-damping functions which are particularly successful:

$$f_\mu = \left[1 - \exp(-0.0165 \text{Re}_y) \right]^2 \left(1 + \frac{20.5}{\text{Re}_t} \right); \quad (3.54)$$

$$f_1 = 1 + \left(\frac{0.05}{f_\mu} \right)^3; \quad f_2 = 1 - \exp(-\text{Re}_t^2)$$

In function f_μ the parameter Re_y is defined by $k^{1/2}y/\nu$. Lam and Bremhorst use $\partial\varepsilon/\partial y = 0$ as a boundary condition.

Assessment of performance

Table 3.4 k - ε model assessment

Advantages

- simplest turbulence model for which only initial and/or boundary conditions need to be supplied
- excellent performance for many industrially relevant flows
- well established; the most widely validated turbulence model

Disadvantages

- more expensive to implement than mixing length model (two extra PDEs)
- poor performance in a variety of important cases such as
 - (i) some unconfined flows
 - (ii) flows with large extra strains (e.g. curved boundary layers, swirling flows)
 - (iii) rotating flows
 - (iv) fully developed flows in non-circular ducts

k - ε Model Implementation Details

The flow equations (3.27) can be written in tensor notation as

$$\begin{aligned} \frac{\partial(\rho U_i)}{\partial t} + \frac{\partial}{\partial x_j}(\rho U_i U_j) &= \frac{\partial}{\partial x_j} \left[\mu \left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i} \right) \right] - \frac{\partial}{\partial x_j} (\rho \overline{u'_i u'_j}) - \frac{\partial P}{\partial x_i} \\ &= \frac{\partial}{\partial x_j} \left[\mu \left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i} \right) - \rho \overline{u'_i u'_j} \right] - \frac{\partial P}{\partial x_i} \end{aligned} \quad (3.54a)$$

Inserting the Boussinesq relation (3.38)

$$\begin{aligned} \tau_{ij} = -\rho \overline{u'_i u'_j} &= \mu_t \left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i} \right) - \frac{2}{3} \rho k \delta_{ij} \\ \frac{\partial(\rho U_i)}{\partial t} + \frac{\partial}{\partial x_j}(\rho U_i U_j) &= \frac{\partial}{\partial x_j} \left[(\mu + \mu_t) \left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i} \right) \right] - \frac{\partial P^*}{\partial x_i} \end{aligned} \quad (3.54b)$$

where $P^* = P + (2/3)\rho k$

Note that Eqn. (3.54b) is the same as that of Navier-Stokes Eqns. given by (2.32a,b,c) with μ replaced by $(\mu + \mu_t)$ and p by P^* .

Rearranging

$$\begin{aligned}
 \frac{\partial(\rho U_i)}{\partial t} + \frac{\partial}{\partial x_j}(\rho U_i U_j) &= \frac{\partial}{\partial x_j} \left[(\mu + \mu_t) \left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i} \right) \right] - \frac{\partial P^*}{\partial x_i} \\
 &= \frac{\partial}{\partial x_j} \left[(\mu + \mu_t) \frac{\partial U_i}{\partial x_j} \right] + \frac{\partial}{\partial x_j} \left[(\mu + \mu_t) \frac{\partial U_j}{\partial x_i} \right] - \frac{\partial P^*}{\partial x_i} \\
 &= \frac{\partial}{\partial x_j} \left[(\mu + \mu_t) \frac{\partial U_i}{\partial x_j} \right] + \frac{\partial}{\partial x_j} \left[(\mu_t) \frac{\partial U_j}{\partial x_i} \right] + \frac{\partial}{\partial x_j} \left[(\mu) \frac{\partial U_j}{\partial x_i} \right] - \frac{\partial P^*}{\partial x_i} \\
 &= \frac{\partial}{\partial x_j} \left[(\mu + \mu_t) \frac{\partial U_i}{\partial x_j} \right] + \underbrace{\frac{\partial}{\partial x_j} \left[(\mu_t) \frac{\partial U_j}{\partial x_i} \right]}_{\text{source term}} - \frac{\partial P^*}{\partial x_i} \quad (3.54c)
 \end{aligned}$$

Note that we have used:

$$\frac{\partial}{\partial x_j} \left[(\mu) \frac{\partial U_j}{\partial x_i} \right] = \mu \frac{\partial}{\partial x_j} \left(\frac{\partial U_j}{\partial x_i} \right) = \mu \frac{\partial}{\partial x_i} \left(\frac{\partial U_j}{\partial x_j} \right) = \mu \frac{\partial}{\partial x_i} (0) = 0$$

(due to continuity if $\mu = \text{constant}$, and the fluid is incompressible)

The terms after the first parenthesis on the RHS of Eqn. (3.54c) are treated as a source .

k - ϵ Model Equations in Generic Form

The low Reynolds number k - ϵ model equations can be written in generic form as

$$\frac{\partial(\rho\phi)}{\partial t} + \text{div}(\rho\mathbf{U}\phi) = \text{div}(\Gamma \text{grad } \phi) + s_\phi$$

Equation	ϕ	Γ	s_ϕ
Continuity	1	0	0
x-Momentum	U	μ_{eff}	$\frac{\partial}{\partial x} \left(\mu_{\text{eff}} \frac{\partial U}{\partial x} \right) + \frac{\partial}{\partial y} \left(\mu_{\text{eff}} \frac{\partial V}{\partial x} \right) + \frac{\partial}{\partial z} \left(\mu_{\text{eff}} \frac{\partial W}{\partial x} \right) - \frac{\partial P^*}{\partial x}$
y-Momentum	V	μ_{eff}	$\frac{\partial}{\partial x} \left(\mu_{\text{eff}} \frac{\partial U}{\partial y} \right) + \frac{\partial}{\partial y} \left(\mu_{\text{eff}} \frac{\partial V}{\partial y} \right) + \frac{\partial}{\partial z} \left(\mu_{\text{eff}} \frac{\partial W}{\partial y} \right) - \frac{\partial P^*}{\partial y}$
z-Momentum	W	μ_{eff}	$\frac{\partial}{\partial x} \left(\mu_{\text{eff}} \frac{\partial U}{\partial z} \right) + \frac{\partial}{\partial y} \left(\mu_{\text{eff}} \frac{\partial V}{\partial z} \right) + \frac{\partial}{\partial z} \left(\mu_{\text{eff}} \frac{\partial W}{\partial z} \right) - \frac{\partial P^*}{\partial z}$
Energy	T	$\frac{\mu}{\sigma} + \frac{\mu_t}{\sigma_t}$	0
Turbulence ke	k	$\mu + \frac{\mu_t}{\sigma_k}$	$P_k - \rho\epsilon$
Dissipation of ke	ϵ	$\mu + \frac{\mu_t}{\sigma_\epsilon}$	$C_{1\epsilon} f_1 \frac{\epsilon}{k} P_k - C_{2\epsilon} f_2 \rho \frac{\epsilon^2}{k}$

$$S_y S_y = \left(\frac{\partial U}{\partial x} \right)^2 + \left(\frac{\partial V}{\partial y} \right)^2 + \left(\frac{\partial W}{\partial z} \right)^2 + \frac{1}{2} \left(\frac{\partial U}{\partial y} + \frac{\partial V}{\partial x} \right)^2 + \frac{1}{2} \left(\frac{\partial W}{\partial x} + \frac{\partial U}{\partial z} \right)^2 + \frac{1}{2} \left(\frac{\partial V}{\partial z} + \frac{\partial W}{\partial y} \right)^2, \quad S_{ij} = \frac{1}{2} \left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i} \right)$$

$$P^* = P + (2/3)\rho k, \quad \mu_{\text{eff}} = \mu + \mu_t, \quad \mu_t = \rho C_\mu f_\mu k^2 / \epsilon, \quad f_1 = 1 + (0.05 / f_\mu)^3, \quad P_k = 2\mu S_y S_y = \mu_t \left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i} \right) \frac{\partial U_i}{\partial x_j}$$

$$f_\mu = \left[1 - \exp(-0.0165 \text{Re}_y) \right]^2 (1 + 20.5 / \text{Re}_y), \quad f_2 = 1 - \exp(-\text{Re}_y^2), \quad \text{Re}_y = \rho k^2 / \epsilon \mu, \quad y = \text{minimum distance to the nearest wall} \\
 \text{Re}_y = \rho k^{1/2} y / \mu, \quad C_\mu = 0.09, \quad \sigma_k = 1.00, \quad \sigma_\epsilon = 1.30, \quad \sigma_t = 0.9, \quad C_{1\epsilon} = 1.44, \quad C_{2\epsilon} = 1.92, \quad \sigma = \text{Pr}$$

Source term linearization in k-ε models

When the source term is linearized as

$$s = s_C + s_P \phi_P$$

- 1) s_P must be negative for a convergent iterative solution, which ensures diagonal dominance of the coefficient matrix. This is a requirement for the boundedness criteria discussed in chapter 5.
- 2) s_C must be positive (and s_P must be negative) to obtain all positive ϕ values. (Patankar, 1980)

Both k and ε are strictly positive quantities. So, to obtain always positive results for k and ε , the source terms should be formulated such that s_C is positive and s_P is negative for k and ε equations.

Source term linearization in k equation

k -equation: $s_k = P_k - \rho\varepsilon$

Consider first term, ρP_k . Inserting $\mu_t = \rho C_\mu f_\mu k^2 / \varepsilon$ into P_k

$$P_k = 2\mu_t S_{ij} S_{ij} = 2\rho C_\mu f_\mu S_{ij} S_{ij} \frac{k^2}{\varepsilon} = C_3 k^2$$

Linearization of ρP_k in the form of $s_k = s_P \phi_P + s_C$ would give $s_P = C_3 k$ and $s_C = 0$. However, since $C_3 > 0$ and $k > 0$, then $s_P > 0$, which violates the boundedness criteria. We can conclude that positive terms (e.g. P_k) cannot be linearized using such an implicit manner.

As a result, P_k should be put into s_C .

Consider the second term, $-\rho\varepsilon$. From $\mu_t = \rho C_\mu f_\mu k^2 / \varepsilon$

$\varepsilon = \rho C_\mu f_\mu k^2 / \mu_t$. Inserting P_k and ε into the source term

$$s_k = P_k - \rho\varepsilon, \text{ gives:}$$

Source term linearization in k -equation

k -equation:

$$s_k = P_k - \rho\varepsilon = P_k - \frac{\rho^2 C_\mu f_\mu}{\mu_t} k^2 = b - ak^2, \quad b = P_k, \quad a = \frac{\rho^2 C_\mu f_\mu}{\mu_t}$$

For the second term, ak^2 , the linearization proposed by Patankar (1980) can be used:

$$s = s^* + \left(\frac{ds}{dk} \right)^* (k_p - k_p^*) = (b - a(k_p^*)^2) - 2ak_p^* (k_p - k_p^*) = [b + a(k_p^*)^2] - 2ak_p^* k_p$$

$$\text{Then, } s_C = b + a(k_p^*)^2 = P_k + \frac{\rho^2 C_\mu f_\mu}{\mu_t} (k_p^*)^2 \quad \text{and} \quad s_P = -2ak_p^* = -2 \frac{\rho^2 C_\mu f_\mu}{\mu_t} k_p^*$$

where the superscript * refers to the previous iteration values. However, this linearization does not yield a robust algorithm. A better approach is to put the coefficient of k to the s_P term. Comparing s_k relation with $s_\phi = s_C + s_P \phi_p$ gives:

$$(s_C)_k = b = P_k$$

$$(s_P)_k = -ak^* = -\frac{\rho^2 C_\mu f_\mu}{\mu_t} k^*$$

Positivity of s_C term in k -equation

- For consistency, the turbulent viscosity, μ_t , should be a positive quantity.
- However,

$$\mu_t = \frac{\rho C_\mu f_\mu k^2}{\varepsilon}$$

- μ_t may become negative if ε becomes negative during iterations.
- μ_t should be enforced to take positive values during iterations. The limiter used for enforcing positive values for μ_t is given later in Eqn. (3.51b).

Source term linearization in ε -equation

ε -equation:

$$s_\varepsilon = C_{1\varepsilon} f_1 P_k \frac{\varepsilon}{k} - C_{2\varepsilon} f_2 \rho \frac{\varepsilon^2}{k} = b - a\varepsilon^2$$

$$\text{where } b = C_{1\varepsilon} f_1 P_k \frac{\varepsilon}{k}, \quad a = C_{2\varepsilon} f_2 \rho / k$$

Note that, since $b > 0$, it cannot be linearized implicitly as it gives a positive s_p .

The second term, $a\varepsilon^2$, can be linearized by transferring the coefficient of the variable ε to the s_p term, in accordance with $s_\phi = s_C + s_p \phi_p$

Then, for $\phi = \varepsilon$ we obtain:

$$(s_C)_\varepsilon = b = C_{1\varepsilon} f_1 P_k \frac{\varepsilon^*}{k^*}$$

$$(s_p)_\varepsilon = -a\varepsilon = -C_{2\varepsilon} f_2 \rho \frac{\varepsilon^*}{k^*}$$

where the superscript * refers to the previous iteration values.

Source term linearization in ε -equation

The source terms s_p and s_C of the ε -equation can be written in terms of eddy viscosity μ_t using

$$\mu_t = \frac{\rho C_\mu f_\mu k^2}{\varepsilon} \rightarrow \frac{\varepsilon}{k} = \frac{\rho C_\mu f_\mu k}{\mu_t}$$

which yields

$$(s_C)_\varepsilon = b = C_{1\varepsilon} f_1 P_k \frac{\rho C_\mu f_\mu k^*}{\mu_t}$$

$$(s_p)_\varepsilon = -a\varepsilon = -C_{2\varepsilon} f_2 \rho \frac{\rho C_\mu f_\mu k^*}{\mu_t}$$

Positivity of s_C term in k - and ε -equations

- s_C term should be positive to obtain positive values during the iteration of a general variable ϕ . Rewriting the s_C term for the k -equation:

$$(s_C)_k = P_k = 2\mu_t S_{ij} S_{ij}$$

- We observe that, $(s_C)_k$ is positive if μ_t is enforced to take positive values since P_k is always positive.
- Rewriting the s_C term for the ε -equation:

$$(s_C)_\varepsilon = C_{1\varepsilon} f_1 P_k \frac{\rho C_\mu f_\mu k^*}{\mu_t}$$

- Again, we observe that, $(s_C)_\varepsilon$ is positive if μ_t is enforced to take positive values.
- Then, enforcing μ_t to take positive values is necessary to obtain positive results during iterations of both k - and ε -equations.

Limiting of eddy viscosity μ_t

- Note that both k and ε are strictly positive quantities.
- However, during iterations, undershoots may develop which will result in negative values for either k or ε which will give negative μ_t values in accordance with Eqn. (3.51):

$$\mu_t = \rho C_\mu f_\mu k^2 / \varepsilon \quad (3.51)$$

- Also if $\varepsilon \rightarrow 0$, division by zero will occur in accordance with Eqn. (3.51).
- Also, s_P and s_C terms of both k and ε -equations contain $1/\mu_t$ term so that if $\mu_t \rightarrow 0$, division by zero will occur. To overcome these difficulties μ_t is limited to a small fraction of laminar viscosity μ_l by using

- $$\mu_t = \max\left(\rho C_\mu f_\mu \frac{k^2}{\varepsilon}, 10^{-4} \mu_l\right) \quad (3.51b)$$

3.5.3 Reynolds stress equation models

- The most complex classical turbulence model is **Reynolds stress equation model (RSM)**,

Also called :

- **second order**
- or **second – moment closure model**

Drawbacks of $k - \varepsilon$ model emerge when it is attempted to predict

- flows with complex strain field
- flows with significant body forces

The exact Reynolds stress transport eqn can account for the directional effects of the Reynolds stress field.

Let $R_{ij} = -\frac{\tau_{ij}}{\rho} = \overline{u'_i u'_j}$ (Reynolds stress)

- The exact solution for the transport of R_{ij} takes the following form

$$\frac{DR_{ij}}{Dt} = P_{ij} + D_{ij} - \varepsilon_{ij} + \Pi_{ij} + \Omega_{ij} \quad (3.55)$$

Rate of change of $R_{ij} = \overline{u'_i u'_j}$	+ Transport of R_{ij} by convection	=	Rate of production of R_{ij}	+ Transport of R_{ij} due to turbulent pressure-strain interactions	+ Transport of R_{ij} by diffusion	+ Transport of R_{ij} due to rotation	- Rate of dissipation of R_{ij}
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- Equation (3.45) describes six partial differential equations: one for the transport of each of the six independent Reynolds stresses

($\overline{u_1'^2}, \overline{u_2'^2}, \overline{u_3'^2}, \overline{u_1' u_2'}, \overline{u_1' u_3'}$ and $\overline{u_2' u_3'}$, since $\overline{u_2' u_1'} = \overline{u_1' u_2'}$, $\overline{u_3' u_1'} = \overline{u_1' u_3'}$ and $\overline{u_3' u_2'} = \overline{u_2' u_3'}$)

- Two new terms appear compared with ke eqn (3.32)

1. Π_{ij} : pressure strain correction term
2. Ω_{ij} : rotation term

- The convective term is

$$C_{ij} = \frac{\partial(\rho U_k \overline{u'_i u'_j})}{\partial x_k} = \text{div}(\rho \overline{u'_i u'_j} \mathbf{U}) \quad (3.56)$$

- The production term is

$$P_{ij} = - \left(R_{im} \frac{\partial U_j}{\partial x_m} + R_{jm} \frac{\partial U_i}{\partial x_m} \right) \quad (3.57)$$

- The rotation term is

$$\Omega_{ij} = -2\omega_k (\overline{u'_j u'_m} e_{ikm} + \overline{u'_i u'_m} e_{jkm}) \quad (3.58)$$

- where ω_k = rotation vector
 $e_{ijk} = +1$ if i, j and k are different and in cyclic order
 $e_{ijk} = -1$ if i, j and k are different and in anti-cyclic order
 $e_{ijk} = 0$ if any two indices are the same

- The diffusion term D_{ij} can be modelled by the assumption that the rate of transport of Reynolds stresses by diffusion is proportional to gradient of Reynolds stress.

$$D_{ij} = \frac{\partial}{\partial x_m} \left(\frac{v_t}{\sigma_k} \frac{\partial R_{ij}}{\partial x_m} \right) = \text{div} \left(\frac{v_t}{\sigma_k} \text{grad} (R_{ij}) \right) \quad (3.59)$$

$$\text{with } v_t = C_\mu \frac{k^2}{\varepsilon}; \quad C_\mu = 0.09 \quad \text{and} \quad \sigma_k = 1.0$$

- The dissipations rate ε_{ij} is modeled by assuming isotropy of the small dissipative eddies. It is set so that it effects the normal Reynolds stresses ($i = j$) only and in equal measure. This can be achieved by

$$\varepsilon_{ij} = \frac{2}{3} \varepsilon \delta_{ij} \quad (3.60)$$

- where ε is the dissipation rate of turbulent kinetic energy defined by (3.43). The Kronecker delta, δ_{ij} is given by $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ if $i \neq j$

- For Π_{ij} term:
 - The pressure – strain interactions are the most difficult to model
 - Their effect on the Reynolds stresses is caused by two distinct physical processes:
 1. Pressure fluctuations due to two eddies interacting with each other
 2. Pressure fluctuations due to the interactions of an eddy with a region of flow of different mean velocity
 - Its effect is to make Reynolds stresses more isotropic and to reduce the Reynolds shear stresses
 - Measurements indicate that;

The wall effect increases the isotropy of normal Reynolds stresses by damping out fluctuations in the directions normal to the wall and decreases the magnitude of the Reynolds shear stresses.

- A comprehensive model that accounts for all these effects is given in Launder *et al* (1975). They also give the following simpler form favoured by some commercial available CFD codes:

$$\Pi_{ij} = -C_1 \frac{\varepsilon}{k} \left(R_{ij} - \frac{2}{3} k \delta_{ij} \right) - C_2 \left(P_{ij} - \frac{2}{3} P \delta_{ij} \right) \quad (3.61)$$

with $C_1 = 1.8$; $C_2 = 0.6$

- Turbulent kinetic energy k is

$$k = \frac{1}{2} (R_{11} + R_{22} + R_{33}) = \frac{1}{2} (\overline{u_1'^2} + \overline{u_2'^2} + \overline{u_3'^2})$$

- The six equations for Reynolds stress transport are solved along with a model equation for the scalar dissipation rate ε . Again a more exact form is found in Launder *et al* (1975), but the equation from the standard $k - \varepsilon$ model is used in commercial CFD for the sake of simplicity

$$\frac{D\varepsilon}{Dt} = \text{div} \left(\frac{\nu_t}{\sigma_\varepsilon} \text{grad } \varepsilon \right) + C_{1\varepsilon} f_1 \frac{\varepsilon}{k} 2\nu_t S_{ij} \cdot S_{ij} - C_{2\varepsilon} \frac{\varepsilon^2}{k} \quad (3.62)$$

where $C_{1\varepsilon} = 1.44$ and $C_{2\varepsilon} = 1.92$

Rate of change of ε	+ of ε by convection	= of ε by diffusion	+ Rate of production of ε	- Rate of destruction of ε
------------------------------------	-------------------------------------	------------------------------------	---	--

- | | |
|------------------------|--|
| • inlet: | specified distributions of R_{ij} and ε |
| • outlet and symmetry: | $\partial R_{ij} / \partial n = 0$ and $\partial \varepsilon / \partial n = 0$ |
| • free stream: | $R_{ij} = 0$ and $\varepsilon = 0$ |
| • solid wall: | wall functions |

Boundary conditions for the RSM model

- The usual boundary conditions for elliptic flows are required for the solution of the Reynolds stress transport equations:

- inlet: specified distributions of R_{ij} and ε
- outlet and symmetry: $\partial R_{ij} / \partial n = 0$ and $\partial \varepsilon / \partial n = 0$
- free stream: $R_{ij} = 0$ and $\varepsilon = 0$ are given
or, $\partial R_{ij} / \partial n = 0$ and $\partial \varepsilon / \partial n = 0$
- solid wall: use wall functions relating R_{ij} to either k or $(u_\tau)^2$
e.g. $\overline{u_1'^2} = 1.1k$, $\overline{u_2'^2} = 0.25k$, $\overline{u_3'^2} = 0.66k$, $-\overline{u_1' u_2'} = 0.26k$

In the absence of any information, inlet distributions are calculated from

$$k = \frac{2}{3} (U_{ref} T_i)^2; \quad \varepsilon = C_\mu^{3/4} \frac{k^{3/2}}{\ell}; \quad \ell = 0.07L;$$

$$\overline{u_1'^2} = k; \quad \overline{u_2'^2} = \overline{u_3'^2} = \frac{1}{2}k; \quad \overline{u_i' u_j'} = 0 \quad (i \neq j)$$

where ℓ is the characteristic length of the equipment (e.g. pipe diameter)

Boundary conditions for RSM model

For computations at high Reynolds numbers wall-function-type boundary conditions can be used which are very similar to those of the k - ε model .

Near wall Reynolds stress values are computed from formulae such as

$$R_{ij} = \overline{u'_i u'_j} = c_{ij} k$$

where c_{ij} are obtained from measurements.

Low Reynolds number modifications to the model can be incorporated to add the effects of molecular viscosity to the diffusion terms and to account for anisotropy in the dissipation rate in the R_{ij} -equations.

Wall damping functions to adjust the constants of the ε -equation and a modified dissipation rate variable

$$\tilde{\varepsilon} (\equiv \varepsilon - 2\nu(\partial k^{1/2} / \partial y)^2)$$

give more realistic modeling near solid walls.

Similar models exist for the 3 scalar transport terms $\overline{u'_i \phi'}$ of eqn (3.32) in the form of PDE's.

In commercial CFD codes a turbulent diffusion coefficient

$$\Gamma_t = \mu_t / \sigma_\phi$$

is added to the laminar diffusion coefficient, where

$$\sigma_\phi = 0.7$$

for all scalars.

Assessment of RSM model

Table 3.6 Reynolds stress equation model assessment.

Advantages

- potentially the most general of all classical turbulence models
- only initial and/or boundary conditions need to be supplied
- very accurate calculation of mean flow properties and *all* Reynolds stresses for many simple and more complex flows including wall jets, asymmetric channel and non-circular duct flows and curved flows

Disadvantages

- very large computing costs (seven extra PDEs)
- not as widely validated as the mixing length and $k-\epsilon$ models
- performs just as poorly as the $k-\epsilon$ model in some flows owing to identical problems with the ϵ -equation modelling (e.g. axisymmetric jets and unconfined recirculating flows)

Assesment of performance of RSM model

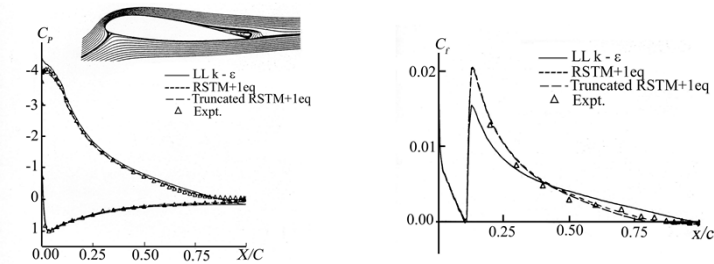


Figure 3.16

Comparison of predictions of RSM and standard $k-\epsilon$ model with measurements on a high-lift Aerospatiale aerofoil: (a) pressure coefficient; (b) skin friction coefficient
 Source: Leschziner, in Peyret and Krause (2000)

Incorrect usage of the terms: High and low Reynolds number

- High or low Reynolds number turbulence models do not necessarily refer to models which are used to simulate flows having high or low speeds. In this usage of the term, the Reynolds number is based on the distance from the wall. In the near-wall region the flow velocity and distance to the wall is low so that the Reynolds number is low. Then, low Reynolds number models refer to turbulence models which can simulate the flow in near-wall region where the viscous effects dominate, by integrating the equations right to the wall, without resorting to wall functions.
- High Reynolds number models on the other hand refer to models which can simulate the flow in the fully turbulent region only, where the Reynolds number is high. In this region the turbulence effects dominate over the viscous effects. The model equations are not valid in the near wall region. As a result, high Reynolds number model equations are not integrated right to the wall. These models use wall functions to find the variables in the near wall region.

Shortcomings of two-equation models such as k - ε model

- **Low Reynolds number flows:** Very rapid changes occur in the distribution of k and ε as we reach the buffer layer between the fully turbulent region and the viscous sublayer. To force the rapid changes to k and ε , the constants of the high Reynolds number model are multiplied by exponential damping functions which require large number of grid points to resolve the changes. As a result, the system of equations become stiff which makes the numerical solution difficult to converge.
- **Rapidly changing flows:** The Reynolds stress $-\rho\overline{u'_i u'_j}$ is proportional to S_{ij} in two-equation models. This only holds in equilibrium flows where the rates of production and dissipation of k are roughly in balance. In rapidly changing flows this is not true.

Shortcomings of two-equation models such as k - ϵ model

- **Stress anisotropy:** Two equation model predicts normal stresses $-\rho \overline{u_i'^2}$ which are all approximately equal to $-\frac{2}{3}\rho k$ if a thin shear layer is simulated. Experimental data presented in section 3.4 showed that this is not correct. In spite of this the k - ϵ model performs well in such flows because the gradients of normal turbulent stresses $-\rho \overline{u_i'^2}$ are small compared with the gradient of the dominant turbulent shear stress $-\rho \overline{u'v'}$. In more complex flows the gradients of normal turbulent stresses are not negligible and can drive significant flows. These effects can not be predicted by the two equation models.
- **Strong adverse pressure gradients and recirculation regions:** This problem is also attributable to the isotropy of the predicted normal stresses of the k - ϵ model. The k - ϵ model overpredicts the shear stress and suppresses separation in flows over curved walls. This is a significant problem in flows over airfoils, e.g. in aerospace applications.

Shortcomings of two-equation models such as k - ϵ model

- **Extra strains:** Streamline curvature, rotation and extra body forces all give rise to additional interactions between the mean strain rate and the Reynolds stresses. These physical effects are not captured by standard two-equation models.
- RSM model addresses most of these problems adequately, but at the cost of additional storage and computer time.
- Below we consider some of the more recent advances in turbulence modeling that seek to address some or all of the above problems.

Advanced turbulence models

Advanced treatment of the near-wall region: two-layer k - ε model

- The numerical instability problems associated with the wall damping functions used in low Reynolds number k - ε models are avoided by subdividing the boundary layer into two regions (Jongen, 1997):

- 1) Fully turbulent region: $Re_d = y\sqrt{k}/\nu \geq Re_d^*$, $Re_d^* = 50 - 200$

In this region the standard k - ε model is used where the eddy viscosity is defined by Eqn. (3.44); $\mu_{t,t} = C_\mu \rho k^2 / \varepsilon$

- 2) Viscous region, $Re_d < Re_d^*$:

Only the momentum equations and the k -equation is solved (Eqn. 3.45). Turbulent viscosity in this viscous region is calculated from

$$\mu_{t,v} = C_\mu \ell_\mu k^{1/2}$$

and ε is calculated from $\varepsilon = k^{3/2} / \ell_\varepsilon$

Two-layer k - ε model

- The length scales ℓ_μ and ℓ_ε contain necessary damping effects in the near wall region and are taken from the one-equation model of Wolfshtein (1969):

$$\ell_\mu = C_l y [1 - \exp(-Re_d / A_\mu)]$$

$$\ell_\varepsilon = C_l y [1 - \exp(-Re_d / A_\varepsilon)]$$

where

$$C_l = \kappa C_\mu^{-3/4}, \quad A_\varepsilon = 2C_l, \quad Re_d = k^{1/2} d / \nu$$

- y is the normal distance to the wall.
- In order to avoid instabilities associated with differences between $\mu_{t,t}$ and $\mu_{t,v}$ at the join between the fully turbulent and viscous regions, a blending formula is used to evaluate the eddy viscosity in

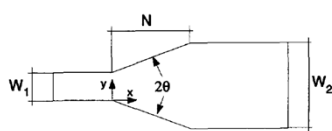
$$\tau_{ij} = -\rho \overline{u'_i u'_j} = 2\mu_t S_{ij} - 2/3 \rho k \delta_{ij}$$

- The turbulent viscosity is calculated from

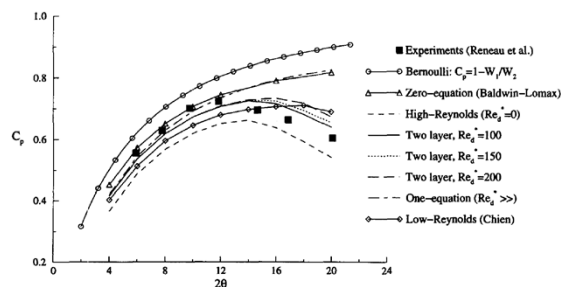
$$\mu_t = \lambda_\varepsilon \mu_{t,t} + (1 - \lambda_\varepsilon) \mu_{t,v}$$

where
$$\lambda_\varepsilon = \frac{1}{2} \left[1 + \tanh \left(\frac{\text{Re}_d - \text{Re}_d^*}{A} \right) \right]$$

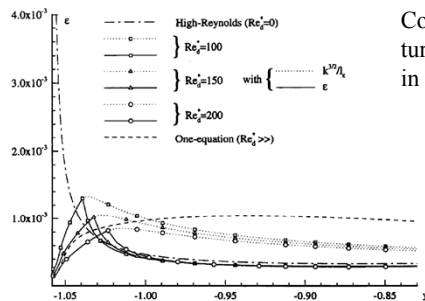
- The blending function λ_ε is zero at the wall and tends to 1 in the fully turbulent region when $\text{Re}_d \gg \text{Re}_d^*$.
- The constant A can be adjusted in order to control the sharpness of the transition from one model to the other. For example $A = 1, \dots, 10$ leads to transitions occurring within a few cells, smaller values giving sharper transitions.



Geometry of the diffuser.
(Jongen, 1997)



Comparison of **two-layer $k-\varepsilon$ model** with various turbulence models in predicting $C_p = (P_2 - P_1) / (\rho U_1^2)$ in a 2-D diffuser, (Jongen, 1997).



Profiles of ε at the exit section of the diffuser ($\theta = 10$), for different values of the switching parameter Re_d^* between the two layers, (Jongen, 1997).

Strain sensitivity: RNG k - ε model

- **RNG k - ε model** (Renormalization Group). The RNG procedure: systematically removes the small scales of motion from the governing equations by expressing their effects in terms of large scale motions and a modified viscosity. (Yakhot *et al* 1992):

RNG k - ε model equations for high Reynolds number flows:

$$\frac{\partial(\rho k)}{\partial t} + \text{div}(\rho k \mathbf{U}) = \text{div}(\alpha_k \mu_{\text{eff}} \text{grad } k) + \tau_{ij} \cdot S_{ij} - \rho \varepsilon \quad (3.65)$$

$$\frac{\partial(\rho \varepsilon)}{\partial t} + \text{div}(\rho \varepsilon \mathbf{U}) = \text{div}(\alpha_\varepsilon \mu_{\text{eff}} \text{grad } \varepsilon) + C_{1\varepsilon}^* f_1 \frac{\varepsilon}{k} \tau_{ij} \cdot S_{ij} - C_{2\varepsilon} f_2 \rho \frac{\varepsilon^2}{k} \quad (3.66)$$

with $\tau_{ij} = -\overline{\rho u_i' u_j'} = 2\mu_t S_{ij} - 2/3 \rho k \delta_{ij}$

and $\mu_{\text{eff}} = \mu + \mu_t; \quad \mu_t = \rho C_\mu \frac{k^2}{\varepsilon} \quad (3.67)$

and $C_\mu = 0.0845; \quad \alpha_k = \alpha_\varepsilon = 1.39; \quad C_{1\varepsilon} = 1.42; \quad C_{2\varepsilon} = 1.68$

and $C_{1\varepsilon}^* = C_{1\varepsilon} - \frac{\eta(1-\eta/\eta_0)}{1+\beta\eta^3}; \quad \eta = \frac{k}{\varepsilon} (2S_{ij} \cdot S_{ij})^{1/2}; \quad \eta_0 = 4.377; \quad \beta = 0.012$

- Only the constant β is adjustable; the above value is calculated from near wall turbulent data. All other constants are explicitly computed as part of the RNG process.
- The ε -equation has long been suspected as one of the main sources of accuracy limitations for the standard version of the k - ε model and the RSM.
- It is, therefore, interesting to note that the model contains a strain-dependent correction term in the constant $C_{1\varepsilon}$ of the production term in the RNG model ε -equation.
- Its performance is better than the k - ε model for an expanding duct, but actually worse for a contraction with the same ratio.

Spalart-Allmaras model

- Has only one transport equation for kinematic eddy viscosity parameter $\tilde{\nu}$. Turbulent eddy viscosity is computed from

$$\mu_t = \rho \tilde{\nu} f_{v1} \quad (3.68)$$

where f_{v1} is the wall damping function given by

$$f_{v1} = \frac{\chi^3}{\chi^3 - c_{v1}^3}; \quad \chi = \frac{\tilde{\nu}}{\nu}$$

which tends to unity for high Reynolds number flows ($\tilde{\nu} \rightarrow \nu_t$), and $f_{v1} \rightarrow 0$ at the wall.

- The Reynolds stresses are computed from

$$\tau_{ij} = -\overline{\rho u_i' u_j'} = 2\mu_t S_{ij} = \rho \tilde{\nu} f_{v1} \left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i} \right) \quad (3.69)$$

- The transport equation for $\tilde{\nu}$ is as follows:

$$\frac{\partial(\rho\tilde{\nu})}{\partial t} + \text{div}(\rho\tilde{\nu}\mathbf{U}) = \text{div} \left[(\mu + \rho\tilde{\nu}) \text{grad}(\tilde{\nu}) + C_{b2}\rho \frac{\partial\tilde{\nu}}{\partial x_k} \frac{\partial\tilde{\nu}}{\partial x_k} \right] + C_{b1}\rho\tilde{\nu}\tilde{\Omega} - C_{w1}\rho \left(\frac{\tilde{\nu}}{\kappa y} \right)^2 f_w \quad (3.70)$$

(I) (II) (III) (IV) (V) (VI)

Rate of change of $\tilde{\nu}$	+	Transport of $\tilde{\nu}$ by convection	=	Transport of $\tilde{\nu}$ by turbulent diffusion	+	Rate of production of $\tilde{\nu}$	-	Rate of dissipation of $\tilde{\nu}$
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$$\tilde{\Omega} = \Omega + \frac{\tilde{\nu}}{(\kappa y)^2} f_{v2}$$

where $\Omega = \sqrt{2\Omega_{ij}\Omega_{ij}}$ = mean vorticity

$$\Omega_{ij} = \frac{1}{2} \left(\frac{\partial U_i}{\partial x_j} - \frac{\partial U_j}{\partial x_i} \right) = \text{mean vorticity tensor}$$

$$f_{v2} = 1 - \frac{\chi}{1 + \chi f_{v1}}, \quad f_w = g \left[\frac{1 + c_{w3}^6}{g^6 + c_{w3}^6} \right]^{1/6} \quad \text{wall damping functions}$$

- In the k - ε model the length scale is $\ell = k^{3/2}/\varepsilon$.
- In a one-equation model ℓ cannot be computed since there is no transport equation for k and ε . However, ℓ should be specified to determine the dissipation rate. Inspection of Eqn. (3.70) reveals that $\ell = \kappa y$ has been used as a length scale. This is also the mixing length used in developing the log-law for wall boundary layers.
- The constants are:

$$\sigma_v = 2/3, \quad \kappa = 0.4187, \quad C_{b1} = 0.1355, \quad C_{b2} = 0.622, \quad C_{w1} = C_{b1} + \kappa^2 \frac{1 + C_{b2}}{\sigma_v}$$

$$g = r + C_{w2}(r^6 - r), \quad r = \min \left[\frac{\tilde{v}}{\tilde{\Omega} \kappa^2 y^2}, 10 \right], \quad C_{w2} = 0.3, \quad C_{w3} = 2$$

- The model gives good performance for flows with adverse pressure gradients. In complex geometries it is difficult to determine the length scale, so the model is unsuitable for more general internal flows.

Wilcox k - ω model

- In the k - ε model the kinematic eddy viscosity is expressed as

$$\mu_t \rightarrow \rho \mathcal{G} \ell, \quad \mathcal{G} = \sqrt{k} \text{ velocity scale}, \quad \ell = k^{3/2}/\varepsilon \text{ length scale}$$
- However, ε is not the only possible length scale determining variable. In fact, k - ω model (Wilcox, 1994) uses $\omega = \varepsilon / k$ as the second variable. Then, the length scale is $\ell = (k)^{1/2} / \varepsilon$ and

$$\mu_t = \rho k / \omega \quad (3.71)$$

- The Reynolds stresses are computed from Boussinesq expression:

$$\tau_{ij} = -\rho \overline{u'_i u'_j} = 2\mu_t S_{ij} - \frac{2}{3} \rho k \delta_{ij} = \mu_t \left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i} \right) - \frac{2}{3} \rho k \delta_{ij} \quad (3.72)$$

- The transport equation for k and ω for turbulent flow at high Reynolds is:

$$\frac{\partial(\rho k)}{\partial t} + \text{div}(\rho k \mathbf{U}) = \text{div} \left[\left(\mu + \frac{\mu_t}{\sigma_k} \right) \text{grad}(k) \right] + P_k - \beta^* \rho k \omega$$

where
and

$$P_k = 2\mu_t S_{ij} \cdot S_{ij} - \frac{2}{3}\rho k \frac{\partial U_i}{\partial x_j} \delta_{ij}$$

$$\frac{\partial(\rho\omega)}{\partial t} + \text{div}(\rho\omega\mathbf{U}) = \text{div}\left[\left(\mu + \frac{\mu_t}{\sigma_\omega}\right)\text{grad}(\omega)\right] + \gamma_1\left(2\rho S_{ij} \cdot S_{ij} - \frac{2}{3}\rho\omega \frac{\partial U_i}{\partial x_j} \delta_{ij}\right) - \beta_1\rho\omega^2 \quad (3.74)$$

Rate of change of k or ω	+	Transport of k or ω by convection	=	Transport of k or ω by turbulent diffusion	+	Rate of production of k or ω	-	Rate of dissipation of k or ω
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■ The model constants are

$$\sigma_k = 2.0, \quad \sigma_\omega = 2.0, \quad \gamma_1 = 0.553, \quad \beta_1 = 0.075, \quad \beta^* = 0.09$$

- The k - ω model initially attracted attention because integration to the wall does not require wall-damping functions in low Reynolds number applications.
- The value of k is set to zero at the wall.
- ω tends to infinity at the wall, \rightarrow use very large value or use $\omega_p = 6\nu / (\beta_1 y_p^2)$ at the near wall grid point P .
- At inlet boundaries k and ω must be specified.
- At outlet boundaries zero gradient conditions are used.
- In a free stream $k \rightarrow 0$ and $\omega \rightarrow 0$, then $\mu_t \rightarrow \infty$ from Eqn. (3.71). This causes problems in free stream regions. So a non-zero value of ω should be specified. Unfortunately, results of the model tend to be dependent on the assumed free stream value of ω .

Menter SST k - ω model

- Menter (1992) noted that the results of the k - ε model are much less sensitive to the (arbitrary) assumed values in the free stream, but its near-wall performance is unsatisfactory for boundary layers with adverse pressure gradients. He suggested a hybrid model using
 - (i) a transformation of the k - ε model into k - ω model in the near-wall region
 - (ii) the standard k - ε model in the fully turbulent region far from the wall.
- The calculation of τ_{ij} and the k -equation are the same as in Wilcox's original k - ω model, but the ε -equation is transformed into an ω -equation by substituting $\varepsilon = k\omega$. This yields

$$\frac{\partial(\rho\omega)}{\partial t} + \text{div}(\rho\omega\mathbf{U}) = \text{div} \left[\left(\mu + \frac{\mu_t}{\sigma_{\omega,1}} \right) \text{grad}(\omega) \right] + \gamma_2 \left(2\rho S_{ij} \cdot S_{ij} - \frac{2}{3} \rho\omega \frac{\partial U_i}{\partial x_j} \delta_{ij} \right) - \beta_2 \rho\omega^2 + 2 \frac{\rho}{\sigma_{\omega,2}\omega} \frac{\partial k}{\partial x_k} \frac{\partial \omega}{\partial x_k}$$

$$\begin{aligned} \underbrace{\frac{\partial(\rho\omega)}{\partial t}}_{(i)} + \underbrace{\text{div}(\rho\omega\mathbf{U})}_{(ii)} &= \underbrace{\text{div} \left[\left(\mu + \frac{\mu_t}{\sigma_{\omega,1}} \right) \text{grad}(\omega) \right]}_{(iii)} & (3.75) \\ &+ \underbrace{\gamma_2 \left(2\rho S_{ij} \cdot S_{ij} - \frac{2}{3} \rho\omega \frac{\partial U_i}{\partial x_j} \delta_{ij} \right)}_{(iv)} - \underbrace{\beta_2 \rho\omega^2}_{(v)} + \underbrace{2 \frac{\rho}{\sigma_{\omega,2}\omega} \frac{\partial k}{\partial x_k} \frac{\partial \omega}{\partial x_k}}_{(vi)} \end{aligned}$$

Comparison with Eqn. (3.74) shows that (3.75) has an extra source term (VI): the cross-diffusion term, which arises during the $\varepsilon = k\omega$ transformation of the diffusion term in the ε -equation.

Revised Menter SST $k-\omega$ model (2003)

■ Model constants:

$$\sigma_k = 1.0, \quad \sigma_{\omega,1} = 2.0, \quad \sigma_{\omega,2} = 1.17, \quad \gamma_2 = 0.44, \quad \beta_2 = 0.083, \quad \beta^* = 0.09$$

■ Blending functions:

The differences between the μ_t values computed from the standard $k-\varepsilon$ model in the far field and the $k-\omega$ model near the wall may cause instabilities. To prevent this, blending functions are introduced in the equation to modify the cross diffusion term. Blending functions are also used for model constants such as σ_ω , γ and β :

$$C = F_1 C_1 + (1 - F_1) C_2 \quad (3.76)$$

where

C_1 = constants in original $k-\omega$ model (inner constants)

C_2 = constants in Menter's transformed $k-\varepsilon$ model (outer constants)

F_1 = a blending function

e.g. $F_C = F_C(\ell_t/y, \text{Re}_y)$ is a function of the ratio of turbulence $\ell_t = (k)^{1/2}/\omega$ and distance y to the wall and a turbulence Reynolds number $\text{Re}_y = y^2\omega/\nu$.

The functional form of F_C is chosen such that

(i) it is zero at the wall

(ii) tends to 1 in the far field

(iii) produces a smooth transition around a distance half way between the wall and the edge of the boundary layer.

In this way, the model combines the good near-wall behaviour of the $k-\omega$ model with the robustness of the $k-\varepsilon$ model in the far field.

Limiters

Eddy viscosity models tend to overpredict k in regions where S_{ij} is high. This problem is sometimes referred to as “stagnation point anomaly” [Durbin, 1996]. To avoid such overprediction, turbulent viscosity μ_t should be limited. The limiter proposed by Medic and Durbin [2002] is

$$\frac{\mu_t}{C_\mu \rho k} = \min \left(\frac{k}{\varepsilon}, \frac{\alpha}{\sqrt{6} C_\mu |S|} \right)$$

where $\alpha = 0.6$

$$|S| = (S_{ij} S_{ij})^{1/2}$$

Limiter in SST k - ω model

■ Limiters:

The eddy viscosity in SST k - ω model is limited to give improved performance in flows with adverse pressure gradients and wake regions:

$$\mu_t = \frac{a_1 \rho k}{\max(a_1 \omega, S F_2)} \quad (3.77a)$$

where $S = \sqrt{2 S_{ij} S_{ij}}$, $a_1 =$ a constant, $F_2 =$ a blending function

The k -production, P_k , is limited to prevent the build-up of turbulence in stagnation regions:

$$P_k = \min \left(10 \beta^* \rho k \omega, 2 \mu_t S_{ij} \cdot S_{ij} - \frac{2}{3} \rho k \frac{\partial U_i}{\partial x_j} \delta_{ij} \right) \quad (3.77b)$$

Menter SST k - ω model: summary

Continuity $\frac{\partial u_i}{\partial x_i} = 0$

momentum $\frac{\partial(\rho u_i u_j)}{\partial x_i} = -\frac{\partial p}{\partial x_i} + \frac{\partial}{\partial x_j} \left[(\mu + \mu_t) \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \right]$

Energy $\frac{\partial(\rho u_i T)}{\partial x_i} = \frac{\partial}{\partial x_i} \left[\left(\frac{\mu}{Pr} + \frac{\mu_t}{Pr_t} \right) \frac{\partial T}{\partial x_i} \right]$

(K) $\frac{\partial(\rho u_j k)}{\partial x_j} = \frac{\partial}{\partial x_j} \left[(\mu + \sigma_k \mu_t) \frac{\partial k}{\partial x_j} \right] + P - \beta^* \rho \omega k$

(ω) $\frac{\partial(\rho u_j \omega)}{\partial x_j} = \frac{\partial}{\partial x_j} \left[(\mu + \sigma_\omega \mu_t) \frac{\partial \omega}{\partial x_j} \right] + 2(1 - F_1) \frac{\rho \sigma_{\omega 2}}{\omega} \frac{\partial k}{\partial x_j} \frac{\partial \omega}{\partial x_j} + \frac{\gamma}{\nu_t} P - \beta \rho \omega^2$

where:

$$P = \tau_{ij} \frac{\partial u_i}{\partial x_j}, \quad \tau_{ij} = \mu_t \left(2S_{ij} - \frac{2}{3} \frac{\partial u_k}{\partial x_k} \delta_{ij} \right) - \frac{2}{3} \rho k \delta_{ij}, \quad S_{ij} = \frac{1}{2} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)$$

Menter SST k - ω model: summary

$$\mu_t = \frac{\rho a_1 k}{\max(a_1 \omega, SF_2)}$$

$$S = \sqrt{2S_{ij}S_{ij}}$$

$$F_1 = \tanh(\arg_1^4)$$

$$\arg_1 = \min \left[\max \left(\frac{\sqrt{k}}{\beta^* \omega y}, \frac{500\nu}{y^2 \omega} \right), \frac{4\rho \sigma_{\omega 2} k}{CD_{k\omega} y^2} \right]$$

$$CD_{k\omega} = \max \left(2\rho \sigma_{\omega 2} \frac{1}{\omega} \frac{\partial k}{\partial x_j} \frac{\partial \omega}{\partial x_j}, 10^{-10} \right)$$

$$F_2 = \tanh(\arg_2^2)$$

$$\arg_2 = \max \left(2 \frac{\sqrt{k}}{\beta^* \omega y}, \frac{500\nu}{y^2 \omega} \right)$$

$$\sigma_{k1} = 0.85, \quad \sigma_{k2} = 1.0, \quad \sigma_{\omega 1} = 0.5, \quad \sigma_{\omega 2} = 0.856$$

$$\beta_1 = 0.075, \quad \beta_2 = 0.0828, \quad \beta^* = 0.09, \quad \kappa = 0.41, \quad a_1 = 0.31$$

$$C_1 = 5/9, C_2 = 0.44$$

Menter SST k - ω model: summary

- Each of the constants is a blend of an inner (1) and outer (2) constants, blended via:

$$C = F_1 C_1 + (1 - F_1) C_2$$

where $C_1 \rightarrow$ inner (1) constants, and $C_2 \rightarrow$ outer (2) constants.

- Note that it is generally recommended to use a production limiter where P in the k -equation is replaced with

$$P = \min(P, 10\beta^* \rho \omega k)$$

- The boundary conditions recommended are:

$$\omega_{wall} = 10 \frac{6\nu}{\beta_1 (\Delta y_1)^2}$$

$$k_{wall} = 0$$

Assessment of turbulence models for aerospace applications

- **External Aerodynamics:**

The Spalart-Allmaras, k - ω and SST k - ω models are all suitable

The SST k - ω model is most general; it gives superior performance for zero pressure gradient and adverse pressure gradient boundary layers.

- **General purpose CFD:**

The Spalart-Allmaras model is unsuitable, but the k - ω and SST k - ω models can both be applied.

They both have a similar range of strengths and weaknesses as the k - ε model and fail to include accounts of more subtle interactions between turbulent stresses and mean flow compared with RSM.

The k - ε - v^2 - f turbulence model

- In the k - ε - v^2 - f model of Durbin, (1991, 1993, 1995) two additional equations, apart from the k and ε -equations, are solved:
 - (i) the normal stress, $\overline{v_2'^2}$
 - (ii) a relaxation function f for the production of k , ($f = P_k / k$)
- In usual eddy-viscosity models wall effects are accounted for through wall functions.
- In the k - ε - v^2 - f model the problem of accounting for the wall damping of $\overline{v_2'^2}$ is simply resolved by solving its transport equation similar to RSM model:

$$\frac{\partial(\rho\overline{v_2'^2})}{\partial t} + \text{div}(\rho\overline{v_2'^2}\mathbf{U}) = \text{div}\left[\left(\mu + \frac{\mu_t}{\sigma_k}\right)\text{grad}\overline{v_2'^2}\right] + \rho kf - \rho\frac{\overline{v_2'^2}}{k}\varepsilon$$

- An equation for the production of k is formulated in terms of a function $f = P_k / k$:

$$L^2 \text{div}(\text{grad}(f)) - f = (1 - C_1) \frac{\left[2/3 - \overline{v_2'^2} / k\right]}{T} - C_2 \frac{P_k}{k}$$

where $T = \max\left[\frac{k}{\varepsilon}, 6\left(\frac{\nu}{\varepsilon}\right)^{1/2}\right]$ time scale

$$L = C_L \ell, \quad \ell^2 = \max\left[\frac{k^3}{\varepsilon^2}, C_\eta^2 \left(\frac{\nu^3}{\varepsilon}\right)^{1/2}\right], \quad \ell = \text{length scale}$$

- Note that: $\text{div grad} = \nabla \cdot \nabla = \nabla^2 = \partial / \partial x^2 + \partial / \partial y^2 + \partial / \partial z^2$
- The eddy viscosity is given by: $\nu_t = \mu / \rho = C_\mu \overline{v_2'^2} T$
- Constants: $C_\mu = 0.19$, $C_1 = 1.4$, $C_2 = 0.3$, $C_L = 0.3$, $C_\eta = 70.0$
- Boundary conditions at the wall:
 - $k = 0$, $\overline{v_2'^2} = 0$, $\varepsilon = 2\nu k / y^2$, $f = -\frac{20\nu^2 \overline{v_2'^2}}{\varepsilon y^4}$, $y = \text{normal distance to the wall}$

A variant of the k - ε - v^2 - f turbulence model: ζ - f model

- In the k - ε - v^2 - f model of Durbin (1995), f is proportional to y^4 near the wall. As a result, the boundary condition for f makes the equation system numerically unstable.
- An eddy-viscosity model based on k - ε - v^2 - f model of Durbin is proposed by Hanjalic et.al. (2004), which solves a transport equation for the velocity scales ratio $\zeta = v_2'^2 / k$ instead of $v_2'^2$
- Thus, the resulting model is more robust and less sensitive to grid non-uniformities.
- Eddy viscosity is defined as

$$\nu_t = C_\mu^\zeta \zeta k T$$

where

$$\zeta = \overline{v_2'^2} / k$$

can be interpreted as the ratio of the wall-normal velocity time scale T_v and the general (scalar) turbulence time scale T :

$$T_v = \overline{v_2'^2} / \varepsilon \quad \text{and} \quad T = k / \varepsilon$$

A variant of the k - ε - v^2 - f turbulence model: ζ - f model

- The complete set of the model equations are:

$$\frac{\partial(\rho)}{\partial t} + \text{div}(\rho \mathbf{U}) = 0$$

$$\frac{\partial(\rho U)}{\partial t} + \text{div}(\rho U \mathbf{U}) = \text{div}(\mu_{\text{eff}} \text{grad } U) + s_{Mx}$$

$$\frac{\partial(\rho V)}{\partial t} + \text{div}(\rho V \mathbf{U}) = \text{div}(\mu_{\text{eff}} \text{grad } V) + s_{My}$$

$$\frac{\partial(\rho W)}{\partial t} + \text{div}(\rho W \mathbf{U}) = \text{div}(\mu_{\text{eff}} \text{grad } W) + s_{Mz}$$

$$\frac{\partial(\rho \zeta)}{\partial t} + \text{div}(\rho \zeta \mathbf{U}) = \text{div} \left[\left(\mu + \frac{\mu_t}{\sigma_\zeta} \right) \text{grad } \zeta \right] + \rho f - \frac{\zeta}{k} P_k$$

$$\frac{\partial(\rho k)}{\partial t} + \text{div}(\rho k \mathbf{U}) = \text{div} \left[\left(\mu + \frac{\mu_t}{\sigma_k} \right) \text{grad } k \right] + P_k - \rho \varepsilon$$

$$\frac{\partial(\rho \varepsilon)}{\partial t} + \text{div}(\rho \varepsilon \mathbf{U}) = \text{div} \left[\left(\mu + \frac{\mu_t}{\sigma_\varepsilon} \right) \text{grad } \varepsilon \right] + \frac{C_{\varepsilon 1} P_k - \rho C_{\varepsilon 2} \varepsilon}{T}$$

$$L^2 \text{div}(\text{grad}(f)) - f = \frac{1}{T} \left(C_1 - 1 + C_2 \frac{P_k}{\varepsilon} \right) \left(\zeta - \frac{2}{3} \right)$$

■ where $\mu_t = \rho C_\mu^{\zeta} \zeta k T$

$$T = \max \left[\min \left(\frac{k}{\varepsilon}, \frac{a}{\sqrt{6} C_\mu^{\zeta} |S| \zeta} \right), C_T \left(\frac{v}{\varepsilon} \right)^{1/2} \right]$$

$$L = C_L \max \left[\min \left(\frac{k^{3/2}}{\varepsilon}, \frac{k^{1/2}}{\sqrt{6} C_\mu^{\zeta} |S| \zeta} \right), C_\eta \left(\frac{v^3}{\varepsilon} \right)^{1/4} \right]$$

$$a = 0.6, \quad C_\mu^{\zeta} = 0.22, \quad C_{\varepsilon 1} = 1.4(1 + 0.012/\zeta), \quad C_{\varepsilon 2} = 1.9, \quad C_2' = 0.65, \quad \sigma_k = 1, \quad \sigma_\varepsilon = 1.3, \\ \sigma_\zeta = 1.2, \quad C_T = 6.0, \quad C_L = 0.36, \quad C_\eta = 85$$

$$s_{M_x} = \frac{\partial}{\partial x} \left(\mu_{eff} \frac{\partial U}{\partial x} \right) + \frac{\partial}{\partial y} \left(\mu_{eff} \frac{\partial V}{\partial x} \right) + \frac{\partial}{\partial z} \left(\mu_{eff} \frac{\partial W}{\partial x} \right) - \frac{\partial P^*}{\partial x}$$

$$s_{M_y} = \frac{\partial}{\partial x} \left(\mu_{eff} \frac{\partial U}{\partial y} \right) + \frac{\partial}{\partial y} \left(\mu_{eff} \frac{\partial V}{\partial y} \right) + \frac{\partial}{\partial z} \left(\mu_{eff} \frac{\partial W}{\partial y} \right) - \frac{\partial P^*}{\partial y}$$

$$s_{M_z} = \frac{\partial}{\partial x} \left(\mu_{eff} \frac{\partial U}{\partial z} \right) + \frac{\partial}{\partial y} \left(\mu_{eff} \frac{\partial V}{\partial z} \right) + \frac{\partial}{\partial z} \left(\mu_{eff} \frac{\partial W}{\partial z} \right) - \frac{\partial P^*}{\partial z}$$

$$\mu_{eff} = \mu + \mu_t, \quad P^* = P + (2/3)\rho k, \quad P_k = 2\mu_t S_{ij} S_{ij}, \quad |S| = \sqrt{2S_{ij} S_{ij}}$$

$$S_{ij} S_{ij} = \left(\frac{\partial U}{\partial x} \right)^2 + \left(\frac{\partial V}{\partial y} \right)^2 + \left(\frac{\partial W}{\partial z} \right)^2 + \frac{1}{2} \left(\frac{\partial U}{\partial y} + \frac{\partial V}{\partial x} \right)^2 + \frac{1}{2} \left(\frac{\partial W}{\partial x} + \frac{\partial U}{\partial z} \right)^2 + \frac{1}{2} \left(\frac{\partial V}{\partial z} + \frac{\partial W}{\partial y} \right)^2$$

■ Boundary conditions at the wall:

$$k = 0, \quad \zeta = 0, \quad \varepsilon = 2\nu k / y^2, \quad f = -\frac{2\nu\zeta}{y^2}, \quad y = \text{normal distance to the wall}$$

■ Note that both the nominator and the denominator of f_w (f at the wall) are proportional to y^2 instead of y^4 as in the Durbin's v^2 - f model.

■ This brings improved stability to computational scheme.

3.5.4 Algebraic stress equation models (ASM)

- ASM is an economical way of accounting for the **anisotropy** of Reynolds stresses.
- Avoids solving the Reynolds stress transport eqns.
- Instead, uses algebraic eqns to model Reynolds stresses.

The Simplest method is:

To neglect the convection and diffusion terms, C_{ij} and D_{ij} in the Reynolds stress equations (3.55).

A more generally applicable method is:

To assume that the sum of the convection and diffusion terms of the Reynolds stresses is proportional to the sum of the convection and diffusion terms of turbulent kinetic energy

$$\begin{aligned} \frac{D\overline{u'_i u'_j}}{Dt} - D_{ij} &\approx \frac{\overline{u'_i u'_j}}{k} \cdot \left(\frac{Dk}{Dt} - [\text{transport of } k \text{ (i.e. div) terms}] \right) \\ &= \frac{\overline{u'_i u'_j}}{k} \cdot (-\overline{u'_i u'_j} \cdot S_{ij} - \varepsilon) \end{aligned} \quad (3.78)$$

Introducing approximation (3.78) into the Reynolds stress transport equation (3.55) with production term P_{ij} (3.57), modeled dissipation rate term (3.60) and pressure – strain correction term (3.61) on the right hand side yields after some arrangement the following **algebraic stress model**:

$$R_{ij} = \overline{u'_i u'_j} = \frac{2}{3} k \delta_{ij} + \left(\frac{C_D}{C_1 - 1 + P/\varepsilon} \right) \left(P_{ij} - \frac{2}{3} P \delta_{ij} \right) \frac{k}{\varepsilon} \quad (3.79)$$

- $\overline{u'_i u'_j}$ appear on both sides (on rhs within P_{ij}). For swirling flows, $C_D=0.55$, $C_1=2.2$
- eqn(3.79) is a set of 6 simultaneous algebraic eqns. for 6 unknowns, $\overline{u'_i u'_j}$
- solved iteratively if k and ε are known.

Algebraic stress model assessment

Advantages

- cheap method to account for Reynolds stress anisotropy
- potentially combines the generality of approach of the RSM (good modelling of buoyancy and rotation effects possible) with the economy of the $k-\varepsilon$ model
- successfully applied to isothermal and buoyant thin shear layers
- if convection and diffusion terms are negligible the ASM performs as well as the RSM

Disadvantages

- only slightly more expensive than the $k-\varepsilon$ model (two PDEs and a system of algebraic equations)
- not as widely validated as the mixing length and $k-\varepsilon$ models
- same disadvantages as RSM apply
- model is severely restricted in flows where the transport assumptions for convective and diffusive effects do not apply – validation is necessary to define the performance limits

Non-linear $k-\varepsilon$ models

- The standard $k-\varepsilon$ model uses the Boussinesq approximation (3.33)

$$\tau_{ij} = -\rho \overline{u'_i u'_j} = \mu_t \left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i} \right) - \frac{2}{3} \rho k \delta_{ij} \quad (3.33)$$

and eddy viscosity expression (3.44).

$$\mu_t = C \rho \mathcal{A} \ell = \rho C_\mu k^2 / \varepsilon \quad (3.44)$$

Hence
$$-\rho \overline{u'_i u'_j} = \tau_{ij} = \tau_{ij}(S_{ij}, k, \varepsilon, \rho) \quad (8.80)$$

- This relationship implies that the turbulence adjusts itself instantaneously as it is convected through the flow domain. However, the viscoelastic analogy holds that the adjustment does not take place immediately. $\rightarrow \tau_{ij}$ should also be a function DS_{ij}/Dt . So,

$$-\rho \overline{u'_i u'_j} = \tau_{ij} = \tau_{ij}\left(S_{ij}, \frac{DS_{ij}}{Dt}, k, \varepsilon, \rho\right)$$

- In RSM, τ_{ij} is actually regarded as a transported quantity.
- Bringing in a dependence on DS_{ij}/Dt can be regarded as a partial account of Reynolds stress transport, which recognises that the state of turbulence lags behind the rapid changes.

Elaborating this idea, a non-linear k - ε model is proposed. This approach involves the derivation of asymptotic expansions for the Reynolds stresses which maintain terms that are quadratic in velocity gradients.

- The nonlinear k - ε model (Speziale, 1987):

$$\tau_{ij} = -\overline{\rho u'_i u'_j} = -\frac{2}{3} \rho k \delta_{ij} + \rho C_\mu \frac{k^2}{\varepsilon} 2S_{ij} \quad (3.82)$$

$$- 4C_D C_\mu^2 \frac{k^3}{\varepsilon^2} \left(S_{im} \cdot S_{mj} - \frac{1}{3} S_{mn} \cdot S_{mn} \delta_{ij} + S_{ij}^o - \frac{1}{3} S_{mn}^o \delta_{ij} \right)$$

where $S_{ij}^o = \frac{\partial S_{ij}}{\partial t} + \mathbf{U} \cdot \text{grad}(S_{ij}) - \left(\frac{\partial U_i}{\partial x_m} \cdot S_{mj} + \frac{\partial U_j}{\partial x_m} \cdot S_{mi} \right)$ and $C_D = 1.68$

- The nonlinear k - ε model accounts for anisotropy.
- Accounts for the secondary flow in non-circular duct flows.

The simplest non-linear eddy viscosity model

- The simplest non-linear eddy viscosity model relates the Reynolds stresses to quadratic tensor products of S_{ij} and Ω_{ij} :

$$\tau_{ij} = -\overline{\rho u'_i u'_j} = 2\mu_t S_{ij} - \frac{2}{3} \rho k \delta_{ij} \quad (3.83)$$

$$\left. \begin{aligned} & -C_1 \mu_t \frac{k}{\varepsilon} \left(S_{ik} \cdot S_{jk} - \frac{1}{3} S_{kl} \cdot S_{kl} \delta_{ij} \right) \\ & -C_2 \mu_t \frac{k}{\varepsilon} \left(S_{ik} \cdot \Omega_{jk} - S_{jk} \cdot \Omega_{ik} \right) \\ & -C_3 \mu_t \frac{k}{\varepsilon} \left(\Omega_{ik} \cdot \Omega_{jk} - \frac{1}{3} \Omega_{kl} \cdot \Omega_{kl} \delta_{ij} \right) \end{aligned} \right\} \text{quadratic terms}$$

where $S_{ij} = 1/2(\partial U_i / \partial x_j + \partial U_j / \partial x_i)$, $\Omega_{ij} = 1/2(\partial U_i / \partial x_j - \partial U_j / \partial x_i)$

- Quadratic terms allow Reynolds stresses to be different.
- The model has the potential to capture anisotropy effects.
- C_1 , C_2 and C_3 are constants in addition to the 5 constants of the original k - ε model.

Cubic k - ε model

- Craft et al. (1996) demonstrated that it is necessary to introduce cubic tensor products to obtain the correct sensitising effect for interactions between Reynolds stress production and streamline curvature.
- They also included:
 - Variable C_μ with functional dependence on S_{ij} and Ω_{ij}
 - Ad hoc modification of the ε -equation to reduce the overprediction of the length scale, leading to poor shear stress predictions in separated flows.
 - Wall damping functions to enable integration of the k - and ε -equations to the wall through the viscous sub-layer.
- The performance of cubic k - ε model is very close to that of RSM.

Large Eddy Simulation (LES)

- Up to now, developing a general-purpose RANS turbulence model suitable for a wide range of practical applications have not been possible.
- This is due to differences in the behaviour of large and small eddies.
- Small eddies are nearly isotropic, but large eddies are anisotropic and their behaviour is dictated by the geometry, boundary conditions and body forces.
- Problem dependence of the largest eddies complicates the search for widely applicable models using RANS.
- An alternative approach to the problem is **large eddy simulation (LES)** which computes the larger eddies with a time dependent simulation, but tries to model only the smaller eddies.
- The universal behaviour of the smaller eddies is easier to capture with a compact model.

- Instead of time averaging, LES uses a spatial filtering operation to separate the larger and smaller eddies.
- To resolve all those eddies with a length scale greater than a certain width, the method selects a filtering function and a certain cutoff width.
- Then, the filtering operation is performed on the time-dependent flow equations.
- During spatial filtering, information relating to the smaller, filtered-out turbulent eddies is destroyed.
- This, and interaction effects between the larger, resolved eddies and the smaller unresolved ones, gives rise to **sub-grid-scale stresses or SGS stresses**.
- Their effect on the resolved flow must be described by means of an **SGS model**.

- In LES,
 - the time-dependent, space filtered flow equations
 - together with the SGS model of the unresolved stressesare solved on a grid of control volumes
- The solution gives:
 - the **mean flow** and
 - **all turbulent eddies at scales larger than the cutoff width**.

Spatial filtering of unsteady Navier-Stokes equations

- Filters are familiar separation devices in electronics that are designed to split an input into a desirable, retained part and an undesirable, rejected part.

- Filtering Functions:**

In LES spatial filtering is done by using a **filter function** $G(x, x', \Delta)$:

$$\bar{\phi}(\mathbf{x}, t) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} G(\mathbf{x}, \mathbf{x}', \Delta) \phi(\mathbf{x}', t) dx'_1 dx'_2 dx'_3 \quad (3.84)$$

where $\bar{\phi}(\mathbf{x}, t)$ = filtered function

$\phi(\mathbf{x}, t)$ = original (unfiltered) function

Δ = filter cutoff width

- In this section overbar indicates spatial filtering, not time averaging.
- Only in LES, the integration is not carried out in time but in three-dimensional space.

Common filtering functions in LES

- Top hat filter:

$$G(\mathbf{x}, \mathbf{x}', \Delta) = \begin{cases} 1/\Delta^3 & \text{if } |\mathbf{x} - \mathbf{x}'| \leq \Delta/2 \\ 0 & \text{if } |\mathbf{x} - \mathbf{x}'| > \Delta/2 \end{cases} \quad (3.85a)$$

- Gaussian filter:

$$G(\mathbf{x}, \mathbf{x}', \Delta) = \left(\frac{\gamma}{\pi \Delta^2} \right)^{3/2} \exp \left(-\gamma \frac{|\mathbf{x} - \mathbf{x}'|^2}{\Delta^2} \right), \quad \gamma = 6 \quad (3.85b)$$

- Spectral cutoff:

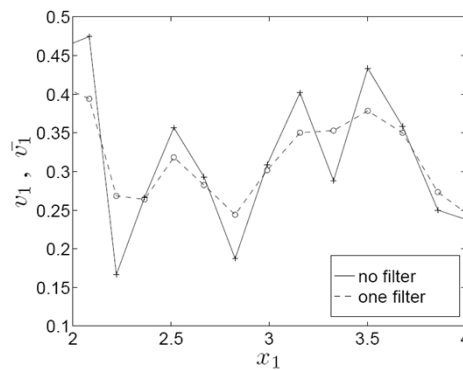
$$G(\mathbf{x}, \mathbf{x}', \Delta) = \prod_{i=1}^3 \frac{\sin[(x_i - x'_i) / \Delta]}{(x_i - x'_i)} \quad (3.85c)$$

- The top-hat filter is used in finite volume implementations of LES.
- Gaussian and spectral cutoff filters are preferred in research.
- Δ is intended as an indicative measure of the size of eddies that are retained in the computations and the eddies that are rejected.
- In principle we can choose Δ to be any size, but in finite volume method it is pointless to choose $\Delta <$ grid size, since only a single nodal value of the variable is retained over the control volume, so all finer detail is lost anyway.
- The most common selection is

$$\Delta = \sqrt[3]{\Delta x \Delta y \Delta z} = (V_{cell})^{1/3} \quad (3.86)$$

where Δx , Δy and Δz are the length of a cubic cell in x , y and z -directions.

Filtering in 1-D



In LES we filter (volume average) the equations. In 1-D we get

$$\bar{\phi}(x, t) = \frac{1}{\Delta x} \int_{x-0.5\Delta x}^{x+0.5\Delta x} \phi(\xi, t) d\xi$$

Filtered unsteady Navier-Stokes equations

- The unsteady Navier-Stokes equations for a fluid with constant viscosity μ are

$$\frac{\partial \rho}{\partial t} + \text{div}(\rho \mathbf{u}) = 0 \quad (2.4)$$

$$\frac{\partial(\rho u)}{\partial t} + \text{div}(\rho u \mathbf{u}) = -\frac{\partial p}{\partial x} + \mu \text{div}(\text{grad}(u)) + S_u \quad (2.37a)$$

$$\frac{\partial(\rho v)}{\partial t} + \text{div}(\rho v \mathbf{u}) = -\frac{\partial p}{\partial y} + \mu \text{div}(\text{grad}(v)) + S_v \quad (2.37b)$$

$$\frac{\partial(\rho w)}{\partial t} + \text{div}(\rho w \mathbf{u}) = -\frac{\partial p}{\partial z} + \mu \text{div}(\text{grad}(w)) + S_w \quad (2.37c)$$

- If the flow is also incompressible $\rightarrow \text{div}(\mathbf{u}) = 0$, $\rightarrow S_u, S_v, S_w = 0$

- Filtering of the equations (2.4) and (2.37) yields:

$$\frac{\partial \bar{\rho}}{\partial t} + \text{div}(\rho \bar{\mathbf{u}}) = 0 \quad (3.87)$$

$$\frac{\partial(\rho \bar{u})}{\partial t} + \text{div}(\rho \bar{u} \mathbf{u}) = -\frac{\partial \bar{p}}{\partial x} + \mu \text{div}(\text{grad}(\bar{u})) \quad (3.88a)$$

$$\frac{\partial(\rho \bar{v})}{\partial t} + \text{div}(\rho \bar{v} \mathbf{u}) = -\frac{\partial \bar{p}}{\partial y} + \mu \text{div}(\text{grad}(\bar{v})) \quad (3.88b)$$

$$\frac{\partial(\rho \bar{w})}{\partial t} + \text{div}(\rho \bar{w} \mathbf{u}) = -\frac{\partial \bar{p}}{\partial z} + \mu \text{div}(\text{grad}(\bar{w})) \quad (3.88c)$$

- The overbar indicates a filtered flow variable.
- The problem is, how to compute $\text{div}(\rho \phi \mathbf{u})$

$$\text{div}(\rho \phi \mathbf{u}) = \text{div}(\bar{\phi} \bar{\mathbf{u}}) + [\text{div}(\rho \phi \mathbf{u}) - \text{div}(\bar{\phi} \bar{\mathbf{u}})]$$

- The second term on the rhs is modeled.

- Substitution of the modeled $div(\overline{\rho\phi\mathbf{u}})$ into (3.88a-c) yields the **LES momentum equations**:

$$\frac{\partial(\overline{\rho u})}{\partial t} + div(\overline{\rho u \mathbf{u}}) = -\frac{\partial \overline{p}}{\partial x} + \mu div(grad(\overline{u})) - (div(\overline{\rho u \mathbf{u}}) - div(\overline{\rho u} \mathbf{u})) \quad (3.89a)$$

$$\frac{\partial(\overline{\rho v})}{\partial t} + div(\overline{\rho v \mathbf{u}}) = -\frac{\partial \overline{p}}{\partial y} + \mu div(grad(\overline{v})) - (div(\overline{\rho v \mathbf{u}}) - div(\overline{\rho v} \mathbf{u}))$$

$$\frac{\partial(\overline{\rho w})}{\partial t} + div(\overline{\rho w \mathbf{u}}) = -\frac{\partial \overline{p}}{\partial z} + \mu div(grad(\overline{w})) - (div(\overline{\rho w \mathbf{u}}) - div(\overline{\rho w} \mathbf{u})) \quad (3.89b)$$

$$(I) \quad (II) \quad (III) \quad (IV) \quad (V)$$

- The filtered momentum eqns. are similar to RANS momentum eqns. (3.26a-c) or (3.27a-c).
- The last terms (V) are caused by the filtering operation similar to the Reynolds stresses in RANS equations that arose as a result of time averaging.
- They can be considered as a divergence of a set of stresses τ_{ij}

- The i -component of these terms can be written as follows:

$$div(\overline{\rho u_i \mathbf{u}} - \overline{\rho u_i} \mathbf{u}) = \frac{\partial(\overline{\rho u_i u} - \overline{\rho u_i} \overline{u})}{\partial x} + \frac{\partial(\overline{\rho u_i v} - \overline{\rho u_i} \overline{v})}{\partial y} + \frac{\partial(\overline{\rho u_i w} - \overline{\rho u_i} \overline{w})}{\partial z}$$

$$= \frac{\partial \tau_{ij}}{\partial x_j} \quad (3.90a)$$

$$\text{where } \tau_{ij} = \overline{\rho u_i \mathbf{u}} - \overline{\rho u_i} \mathbf{u} = \overline{\rho u_i u_j} - \overline{\rho u_i} \overline{u_j} \quad (3.90b)$$

- τ_{ij} are termed as **sub-grid-scale stresses**.
- Remembering that a flow variable $\phi(\mathbf{x}, t)$ can be decomposed as the sum of
 - (i) the filtered function $\overline{\phi}(\mathbf{x}, t)$ with spatial variations that are larger than the cutoff width and are resolved by the LES computations and
 - (ii) $\phi'(\mathbf{x}, t)$, which contains unresolved spatial variations at length scales smaller than the filter cutoff width,
 we can write:

$$\phi(\mathbf{x}, t) = \bar{\phi}(\mathbf{x}, t) + \phi'(\mathbf{x}, t) \quad (3.91)$$

- Using this decomposition in eqn. (3.90b) we can write the first term on the far rhs as

$$\begin{aligned} \overline{\rho u_i u_j} &= \overline{\rho(\bar{u}_i + u'_i)(\bar{u}_j + u'_j)} = \overline{\rho \bar{u}_i \bar{u}_j} + \overline{\rho \bar{u}_i u'_j} + \overline{\rho u'_i \bar{u}_j} + \overline{\rho u'_i u'_j} \\ &= \overline{\rho \bar{u}_i \bar{u}_j} + (\overline{\rho \bar{u}_i u'_j} - \overline{\rho \bar{u}_i \bar{u}_j}) + \overline{\rho \bar{u}_i u'_j} + \overline{\rho u'_i \bar{u}_j} + \overline{\rho u'_i u'_j} \end{aligned}$$

- Then, we can write SGS stresses as

$$\tau_{ij} = \overline{\rho u_i u_j} - \overline{\rho \bar{u}_i \bar{u}_j} = \underbrace{(\overline{\rho \bar{u}_i u'_j} - \overline{\rho \bar{u}_i \bar{u}_j})}_{\text{Leonard stresses, } L_{ij}} + \underbrace{\overline{\rho \bar{u}_i u'_j} + \overline{\rho u'_i \bar{u}_j}}_{\text{cross-stresses, } C_{ij}} + \underbrace{\overline{\rho u'_i u'_j}}_{\text{LES Reynolds stresses, } R_{ij}} \quad (3.92)$$

- Leonard stresses, L_{ij} , are due to effects at resolved scale. They are caused due to the fact that $\overline{\bar{\phi}} \neq \bar{\phi}$ for space filtered variables, unlike in time averaging, where $\overline{\varphi(t)} = \bar{\Phi} = \Phi = \overline{\varphi(t)}$

- The cross stresses C_{ij} are due to interactions between the SGS eddies and the resolved flow.
- LES Reynolds stresses R_{ij} are caused by convective momentum transfer due to interactions of SGS eddies and are modeled with a so-called SGS turbulence model.
- The SGS stresses (3.92) must be modeled.

Smagorinski-Lilly SGS model

- Smagorinski (1963) suggested that, since the smallest turbulent eddies are almost isotropic, we expect that the Boussinesq hypothesis (3.33) might provide a good description of the effects of the unresolved eddies on the resolved flow.

- Thus in Smagorinsky's model the **local SGS stresses R_{ij} are taken to be proportional to the local rate of strain of the resolved flow:**

$$R_{ij} \sim \bar{S}_{ij} = 1/2(\partial\bar{u}_i/\partial x_j + \partial\bar{u}_j/\partial x_i):$$

$$R_{ij} = -2\mu_{SGS}\bar{S}_{ij} + \frac{1}{3}R_{kk}\delta_{ij} = -\mu_{SGS}\left(\frac{\partial\bar{u}_i}{\partial x_j} + \frac{\partial\bar{u}_j}{\partial x_i}\right) + \frac{1}{3}R_{kk}\delta_{ij} \quad (3.93)$$

- $\frac{1}{3}R_{kk}\delta_{ij}$ performs the same function as the term $\frac{2}{3}\rho k\delta_{ij}$ in eqn (3.33).
- It ensures that the sum of the modeled normal SGS stresses is equal to the kinetic energy of the of the SGS eddies.
- Above model is used **together with approximate forms of Leonard stresses L_{ij} and cross stresses C_{ij} .**

- In spite of the different nature of L_{ij} and C_{ij} , they are lumped together with R_{ij} in the current versions of the finite volume method.

- Then, the whole stress τ_{ij} is modeled as a single entity by means of a single **SGS turbulence model:**

$$\tau_{ij} = -2\mu_{SGS}\bar{S}_{ij} + \frac{1}{3}\tau_{kk}\delta_{ij} = -\mu_{SGS}\left(\frac{\partial\bar{u}_i}{\partial x_j} + \frac{\partial\bar{u}_j}{\partial x_i}\right) + \frac{1}{3}\tau_{kk}\delta_{ij} \quad (3.94)$$

- Smagorinsky-Lilly model is based on Prandtl's mixing length model:

$$\nu_{SGS} = C\mathcal{G}\ell$$

- Length scale: $\ell = \Delta$ is used since Δ fixes the size of SGS eddies.

- Velocity scale: $\mathcal{G} = \Delta \times |S|$ where $|S| = \sqrt{2\bar{S}_{ij}\bar{S}_{ij}}$

- Substituting ℓ and \mathcal{G} into ν_{SGS} and defining $\mu_{SGS} = \rho\nu_{SGS}$

$$\mu_{SGS} = \rho(C_{SGS}\Delta)^2 |S| = \rho(C_{SGS}\Delta)^2 \sqrt{2\bar{S}_{ij}\bar{S}_{ij}} \quad (3.95)$$

where
$$\bar{S}_{ij} = \frac{1}{2}\left(\frac{\partial\bar{u}_i}{\partial x_j} + \frac{\partial\bar{u}_j}{\partial x_i}\right)$$

- Near the wall, μ_{SGS} becomes quite large because velocity gradients are high there.
- However, since the SGS turbulent fluctuations near a wall go to zero, so must μ_{SGS} .
- To ensure this, μ_{SGS} is multiplied with a damping function f_μ , so that

$$\mu_{SGS} = \rho (f_\mu C_{SGS} \Delta)^2 |\bar{S}|$$

where

$$f_\mu = 1 - \exp(-y^+ / 26)$$

- A more convenient way to dampen μ_{SGS} near the wall is simply to use the RANS length scale as an upper limit, i.e.

$$\Delta = \min \left[(V_{cell})^{1/3}, \kappa n \right]$$

n = distance to the nearest wall

- Disadvantage of the Smagorinsky model: the “constant” C_{SGS} is not constant, but it is flow dependent. It is found to vary in the range from $C_{SGS} = 0.065$ (Moin, 1982) to $C_{SGS} = 0.25$ (Jones, 1995).

- The difference in C_{SGS} values is attributable to the effect of the mean flow strain or shear.
- This indicates that the behavior of the small eddies is not as universal as was surmised at first.
- → Successful LES turbulence modeling might require
 - (i) a case by case adjustment of C_{SGS} , or
 - (ii) a more sophisticated approach.

Higher order SGS models

- Boussinesq eddy viscosity hypothesis for Reynolds stresses given by Eqn. (3.93) assumes that changes in the resolved flow take place sufficiently slowly that the SGS eddies can adjust themselves instantaneously to the rate of strain of the resolved flow field.
- Instead of using a case by case adjustment of C_{SGS} for different applications, another approach is to use the idea of RANS modeling to account for the transport effects. In this approach we define

Length scale: $\ell = \Delta$

Velocity scale: $\mathcal{G} = \sqrt{k_{SGS}}$ where k_{SGS} = SGS turbulent kinetic energy

Then $\mu_{SGS} = \rho C'_{SGS} \Delta \sqrt{k_{SGS}}$ (3.96)

where $C'_{SGS} = \text{constant}$

- To account for the effects of convection, diffusion, production and destruction on the SGS velocity scale we solve a transport equation to determine the distribution of k_{SGS} :

$$\frac{\partial(\rho k_{SGS})}{\partial t} + \text{div}(\rho k_{SGS} \mathbf{u}) = \text{div} \left(\frac{\mu_{SGS}}{\sigma_k} \text{grad}(k_{SGS}) \right) + \underbrace{2\mu_{SGS} \bar{S}_{ij} \cdot \bar{S}_{ij}}_{P_{k_{SGS}}} - \rho \varepsilon_{SGS} \quad (3.96)$$

- Dimensional analysis shows that the rate of dissipation ε_{SGS} of SGS turbulent kinetic energy is related to length and velocity scales as

$$\varepsilon_{SGS} = C_\varepsilon \frac{k_{SGS}^{3/2}}{\Delta} \quad (3.97)$$

$C_\varepsilon = \text{constant}$

- This model is implemented in commercial CFD code STAR-CD.

Advanced SGS models

- The Smagorinsky model is purely dissipative: the direction of energy flow is from eddies at the resolved scales towards the sub-grid scales.
- Quarini (1979) have shown that there is also energy flow in reverse direction.
- Furthermore, modeled SGS stresses using the Smagorinsky-Lilly model do not correlate strongly with actual SGS stresses computed by accurate DNS. (Clark et al. (1979), McMillan and Ferziger, (1979)).
- These authors suggested that the SGS stresses should not be taken as proportional to the strain rate of the whole resolved flow field, but should be estimated from the strain rate of the smallest resolved eddies.
- Bardina et al. (1980) proposed a method to compute local values of C_{SGS} based on the application of two filtering operations, taking the SGS stresses to be proportional to the stresses due to eddies at the smallest resolved scale. They proposed:

The Bardina model

$$\tau_{ij} = \rho C' (\overline{\overline{u_i u_j}} - \overline{u_i} \overline{u_j})$$

where C' is an adjustable constant.

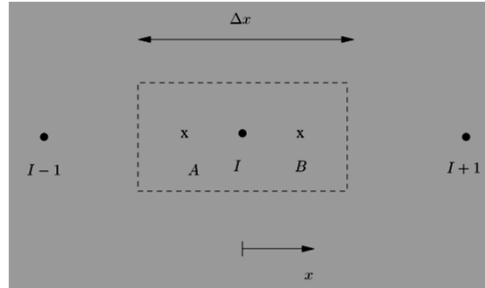
- The term in the brackets can be evaluated from twice-filtered resolved flow field information. The cutoff width of the test filter may be equal to that of first filter or it may be coarser.
- Bardina model has improved performance. But, the appearance of negative viscosities generated stability problems.
- They proposed adding a damping term with the form of the Smagorinsky model (3.94)-(3.95) to stabilise calculations, which yields a **mixed model**:

$$\tau_{ij} = \rho C' (\overline{\overline{u_i u_j}} - \overline{u_i} \overline{u_j}) - 2\rho C_{SGS}^2 \Delta^2 |\overline{S}| \overline{S}_{ij} \quad (3.99)$$

- C' depends on the cutoff width used for the second filtering, but $C' \approx 1$

Twice filtering

Let's filter velocity component \bar{u} once more at a node I using a cutoff width Δx . For simplicity we do it in 1D.



Box filter for a control volume.
(Lars Davidson, Lecture notes
Chalmers Univ.)

$$\bar{\bar{u}}_I = \frac{1}{\Delta x} \int_{-\Delta x/2}^{\Delta x/2} \bar{u}(\xi) d\xi = \frac{1}{\Delta x} \left(\int_{-\Delta x/2}^0 \bar{u}(\xi) d\xi + \int_0^{\Delta x/2} \bar{u}(\xi) d\xi \right) = \frac{1}{\Delta x} \left(\frac{\Delta x}{2} \bar{u}_A + \frac{\Delta x}{2} \bar{u}_B \right)$$

Estimating \bar{u} at locations A and B by linear interpolation gives

$$\bar{\bar{u}}_I = \frac{1}{2} \left[\left(\frac{1}{4} \bar{u}_{I-1} + \frac{3}{4} \bar{u}_I \right) + \left(\frac{3}{4} \bar{u}_I + \frac{1}{4} \bar{u}_{I+1} \right) \right] = \frac{1}{8} (\bar{u}_{I-1} + 6\bar{u}_I + \bar{u}_{I+1}) \neq \bar{u}_I$$

Dynamic SGS model

- Germano (1986) proposed a different decomposition of the turbulent stresses for 2 different filtering operations with cutoff widths Δ_1 and Δ_2

$$\tau_{ij}^{(2)} - \tau_{ij}^{(1)} = \rho L_{ij} \equiv \rho \bar{\bar{u}}_i \bar{\bar{u}}_j - \bar{\bar{u}}_i \bar{\bar{u}}_j \quad (3.100)$$

Second filter is coarser and is called the **test filter** with $\Delta_2 = 2\Delta_1$

- Eqn. (3.100) is evaluated from the resolved flow data.
- The SGS stresses are modeled using Smagorinsky's model (3.94) – (3.95) assuming same C_{SGS} for both filtering operations. This yields:

$$L_{ij} - \frac{1}{3} L_{kk} \delta_{ij} = C_{SGS}^2 M_{ij} \quad (3.101a)$$

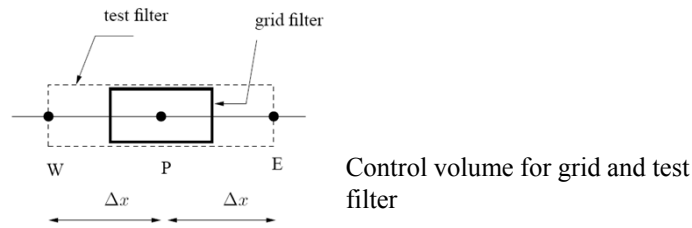
where
$$M_{ij} = -2\Delta_2^2 \left| \bar{\bar{S}} \right| \bar{\bar{S}}_{ij} + 2\Delta_1^2 \left| \bar{S} \right| \bar{S}_{ij} \quad (3.101b)$$

- Lilly suggested a least squares approach to evaluate local values of C_{SGS} :

$$C_{SGS}^2 = \frac{L_{ij} L_{ij}}{M_{ij} M_{ij}} \quad (3.102)$$

Note that the product of two tensors is a scalar.

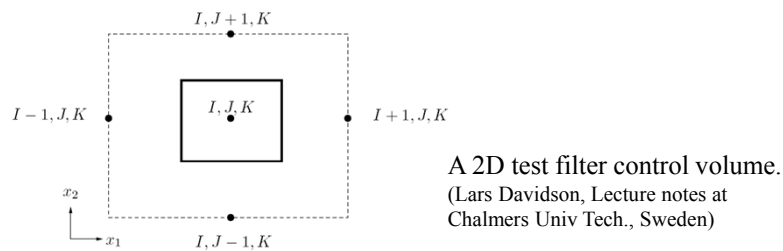
The test filter



The test-filtered variables are computed by integration over the test filter. The test filter is twice the size of grid filter, i.e. $\Delta_2 = 2\Delta_1$. For example in 1D, $\bar{\bar{u}}$ is computed as

$$\begin{aligned}\bar{\bar{u}} &= \frac{1}{2\Delta x} \int_W^E \bar{u} dx = \frac{1}{2\Delta x} \left(\int_W^P \bar{u} dx + \int_P^E \bar{u} dx \right) \\ &= \frac{1}{2\Delta x} (\bar{u}_w \Delta x + \bar{u}_e \Delta x) = \frac{1}{2} \left(\frac{\bar{u}_w + \bar{u}_p}{2} + \frac{\bar{u}_p + \bar{u}_e}{2} \right) \\ &= \frac{1}{4} (\bar{u}_w + 2\bar{u}_p + \bar{u}_e)\end{aligned}$$

The test filter in 3D



In 3D, filtering at the test level is carried out in the same way by integrating over the test cell assuming linear variation of the variables (Zang, et. al., 1993)

$$\begin{aligned}\bar{\bar{\bar{u}}}_{I,J,K} &= \frac{1}{8} (\bar{\bar{u}}_{I-1/2, J-1/2, K-1/2} + \bar{\bar{u}}_{I+1/2, J-1/2, K-1/2} \\ &\quad + \bar{\bar{u}}_{I-1/2, J+1/2, K-1/2} + \bar{\bar{u}}_{I+1/2, J+1/2, K-1/2} \\ &\quad + \bar{\bar{u}}_{I-1/2, J-1/2, K+1/2} + \bar{\bar{u}}_{I+1/2, J-1/2, K+1/2} \\ &\quad + \bar{\bar{u}}_{I-1/2, J+1/2, K+1/2} + \bar{\bar{u}}_{I+1/2, J+1/2, K+1/2})\end{aligned}$$

Initial and boundary conditions for LES

■ Initial conditions

For steady flows it is adequate to specify an initial field that conserves mass with superimposed Gaussian random fluctuations with the correct turbulence level or spectral content.

■ Solid walls

- No-slip b.c. is used if equations are integrated to the wall. This requires fine grids with near-wall grid points $y^+ \leq 1$
- For high Re flows with thin boundary layers non-uniform grids clustered near the walls is necessary.
- Alternatively, wall functions can be used.

Implementation of Wall functions in LES

■ The wall function models used in LES are:

1) The universal near wall model

- If the first near-wall node P is in the laminar sublayer use

$$\frac{\bar{u}}{u_\tau} = \frac{\rho u_\tau y}{\mu} \quad \text{for } y^+ < 5 \quad \text{where } u_\tau = (\tau_w / \rho)^{1/2} \text{ friction velocity, } y^+ = \frac{\rho u_\tau y}{\mu}$$

- If node P is in the fully turbulent region use

$$\frac{\bar{u}}{u_\tau} = \frac{1}{\kappa} \ln E \left(\frac{\rho u_\tau y}{\mu} \right), \quad 30 < y^+ < 500, \quad E = 9.793$$

- If node P is within the buffer region the two above layers are blended using a function suggested by Kader (1981)

$$u^+ = e^\Gamma u_{lam}^+ + e^{1/\Gamma} u_{turb}^+, \quad u^+ = \frac{\bar{u}}{u_\tau}$$

where

$$\Gamma = -\frac{a(y^+)^4}{1 + by^+}, \quad a = 0.01, \quad b = 5$$

- Similarly, the general equation for the derivative du^+/dy^+ is

$$\frac{du^+}{dy^+} = e^\Gamma \frac{du_{lam}^+}{dy^+} + e^{1/\Gamma} \frac{du_{turb}^+}{dy^+}$$

- This approach allows the fully turbulent law to be easily modified and extended to take into account other effects such as pressure gradients or variable properties.
- This formula guarantees reasonable representation of velocity profiles in the cases where y^+ falls inside the wall buffer region ($3 < y^+ < 10$).
- Friction velocity can be calculated from $u_\tau = k_p^{1/2} C_\mu^{1/4}$
- However, in most of the LES models k equation is not solved and k_p is not available. Since u_τ appears on both sides of the non-linear logarithmic law equation, then an iterative method should be used to solve for u_τ .

2) The Werner-Wengle model

2) The Werner-Wengle model (Werner,1993)

- To avoid the iterative solution procedure required for determining u_τ , Werner and Wengle proposed analytical integration of **power-law near-wall velocity distribution** resulting in the following expressions for the wall shear stress τ_w :

$$|\tau_w| = \begin{cases} \frac{2\mu|u_p|}{\Delta z} & \text{for } |u_p| \leq \frac{\mu}{2\rho\Delta z} A^{2/(1-B)} \\ \rho \left[\frac{1-B}{2} A^{1/B} \left(\frac{\mu}{\rho\Delta z} \right)^{1+B} + \frac{1+B}{A} \left(\frac{\mu}{\rho\Delta z} \right)^B |u_p| \right]^{2/1+B} & \text{for } |u_p| \leq \frac{\mu}{2\rho\Delta z} A^{2/(1-B)} \end{cases}$$

u_p = velocity parallel to wall, $A = 8.3$, $B = 1/7$, Δz = near-wall control volume length scale (height if CV is rectangular).

- Then u_τ is calculated from $u_\tau = (\tau_w/\rho)^{1/2}$

3) Spalding's law of the wall method

- The universal velocity profile proposed by Spalding is a fit of the laminar, buffer and logarithmic regions of the equilibrium boundary layer into a single equation:

$$y^+ = u^+ + \frac{1}{E} \left[e^{\kappa u^+} - 1 - \kappa u^+ - \frac{1}{2} (\kappa u^+)^2 - \frac{1}{6} (\kappa u^+)^3 \right]$$

where $\kappa = 0.42$ and $E = 9.1$.

- If $y^+ = y_p \rho u_\tau / \mu$ and $u^+ = u_p / u_\tau$ are inserted into the above equation, an equation of only one unknown u_τ is obtained. This non-linear equation can be solved using the Newton-Raphson method (Villiers, 2006):

$$u_\tau = u_\tau^{n-1} - f / f'$$

where $(n-1) \rightarrow$ previous iteration value

$$f = u^+ - y^+ + \frac{1}{E} \left[e^{\kappa u^+} - 1 - \kappa u^+ - \frac{1}{2} (\kappa u^+)^2 - \frac{1}{6} (\kappa u^+)^3 \right]$$

$$f' = \frac{\partial f}{\partial u_\tau} = -\frac{u^+}{u_\tau} - \frac{y^+}{u_\tau} + \frac{1}{E} \left[-\frac{\kappa u^+}{u_\tau} e^{\kappa u^+} + \frac{\kappa u^+}{u_\tau} + \frac{1}{u_\tau} (\kappa u^+)^2 + \frac{1}{2u_\tau} (\kappa u^+)^3 \right]$$

- The Newton-Raphson method converges rapidly to a tight tolerance when applied in the above procedure.
- By using initial values from the previous timestep, only 1 or 2 iterations are sufficient.
- The predicted wall shear is then calculated from

$$\tau_w = \rho u_\tau^2$$

Implementation of LES

- For the most general case incompressible fluids where μ is not constant, LES momentum Eqns. (3.89) can be written as

$$\frac{\partial(\rho\bar{u}_i)}{\partial t} + \frac{\partial}{\partial x_j}(\rho\bar{u}_i\bar{u}_j) = \frac{\partial}{\partial x_j} \left[\mu \left(\frac{\partial\bar{u}_i}{\partial x_j} + \frac{\partial\bar{u}_j}{\partial x_i} \right) \right] - \frac{\partial}{\partial x_j}(\tau_{ij}) - \frac{\partial\bar{p}}{\partial x_i} \quad (3.105)$$

where $\tau_{ij} = \overline{\rho u_i u_j} - \rho\bar{u}_i\bar{u}_j$

- The SGS turbulence model for τ_{ij} is

$$\tau_{ij} = -2\mu_{SGS}\bar{S}_{ij} + \frac{1}{3}\tau_{kk}\delta_{ij} = -\mu_{SGS} \left(\frac{\partial\bar{u}_i}{\partial x_j} + \frac{\partial\bar{u}_j}{\partial x_i} \right) + \frac{1}{3}\tau_{kk}\delta_{ij} \quad (3.94)$$

- Then, the second term on the RHS of Eqn. (3.105) becomes

$$-\frac{\partial}{\partial x_j}(\tau_{ij}) = \frac{\partial}{\partial x_j} \left[\mu_{SGS} \left(\frac{\partial\bar{u}_i}{\partial x_j} + \frac{\partial\bar{u}_j}{\partial x_i} \right) \right] - \frac{\partial}{\partial x_i} \left(\frac{1}{3}\tau_{kk} \right) \quad (3.106)$$

- Substituting (3.106) into (3.105) gives

$$\frac{\partial(\rho\bar{u}_i)}{\partial t} + \frac{\partial}{\partial x_j}(\rho\bar{u}_i\bar{u}_j) = -\frac{\partial\bar{p}^*}{\partial x_i} + \frac{\partial}{\partial x_j} \left[(\mu + \mu_{SGS}) \left(\frac{\partial\bar{u}_i}{\partial x_j} + \frac{\partial\bar{u}_j}{\partial x_i} \right) \right] \quad (3.106)$$

where $\bar{p}^* = p + \frac{1}{3}\tau_{kk}$, ($k=1,3$)

- Note that Eqn. (3.106) is the same as that of the Navier-Stokes Eqns. given by (2.32a,b,c) with

μ replaced by $(\mu + \mu_{SGS})$ and

p replaced by \bar{p}^*

- After the solution, τ_{11} , τ_{22} , τ_{33} can be obtained by solving the system of equations (3.107), which are obtained from Eqn. (3.94):

$$\begin{bmatrix} 2/3 & -1/3 & -1/3 \\ -1/3 & 2/3 & -1/3 \\ -1/3 & -1/3 & 2/3 \end{bmatrix} \begin{bmatrix} \tau_{11} \\ \tau_{22} \\ \tau_{33} \end{bmatrix} = \begin{bmatrix} -2\mu\bar{S}_{11} \\ -2\mu\bar{S}_{22} \\ -2\mu\bar{S}_{33} \end{bmatrix} \quad (3.107)$$

Smagorinsky-Lilly SGS Model Equations in Generic Form

The Smagorinsky-Lilly SGS model equations can be written in generic form as

$$\frac{\partial(\rho\phi)}{\partial t} + \text{div}(\rho\mathbf{U}\phi) = \text{div}(\Gamma \text{grad } \phi) + s_\phi$$

Equation	ϕ	Γ	s_ϕ
Continuity	1	0	0
x-Momentum	\bar{u}	μ_{eff}	$\frac{\partial}{\partial x} \left(\mu_{\text{eff}} \frac{\partial \bar{u}}{\partial x} \right) + \frac{\partial}{\partial y} \left(\mu_{\text{eff}} \frac{\partial \bar{v}}{\partial x} \right) + \frac{\partial}{\partial z} \left(\mu_{\text{eff}} \frac{\partial \bar{w}}{\partial x} \right) - \frac{\partial \bar{p}^*}{\partial x}$
y-Momentum	\bar{v}	μ_{eff}	$\frac{\partial}{\partial x} \left(\mu_{\text{eff}} \frac{\partial \bar{u}}{\partial y} \right) + \frac{\partial}{\partial y} \left(\mu_{\text{eff}} \frac{\partial \bar{v}}{\partial y} \right) + \frac{\partial}{\partial z} \left(\mu_{\text{eff}} \frac{\partial \bar{w}}{\partial y} \right) - \frac{\partial \bar{p}^*}{\partial y}$
z-Momentum	\bar{w}	μ_{eff}	$\frac{\partial}{\partial x} \left(\mu_{\text{eff}} \frac{\partial \bar{u}}{\partial z} \right) + \frac{\partial}{\partial y} \left(\mu_{\text{eff}} \frac{\partial \bar{v}}{\partial z} \right) + \frac{\partial}{\partial z} \left(\mu_{\text{eff}} \frac{\partial \bar{w}}{\partial z} \right) - \frac{\partial \bar{p}^*}{\partial z}$

$$\mu_{\text{eff}} = \mu + \mu_{\text{SGS}}, \quad \mu_{\text{SGS}} = \rho(C_{\text{SGS}}\Delta)^2 |\bar{S}| = \rho(C_{\text{SGS}}\Delta)^2 \sqrt{2\bar{S}_{ij}\bar{S}_{ij}}, \quad \bar{S}_{ij} = \frac{1}{2} \left(\frac{\partial \bar{u}_i}{\partial x_j} + \frac{\partial \bar{u}_j}{\partial x_i} \right),$$

$$\bar{S}_{ij}\bar{S}_{ij} = \left(\frac{\partial \bar{u}}{\partial x} \right)^2 + \left(\frac{\partial \bar{v}}{\partial y} \right)^2 + \left(\frac{\partial \bar{w}}{\partial z} \right)^2 + 2 \left(\frac{\partial \bar{u}}{\partial y} + \frac{\partial \bar{v}}{\partial x} \right)^2 + 2 \left(\frac{\partial \bar{w}}{\partial x} + \frac{\partial \bar{u}}{\partial z} \right)^2 + 2 \left(\frac{\partial \bar{v}}{\partial z} + \frac{\partial \bar{w}}{\partial y} \right)^2,$$

$$\bar{p}^* = p + \frac{1}{3} \tau_{kk} \quad (k=1,3), \quad C_{\text{SGS}} = 0.065 - 0.25, \quad \Delta = (V_{\text{cell}})^{1/3}$$

References (in addition to that given in textbook)

- G. Medic and P. A. Durbin (2002) . Towards improved prediction of heat transfer on turbine blades, *Journal of turbomachinery*, 124:187-192.
- P. A. Durbin(1996). On the k - ϵ stagnation point anomaly. *International journal of heat and fluid flow*, 17(1):89–90.
- T. Jongen and Y. P. Marx (1997). Design of an unconditionally stable, positive scheme for the k-e and two-layer turbulence models, *Computers and Fluids*, vol. 26, No 5, pp. 469-487.
- Y. Zang, R.L., Street and J.R. Hoseff, 1993 . A dynamic subgrid-scale eddy viscosity model, *Physics of fluids A*, 5:3186-3196.
- B. Kader, Temperature and Concentration Profiles in Fully Turbulent Boundary Layers. *Int. J. Heat Mass Transfer*, 24(9):1541-1544, 1981
- H. Werner and H. Wengle. Large-Eddy Simulation of Turbulent Flow Over and Around a Cube in a Plate Channel. In *Eighth Symposium on Turbulent Shear Flows*, Munich, Germany, 1991.
- D.B. Spalding (1961). A single formula for the law of the wall. *Journal of Applied Mechanics, Trans ASME, Series E*, Vol. 28, pp. 455-458.
- E. Villiers (2006). The Potential of Large Eddy Simulation for the Modeling of Wall Bounded Flows. PhD thesis, Imperial College of Science, Technology and Medicine, Department of Mechanical Engineering.

- P. A. Durbin (1991). Near-wall turbulence closure modeling without damping functions, *Theoretical and Computational Fluid Dynamics*, vol. 3, No. 1, pp. 1-13.
- P. A. Durbin (1993). Application of a near-wall turbulence model to boundary layers and heat transfer, *Int. J. Heat and Fluid Flow*, vol. 14, no. 4, pp. 316-323.
- P. A. Durbin (1995). Separated flow computations with the $k-\varepsilon-v^2$ model, *AIAA Journal*, vol. 33, no. 4, pp.659-664.
- K. Hanjalic , M. Popovac and M. Hadziabdic (2004). A robust near-wall elliptic-relaxation eddy-viscosity turbulence model for CFD, *Int. J. Heat and Fluid Flow*, vol. 25, pp. 1047-1051.
- P. Moin and J. Kim (1982). Numerical investigation of turbulent channel flow. *Journal of Fluid Mechanics*, vol. 118, pp.341–377.
- W.P. Jones and M. Wille (1995). Large eddy simulation of a jet in a cross-flow. In *10th Symp. on Turbulent Shear Flows, pages 4:1 – 4:6, The Pennsylvania State University*.