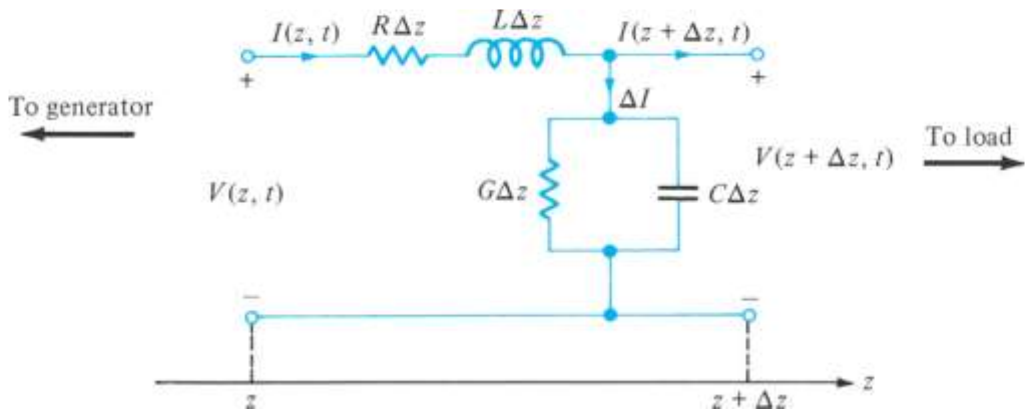
TRANSMISSION LINE THEORY

Transmission Lines (TL) hold Transverse Electromagnetic Fields (TEM).

Circuit Representation of TL's

A uniform TL may be modeled by the following circuit representation:



R: Series resistance per unit length of line (for both conductors (ohm/m)).

L: Series inductance per unit length of line (Henry/m).

G: Shunt conductance per unit length of line (mho/m).

C: Shunt capacitance per unit length of line (Farad/m).

Applying the Kirchhoff's laws, we can get the TL equations or telegrapher equations for the instantaneous voltage and current:

$$\frac{\partial v(z,t)}{\partial z} = -R\, i(z,t) - L\frac{\partial i(z,t)}{\partial t}$$

$$\frac{\partial i(z,t)}{\partial z} = -G\, v(z,t) - C\frac{\partial v(z,t)}{\partial t}$$

The solution of these equations, together with the electrical properties of the generator and load, allow us to determine the instantaneous voltage and current at any time t and any place z along the uniform TL.

Lossless Line: For the case of perfect conductors (R=0) and insulators (G=0), the telegrapher equations reduce to the following form:

$$\frac{\partial v(z,t)}{\partial z} = -L\frac{\partial i(z,t)}{\partial t}$$

$$\frac{\partial i(z,t)}{\partial z} = -C\frac{\partial v(z,t)}{\partial t}$$

Wave equation's for voltage and current on a lossless TL:

$$\frac{\partial^2 v(z,t)}{\partial z^2} - LC\frac{\partial^2 i(z,t)}{\partial t^2} = 0$$

$$\frac{\partial^2 i(z,t)}{\partial z} - LC\frac{\partial^2 i(z,t)}{\partial t^2} = 0$$

Although real lines are never lossless, lolessness approximation for practical TL's is very usefull.

TRANSMISSION LINES WITH SINUSOIDAL EXCITATION

Under sinusoidal steady state conditions, the TL equations take the form:

$$\frac{dV(z)}{dz} = -(R + j\omega L)I(z)$$

$$\frac{dI(z)}{dz} = -(G + j\omega C)V(z)$$

Where $V(z)$ and $I(z)$ are <u>voltage and current phasors.</u>

The real sinusoidal voltage and current waveforms are obtained from:

$$v(z,t) = \mathrm{Re}\left[V(z)e^{j\omega t}\right]$$

$$i(z,t) = \mathrm{Re}\left[I(z)e^{j\omega t}\right]$$

Wave Propagation on a TL

The second order differential equations for $V(z)$ and $I(z)$ are:

$$\frac{d^2V(z)}{dz^2} - \gamma^2 V(z) = 0$$

$$\frac{d^2I(z)}{dz^2} - \gamma^2 I(z) = 0$$

$$\gamma = \alpha + j\beta = \left[(R + j\omega L)(G + j\omega C)\right]^{1/2}$$

Where,

$\gamma =$ complex propagation constant.

$\alpha =$ attenuation constant (Np/m).

$\beta =$ phase constant (rad/m).

The solution for $V(z)$ is:

$$V(z) = V_0^+ e^{-\gamma z} + V_0^- e^{+\gamma z}$$

$$V(z) = V^+(z) + V^-(z)$$

Where V_0^+ and V_0^- are constants independent of *z*, and

$$V^+(z) = V_0^+ e^{-\gamma z}$$

$$V^-(z) = V_0^- e^{+\gamma z}$$

Represent voltage waves traveling on the line in the positive and negative z directions respectively.

WAVE PROPAGATION ON A TL

The second order equations for *V(z)* and *I(z)* are:

$$\frac{d^2V(z)}{dz^2} - \gamma^2 V(z) = 0$$

$$\frac{d^2I(z)}{dz^2} - \gamma^2 I(z) = 0$$

The solution for *V(z)* is:

$$V(z) = V_0^+ e^{-\gamma z} + V_0^- e^{+\gamma z}$$

Where V_0^+ is a constant of voltage waves traveling in the forward direction, V_0^- is a constant of voltage waves traveling in the reverse direction, independent of z.

Define

$$Z_0 = \left(\frac{(R + j\omega L)}{(G + j\omega C)} \right)^{1/2}$$

the characteristic impedance of the TL.

with,

$$I_0^+ = \frac{V_0^+}{Z_0} \qquad I_0^- = \frac{V_0^-}{Z_0}$$

Constants independent of z.

$$I^+(z) = \frac{V_0^+}{Z_0} e^{-\gamma z} \quad I^-(z) = \frac{V_0^-}{Z_0} e^{+\gamma z}$$

Or,

$$\mathrm{I}(z) = \frac{1}{Z_0}\left(V_0^+ e^{-\gamma z} - V_0^- e^{+\gamma z}\right)$$

Lossless Line:

For the lossless line R=0, G=0.

i) $Z_0 = \sqrt{\frac{L}{C}} \quad \Omega$

ii)

$$\gamma = \sqrt{(R + j\omega L)(G + j\omega C)}$$
$$= \sqrt{(j\omega L)(j\omega C)} = j\omega LC$$
$$= j\beta$$

iii) $\beta = \omega\sqrt{LC}$ Propagation constant

$\alpha = 0$ (no loss)

iv)

$$V(z) = V_0^+ e^{-j\beta z} + V_0^- e^{+j\beta z}$$

$$I(z) = \frac{1}{Z_0}(V_0^+ e^{-j\beta z} - V_0^- e^{+j\beta z})$$

v) If $V_0^+ = \left|V_0^+\right| e^{j\theta_+}$ $V_0^- = \left|V_0^-\right| e^{j\theta_-}$ then

$$v(z,t) = \left|V_0^+\right| \cos\left(\omega t - \beta z + \theta_+\right) + \left|V_0^-\right| \cos\left(\omega t + \beta z + \theta_-\right)$$

$$i(z,t) = \frac{1}{Z_0}\left[\left|V_0^+\right| \cos\left(\omega t - \beta z + \theta_+\right) - \left|V_0^-\right| \cos\left(\omega t + \beta z + \theta_-\right)\right]$$

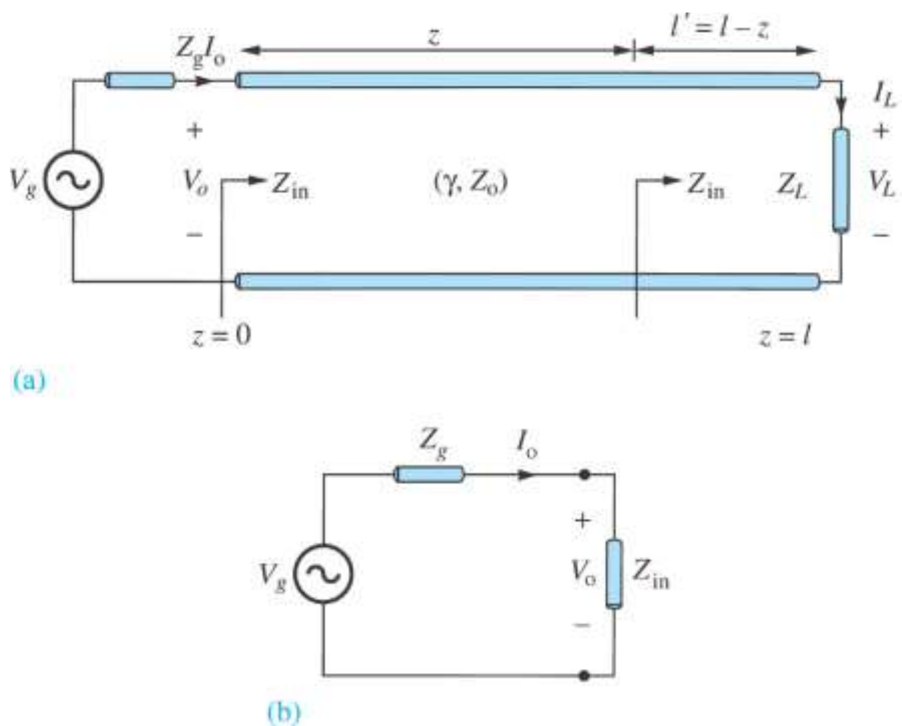
vi) $\lambda = \frac{2\pi}{\beta} = \frac{2\pi}{2\pi f\sqrt{LC}} = \frac{1}{f\sqrt{LC}}$

vii) $u_p = \frac{\omega}{\beta} = \frac{\omega}{\omega\sqrt{LC}} = \frac{1}{\sqrt{LC}}$ So,

$$u_p = \frac{1}{\sqrt{LC}} \qquad \lambda = \frac{v_p}{f}$$

$$u_p = \lambda f$$

TERMINATED TRANSMISSION LINE



Infinitely Long TL

For the infinite line, only forward traveling waves exist and therefore at z=0,

$$V = V_0^+ \qquad I = \frac{V_0^+}{Z_0} = I_0^+$$

$$Z_{in} = \frac{V}{I} = Z_0$$

Suppose now that the infinite line is broken at $z=\ell$. Since the line to the right of $z=\ell$ is still infinite, its input impedance is Z_0 and therefore replacing it by a load impedance of the same value does not change any of the conditions to the left of $z=\ell$. This means that a finite line terminated in its characteristic impedance is equivalent to an infinitely long line. Like the infinite line, a finite length line, terminated in Z_0 has no reflections. Also, its input impedance is equal to Z_0 and independent of the line length.

$$V_{in}=V_0=\frac{Z_0}{Z_0+Z_g}V_g$$

$$I_{in}=I_0=\frac{V_g}{Z_0+Z_g}$$

$$V_{in}=V_0^+$$

$$I_{in}=I_0^+=\frac{V_0^+}{Z_0}$$

So,

$$V_0^+ = \frac{Z_0}{Z_0 + Z_g} V_g$$

And

$$V(z) = \frac{Z_0}{Z_0 + Z_g} V_g \, e^{-\alpha z} e^{-j\beta z}$$

$$I(z) = \frac{V_g}{Z_0 + Z_g} e^{-\alpha z} e^{-j\beta z}$$

The time-averaged power absorbed by the load is:

$$P_L = \frac{1}{2}\mathrm{Re}\left(V_L I^*_L\right) = \frac{1}{2}\mathrm{Re}\left[\frac{Z_0}{Z_0 + Z_g} V_g e^{-\alpha l} e^{-j\beta l} \left(\frac{V_g}{Z_0 + Z_g}\right)^* e^{-\alpha l} e^{j\beta l}\right]$$

$$P_L = \frac{Z_0}{2} e^{-2\alpha l} \frac{\left|V_g\right|^2}{\left|Z_0 + Z_g\right|^2} = \frac{1}{2} Z_0 e^{-2\alpha l} \left|\frac{V_g}{Z_0 + Z_g}\right|^2$$

If $\alpha = 0$ (lossless line) $\quad P_L = \frac{1}{2} Z_0 \left|\frac{V_g}{Z_0 + Z_g}\right|^2 = P_{in}$