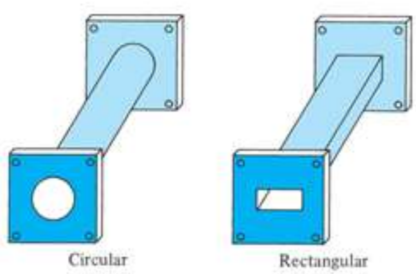
TRANSMISSION LINES AND WAVEGUIDES

Definition: Transmission Line

TL is the term to describe transmission systems with two or more metallic conductors electrically insulated from each other.

Definition: Waveguide

The propagation in a waveguide is generally ensured by successive reflections on the guide boundaries. These are conducting walls in the case of metallic WG's. Dielectric WG's and optical fibers utilize the total internal reflection.



Clasification of the Modes of Propagation

The presence and apsence of longitudial field components affects the propagation behaviour of the modes. Four mode categories can exist:

E_z	H_z	Name	Mode
0	0	Transverse Electromagnetic	TEM
0	$\neq 0$	Transverse Electric	**TE**
$\neq 0$	0	Transverse Magnetic	**TM**
$\neq 0$	$\neq 0$	Hybrid	

Separation of Maxwell's Equations into Longitudinal and Transverse Components

The del operator ∇ can be expressed as:

$$\nabla = \nabla_t + \hat{a}_z \frac{\partial}{\partial z}$$

Where ∇_t is the transverse del operator and is given in the Cartesian coordinates by

$$\nabla_t = \hat{a}_x \frac{\partial}{\partial x} + \hat{a}_y \frac{\partial}{\partial y}$$

Assuming now, time-harmonic fields with an $e^{j\omega t}$ time dependence and wave propagation along the + z-axis, the field vectors can be written as:

$$\bar{E}(x,y,z)=\left[\bar{e}(x,y)+\hat{a}_z e_z(x,y)\right]e^{-j\beta z}$$

$$\bar{H}(x,y,z)=\left[\bar{h}(x,y)+\hat{a}_z h_z(x,y)\right]e^{-j\beta z}$$

GENERAL SOLUTIONS FOR THE TEM, TE AND TM WAVES

The above six equations can be solved for the four transverse field components in terms of e_z and h_z as:

$$h_x=\frac{j}{k_c^2}\left(\omega\varepsilon\frac{\partial e_z}{\partial y}-\beta\frac{\partial h_z}{\partial x}\right)$$

$$h_y=-\frac{j}{k_c^2}\left(\omega\varepsilon\frac{\partial e_z}{\partial x}+\beta\frac{\partial h_z}{\partial y}\right)$$

$$e_x=-\frac{j}{k_c^2}\left(\beta\frac{\partial e_z}{\partial x}+\omega\mu\frac{\partial h_z}{\partial y}\right)$$

$$e_y=\frac{j}{k_c^2}\left(-\beta\frac{\partial e_z}{\partial y}+\omega\mu\frac{\partial h_z}{\partial x}\right)$$

Where $k_c^2=k^2-\beta^2$ and $k=\omega\sqrt{\varepsilon\mu}$.

The wave impedance of a TEM mode is defined as:

$$Z_{TEM} = \frac{e_x}{h_y} = \frac{\omega\mu}{\beta} = \sqrt{\frac{\mu}{\varepsilon}} = \eta$$

$$Z_{TEM} = -\frac{e_y}{h_x} = \sqrt{\frac{\mu}{\varepsilon}} = \eta$$

The propagation of a TEM mode on a homogeneous multi-conductor line only depends on the propagation medium; it is independent of the geometry and the line dimensions.
The TEM mode can propagate at any frequency on a multi-conductor.

For a TEM mode, we have,

$$\bar{h}(x,y) = \frac{1}{Z_{TEM}} \hat{a}_z X \, \bar{e}(x,y)$$

TE WAVES

$E_z = 0$ (by definition), $H_z \neq 0$. Then,

$$h_x = -\frac{j\beta}{k_c^2}\frac{\partial h_z}{\partial x}$$

$$h_y = -\frac{j\beta}{k_c^2}\frac{\partial h_z}{\partial y}$$

$$e_x = -\frac{j\omega\mu}{k_c^2}\frac{\partial h_z}{\partial y}$$

$$e_y = \frac{j\omega\mu}{k_c^2}\frac{\partial h_z}{\partial x}$$

Where, h_z satisfies:

$$\left(\frac{\partial}{\partial x^2} + \frac{\partial}{\partial x^2} + k_c^2\right)h_z = 0, \qquad k_c^2 = k^2 - \beta^2$$

The solutions of the above equation **are only found for particular values of** k_c, when we apply the boundary conditions. These are the eigenvalues of the TE mode problem. The transverse wavenumber k_c is specified by the guide cross-section (shape and size) and by the transverse distribution of the fields for the mode considered; it is independent of the medium filling the guide. In a homogeneous guide, the transverse wave number is always real and positive.

The TE wave impedance is:

$$Z_{TE} = \frac{E_x}{H_y} = -\frac{E_y}{H_x} = \frac{\omega\mu}{\beta} = \frac{k\eta}{\beta}$$

Since β is frequency dependent, Z_{TE} depends on the frequency. TE waves can be supported inside closed conductors, as well as between two or more conductors.

TM WAVES

$E_z \neq 0$ (by definition), $H_z = 0$. Then,

$$h_x = \frac{j\omega\varepsilon}{k_c^2}\frac{\partial e_z}{\partial y}$$

$$h_y = -\frac{j\omega\varepsilon}{k_c^2}\frac{\partial e_z}{\partial x}$$

$$e_x = -\frac{j\beta}{k_c^2}\frac{\partial e_z}{\partial x}$$

$$e_y = -\frac{j\beta}{k_c^2}\frac{\partial e_z}{\partial y}$$

Where e_z satisfies:

$$\left(\frac{\partial}{\partial x^2}+\frac{\partial}{\partial x^2}+k_c^2\right)e_z=0, \qquad k_c^2=k^2-\beta^2$$

The TM Wave Impedance:

$$Z_{TM}=\frac{E_x}{H_y}=-\frac{E_y}{H_x}=\frac{\beta}{\omega\varepsilon}=\frac{\beta\eta}{k} \ (Frequency\ dependent)$$

TM waves can be supported inside hollow conductors, as well as between two conductors.

RECTANGULAR WAVEGUIDES

Hollow WG's are commonly used as TL's at frequencies above *5GHz*. Compared to coaxial lines; WG's have the following advantages:

1) Higher power handling capability
2) Lower loss per unit length
3) A simpler, lower cost mechanical structure
4) The reflections caused by the flanges used in connecting WG sections is usually less than that associated with coaxial connectors.

The disadvantages are:

1) Lager cross-sectional dimensions

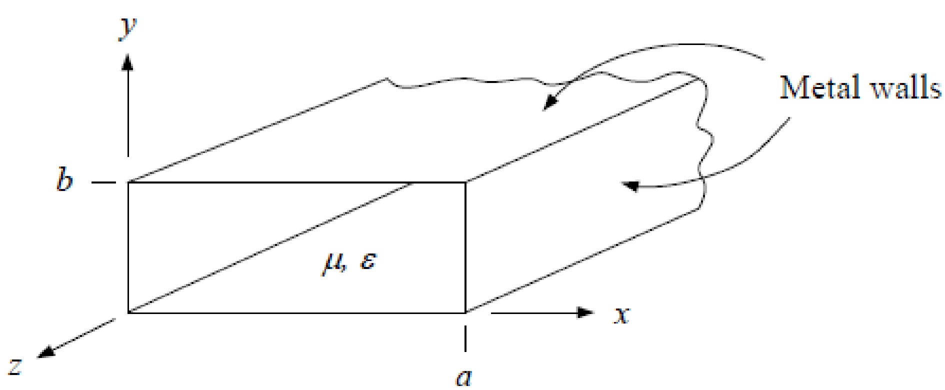
2) Lower usable bandwidth

A large variety of components such as couplers, detectors, isolators, attenuators and slotted lines are commercially available for various WG bands from *1GHz* to over *220GHz.*

The hollow WG's can support TM and TE modes but not TEM modes.

RECTANGULAR WAVEGUIDES

Conventionally, the longer side of the WG is located along the x-axis, so $a > b$.



<u>TE MODES</u>

$E_z = 0$ by definition and h_z satisfies:

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + k_c^2\right) h_z(x, y) = 0 \text{ ....(*)}$$

The complete expression for $H_z(x, y)$ is:

$$H_z(x,y,z) = h_z(x,y)e^{-j\beta z}$$

And $k_c^2 = k^2 - \beta^2$ with $k^2 = \omega^2 \varepsilon \mu$.

A Separation of Variables Solution for $h_z(x,y)$

Assume $h_z(x,y) = X(x)Y(y)$

Substituting this into (*) and dividing each term by XY :

$$\frac{1}{X}\frac{d^2X}{dx^2} + \frac{1}{Y}\frac{d^2Y}{dy^2} + k_c^2 = 0$$

This equation can be satisfied for all x and y only if:

$$\frac{1}{X}\frac{d^2X}{dx^2} + k_x^2 = 0 \qquad \frac{1}{Y}\frac{d^2Y}{dy^2} + k_y^2 = 0$$

With:

$$k_x^2 + k_y^2 = k_c^2$$

The general solution for $h(x,y)$ can then is written as:

$$h_z(x,y) = \left(A\cos k_x x + B\sin k_x x\right)\left(C\cos k_y y + D\sin k_y y\right)$$

<u>The Boundary Conditions:</u>

We must have:

$e_x(x,y)=0 \quad at \;\; y=0$ and $y=b$

$e_y(x,y)=0 \quad at \;\; x=0$ and $x=a$

But we had,

$$e_x=-\frac{j\omega\mu}{k_c^2}\frac{\partial h_z}{\partial y} \qquad e_y=\frac{j\omega\mu}{k_c^2}\frac{\partial h_z}{\partial x}$$

Therefore it is necessary that:

We must have

$\frac{\partial h_z}{\partial y}=0 \quad at \;\; y=0$ and $y=b$

$\frac{\partial h_z}{\partial x}=0 \quad at \;\; x=0$ and $x=a$

Or

$-Ak_x\sin k_x x+Bk_x\cos k_x x=0 \; at \;\; x=0$ and $x=a$ which gives $B=0$ for $k_x\neq 0$ and

$$-Ak_x\sin k_x a=0$$

Which gives,

$k_x a=m\pi \quad (m=0,1,2,\ldots)$

Similarly,

$-Ck_y \sin k_y y + Dk_y \cos k_y y = 0 \; at \;\; y = 0$ and $y = b$ which gives $D = 0$ for $k_y \neq 0$ and

$$-Ck_y \sin k_y b = 0$$

Which gives,

$$k_y b = n\pi \quad (n = 0, 1, 2, \dots)$$

So, $k_x = \dfrac{m\pi}{a} \qquad k_y = \dfrac{n\pi}{b} \quad m = 0, 1, 2, \dots \quad n = 0, 1, 2, \dots$

Then,

$$h_z(x, y) = A_{mn} \cos\left(\frac{m\pi}{a} x\right) \cos\left(\frac{n\pi}{b} y\right)$$

And

$$H_z(x, y, z) = A_{mn} \cos\left(\frac{m\pi}{a} x\right) \cos\left(\frac{n\pi}{b} y\right) e^{-j\beta z}$$

Where A_{mn} : Constant depending on the excitation strength.

The other field components are found as:

$$E_x = \frac{j\omega\mu\pi}{k_c^2 b} nA_{mn} \cos\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right)e^{-j\beta z}$$

$$E_y = -\frac{j\omega\mu\pi}{k_c^2 a} mA_{mn} \sin\left(\frac{m\pi}{a}x\right)\cos\left(\frac{n\pi}{b}y\right)e^{-j\beta z}$$

$$H_x = \frac{j\beta\pi}{k_c^2 a} mA_{mn} \sin\left(\frac{m\pi}{a}x\right)\cos\left(\frac{n\pi}{b}y\right)e^{-j\beta z}$$

$$H_y = \frac{j\beta\pi}{k_c^2 b} nA_{mn} \cos\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right)e^{-j\beta z}$$

<u>The propagation constant</u>

$$\beta = \sqrt{k^2 - k_c^2}$$

Since $k_c^2 = k_x^2 + k_y^2 = \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2$

$$\beta = \left[k^2 - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2\right]^{\frac{1}{2}}$$

$$\beta = \left[\omega^2\varepsilon\mu - \left\{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2\right\}\right]^{\frac{1}{2}}$$

Consider two cases of interest:

A) $k > k_c$

If the frequency *f* is high enough so that, for a given set of values of *a, b, m* and *n*,

$$k=\omega\sqrt{\varepsilon\mu}=2\pi f\sqrt{\varepsilon\mu}>\left[\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2\right]^{1/2}$$

then β is real. Real β corresponds to propagation. k_c is the cutoff wavenumber. Each mode has a cutoff frequency $f_{c_{mn}}$ given by:

$$f_{c_{mn}}=\frac{k_c}{2\pi\sqrt{\varepsilon\mu}}=\frac{1}{2\pi\sqrt{\varepsilon\mu}}\left[\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2\right]^{1/2}$$

Let $v=\frac{1}{\sqrt{\varepsilon\mu}}$ =Phase velocity for an unbounded medium filled with material having ε and μ. Then,

$$f_{c_{mn}}=\frac{1}{2\pi\sqrt{\varepsilon\mu}}\left[\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2\right]^{1/2}=\frac{v}{2}\left[\left(\frac{m}{a}\right)^2+\left(\frac{n}{b}\right)^2\right]^{1/2}$$

The mode with the lowest cutoff frequency is called the **dominant mode.**

Since *a>b*, the lowest f_c occurs for m=1 and *n=0*. So,

$$f_{c_{10}}=\frac{1}{2\pi\sqrt{\varepsilon\mu}}\frac{\pi}{a}=\frac{1}{2a\sqrt{\varepsilon\mu}}=\frac{v}{2a}$$

TE_{10} mode is the dominant mode.

Since E_x, E_y, H_x and H_y are all zero for *m*=*n*=0, there is no TE_{00} mode.

B) $k<k_c$

In this case β becomes purely imaginary,

$$\beta = -j\left[\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 - k^2\right]^{1/2} = -jq$$, q is real. Then, the term $e^{-j\beta z}$ becomes $e^{-j(-jq)z} = e^{-qz}$ which corresponds to the attenuation of fields exponentially. Such modes are non-propagating or evanescent. Note that this attenuation is not associated with the dissipative losses.

The Wave Impedance:

$$Z_{TE} = \frac{E_x}{H_y} = -\frac{E_y}{H_x} = \frac{k\eta}{\beta}$$

$$k = \omega\sqrt{\varepsilon\mu}, \qquad \eta = \sqrt{\frac{\mu}{\varepsilon}}$$

i) When $k > k_c$ $(f > f_c)$, β and Z_{TE} are both real.

ii) When $k < k_c$ $(f < f_c)$, β and Z_{TE} are both purely imaginary.

$$Z_{TE_{10}} = \frac{k\eta}{\beta_{10}} = \frac{k\eta}{\left[\omega^2\varepsilon\mu - \left(\frac{\pi}{a}\right)^2\right]^{1/2}} = \frac{\omega\mu}{\left[\omega^2\varepsilon\mu - \left(\frac{\pi}{a}\right)^2\right]^{1/2}}$$

The guide wavelength:

$$\lambda_g = \frac{2\pi}{\beta}$$

For a propagating mode $\beta = \sqrt{k^2 - k_c^2}$ so $\beta < k$ and

$$\lambda_g = \frac{2\pi}{\beta} > \lambda = \frac{2\pi}{k}$$

$$\lambda_g = 2\pi \left[k^2 - \left(\frac{m\pi}{a} \right)^2 - \left(\frac{n\pi}{b} \right)^2 \right]^{-1/2}$$

Let, $\lambda_c = \frac{2\pi}{k_c},$, cutoff wavelength. Then,

$$\lambda_g = \frac{2\pi}{\sqrt{k^2 - k_c^2}} = \frac{1}{\left[\left(\frac{k}{2\pi} \right)^2 - \left(\frac{k_c}{2\pi} \right)^2 \right]^{1/2}}$$

Or,

$$\lambda_g = \frac{1}{\left(\frac{1}{\lambda^2} - \frac{1}{\lambda_c^2} \right)^{1/2}} = \frac{1}{\frac{1}{\lambda} \left(1 - \frac{\lambda^2}{\lambda_c^2} \right)^{1/2}} = \frac{\lambda}{\sqrt{1 - (\lambda / \lambda_c)^2}}$$

$$\lambda_g = \frac{\lambda}{\sqrt{1 - (\lambda / \lambda_c)^2}} = \frac{v / f}{\sqrt{1 - (f_c / f)^2}} , \qquad v = \frac{1}{\sqrt{\varepsilon\mu}}$$

We have the following relationship:

$$\frac{1}{\lambda_g^2} + \frac{1}{\lambda_c^2} = \frac{1}{\lambda^2}$$

For the dominant mode:

$k_c = \frac{\pi}{a}$, $\lambda_c = \frac{2\pi}{\pi / a}$, so $\lambda_c = 2a$

$$\frac{1}{\lambda_g^2} = \frac{1}{\lambda^2} - \frac{1}{(2a)^2}$$

The phase velocity:

$$v_p = \frac{\omega}{\beta} = \frac{\omega}{\sqrt{k^2 - k_c^2}} > \frac{\omega}{k} = \frac{1}{\sqrt{\varepsilon\mu}}$$

$$v_p > \frac{1}{\sqrt{\varepsilon\mu}}$$

$$f_c = \frac{k_c}{2\pi\sqrt{\varepsilon\mu}}$$

$$v_p = \omega\left[\omega^2\varepsilon\mu - \omega_c^2\varepsilon\mu\right]^{-1/2} = \frac{\omega}{\sqrt{\varepsilon\mu}} \frac{1}{\omega\sqrt{1 - \left(\omega_c/\omega\right)^2}}$$

$$v_p = \frac{1/\sqrt{\varepsilon\mu}}{\sqrt{1 - \left(f_c/f\right)^2}}$$

TM MODES

We have $H_z = 0$ and $e_z(x,y)$ must satisfy $\left(\frac{\partial^2 e_z}{\partial x^2} + \frac{\partial^2 e_z}{\partial y^2} + k_c^2 e_z\right) = 0$, $k_c^2 = k^2 - \beta^2$

The general solution is,

$$e_z(x,y) = (A\cos k_x x + B\sin k_x x)(C\cos k_y y + D\sin k_y y)$$

The boundary conditions are:

$e_z(x,y) = 0$ for $x = 0$ *and* $x = a$

$e_z(x,y) = 0$ for $y = 0$ *and* $y = b$

Using these B.C.'s we obtain:

A=0 and $k_x = \frac{m\pi}{a}, \ m = 1,2,3,...$

C=0, and $k_y = \frac{n\pi}{b}, \ n = 1,2,3,...$

So,

$$E_z(x,y,z) = B_{mn}\sin\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right)e^{-j\beta z}$$

The other field components are:

$$E_x(x,y,z) = -\frac{j\beta m\pi}{ak_c^2}B_{mn}\cos\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right)e^{-j\beta z}$$

$$E_y(x,y,z) = -\frac{j\beta n\pi}{bk_c^2}B_{mn}\sin\left(\frac{m\pi}{a}x\right)\cos\left(\frac{n\pi}{b}y\right)e^{-j\beta z}$$

$$H_x(x,y,z) = \frac{j\omega\varepsilon\pi n}{bk_c^2}B_{mn}\sin\left(\frac{m\pi}{a}x\right)\cos\left(\frac{n\pi}{b}y\right)e^{-j\beta z}$$

$$H_y(x,y,z) = -\frac{j\omega\varepsilon\pi m}{ak_c^2}B_{mn}\cos\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right)e^{-j\beta z}$$

$$\beta = \sqrt{k^2 - k_c^2} = \left[k^2 - \left(\frac{m\pi}{a} \right)^2 - \left(\frac{n\pi}{b} \right)^2 \right]^{1/2}$$

If either *m* or *n* is zero, the fields vanish identically. So there are no TM_{00}, TM_{01} or TM_{01} modes.

The lowest order TM mode is the TM_{11} mode with:

$$f_{c_{11}} = \frac{1}{2\pi\sqrt{\varepsilon\mu}} \left[\left(\frac{\pi}{a} \right)^2 + \left(\frac{\pi}{b} \right)^2 \right]^{1/2} > f_{c_{TE10}}$$

Since f_c of the lowest order TM mode is greater than the f_c of the lowest order TE mode, TE_{10} is the lowest among all modes.

<u>The wave-impedance</u>

$$Z_{TM} = \frac{E_x}{H_y} = -\frac{E_y}{H_x} = \frac{\eta}{k}\beta$$