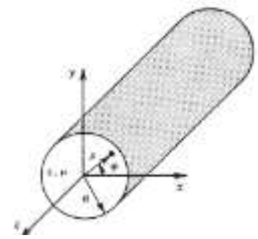
CIRCULAR WAVEGUIDES

A circular waveguide is a circular metalic guiding structure. As in the case of the rectangular waveguides, circular waveguides also can hold TE and TM modes.



For a circular waveguide of radius 'a' we can perform the same sequence of steps in cylindrical coordinates as we did in rectangular coordinates to find the transverse field components in terms of the longitudinal (i.e. Ez, Hz) components.

$$H_\rho = \frac{j}{k_c^2}\left(\frac{\varepsilon\omega}{\rho}\frac{\partial Ez}{\partial\phi} - \beta\frac{\partial Hz}{\partial\rho}\right)$$

$$H_\phi = \frac{-j}{k_c^2}\left(\varepsilon\omega\frac{\partial Ez}{\partial\rho} + \frac{\beta}{\rho}\frac{\partial Hz}{\partial\phi}\right)$$

$$E_\rho = \frac{-j}{k_c^2}\left(\beta\frac{\partial Ez}{\partial\rho} + \frac{\omega\mu}{\rho}\frac{\partial Hz}{\partial\phi}\right)$$

$$E_\phi = \frac{-j}{k_c^2}\left(\frac{\beta}{\rho}\frac{\partial Ez}{\partial\phi} - \omega\mu\frac{\partial Hz}{\partial\rho}\right)$$

where $k_c^2 = k^2 - \beta^2$ as before. Please note that here (as well as in rectangular waveguide derivation), we have assumed $e^{-j\beta z}$ propagation.

TE Modes

For TE modes $E_z = 0$ and and H_z is the solution of the wave equation:

$$\nabla^2 H_z + k^2 H_z = 0$$

If $H(\rho,\phi,z) = h_z(\rho,\phi)e^{-j\beta z}$, the wave equation for h_z in the cylindrical coordinates is:

$$\left(\frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2} + \frac{\partial^2}{\partial z^2} + k_c^2\right) \mathrm{h}_z(\rho, \phi, z) = 0$$

As in the case of the rectangular waveguides the solution can be obtained by using the method of separation of variables and the boundary conditions.

The solution obtained for H_z is:

$$H_z(\rho,\ \phi,\ z) = \left[A\, sin(n\phi) + B\, cos(n\phi)\right] J_n(k_c\rho)e^{-j\beta z}$$

The relative values of A and B have to do with the absolute coordinate frame we use to define the waveguide.

For example, let $A = F\ \sin(n\phi_0)$ and $B = F\ \cos(n\phi_0)$ (you can find a value of F and ϕ_0 for this purpose). Then:

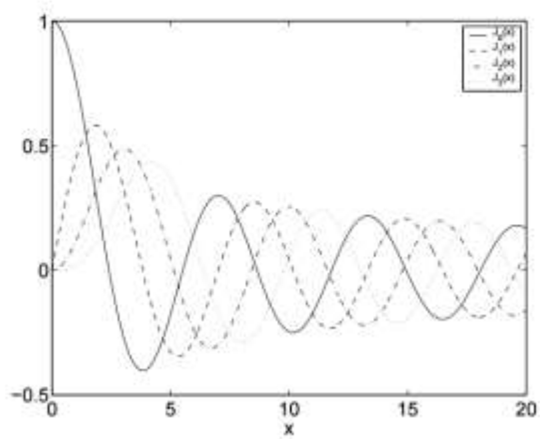
$$A\, sin(n\phi) + B\, cos(n\phi) = F\ \cos\left[n(\phi - \phi_0)\right]$$

We can think that ϕ_0 is the reference for measuring, ϕ.

So,

$$H_z(\rho,\ \phi,\ z) = A_{nm} cos(n\phi) J_n(k_c\rho)e^{-j\beta z}$$

where J_n is the n[th] order Bessel function of the first kind with $x = k_c\rho$.



We still need to determine k_c. The boundary condition that we can apply is $E_\phi(a,\ \phi,\ z) = 0$ where $\rho = a$ represents the waveguide boundary. Since,

$$E_\phi(\rho,\ \phi,\ z) = \frac{j\omega\mu}{k_c^2}\ \frac{\partial Hz}{\partial\rho}$$

Then,

$$E_{\phi}(\rho,\ \phi,\ z) = \frac{j\omega\mu}{k_c^2} A_{nm} cos(n\phi)\ J_n^1(k_c\rho) e^{-j\beta z}$$

$$J'_n\ (k_c\rho) = \frac{d}{d\rho} J_n(k_c\rho)$$

$$J'_n\ (k_c a) = 0$$

$$k_c a = p'_{nm}$$

$$k_{c_{nm}} = \frac{p'_{nm}}{a}$$

p'_{nm} is the m[th] root of derivative of J_n (J'_n).

Values of p'_{nm} for TE mode of a circular waveguide.

n, order of Bessel Function	First Root of Derivative, m=1	Second Root of Derivative, m=2	Third Root of Derivative, m=3
0	3.832	7.016	10.174
1	1.841	5.331	8.536
2	3.054	6.706	9.970

Cutoff frequency of TE mode:

$$f_{c_{nm}} = \frac{c\ p'_{nm}}{2\ \pi\ a}$$

The first mode to propagate is the mode with the smallest p'_{nm}, which is TE_{11} mode. This is the dominant mode of the circular waveguide. Since $m \geq 1$ there is no TE_{10} mode but there is a TE_{01} mode.

The other parameters:

$$\lambda_g = \frac{\lambda}{\sqrt{1-\left(\frac{f_c}{f}\right)^2}}$$

$$\beta = \frac{2\,\pi}{\lambda_g}$$

$$Z_{TE} = \frac{\eta}{\sqrt{1-\left(\frac{f_c}{f}\right)}}$$

TM Modes

The derivation is the same except that we are solving for E_z where H_z=0

$$\left(\frac{\partial^2}{\partial\rho^2}+\frac{1}{\rho}\frac{\partial}{\partial\rho}+\frac{1}{\rho^2}\,\frac{\partial^2}{\partial\phi^2}+\,\frac{\partial^2}{\partial z^2}\,+\,k_c^{\,2}\right) e_z(\rho,\ \phi,\ z)\ =\ 0$$

n, order of Bessel Function	First Root , m=1	Second Root, m=2	Third Root of, m=3
0	2.405	5.520	8.654
1	3.832	7.016	10.174
2	5.135	8.417	11.620

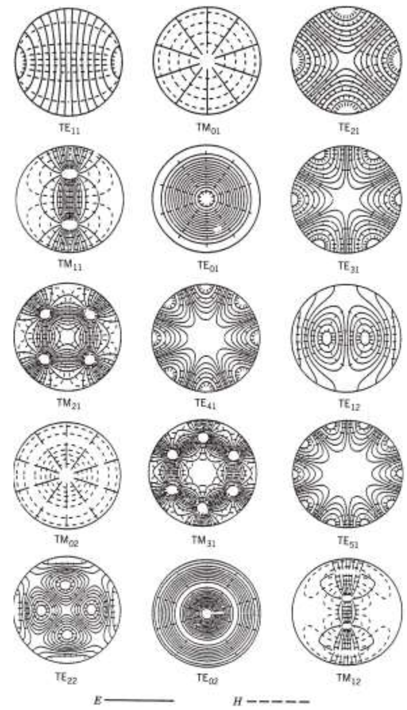
$$E_\phi(\rho,\ \phi)=0 \quad at \quad \rho=a$$

$$J_n\ (k_c a)\ =0$$

$$k_c = \frac{p_{nm}}{a}$$

$$f_{c_{nm}} = \frac{c\ p_{nm}}{2\ \pi\ a}$$

The first TM mode is TM_{01} mode with p_{01}=2.405. Sicne, p'_{11}=1.841 of the lowest order, then TE_{11} is the dominant mode of the circular waveguide.



TE_{11}
TM_{01}
TE_{21}
TM_{11}
TE_{01}
TE_{31}
TM_{21}
TE_{41}
TE_{12}
TM_{02}
TM_{31}
TE_{51}
TE_{22}
TE_{02}
TM_{12}
E ———
H - - - - -