MICROWAVE RESONATORS

Electromagnetic components that exhibit a resonant behavior with frequency find many application areas. They are used in:

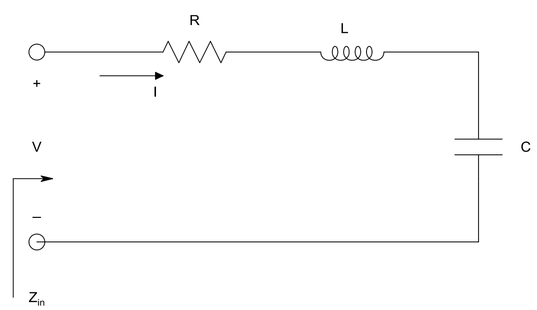
a) Filters,
b) Oscillators,
c) Tuned amplifiers,
d) Phase Equalizers,
e) Frequency meters,
f) Impedance matching,
g) Production of high-field strengths with limited input signals.

Resonators exhibit strong frequency-dependent behavior within narrow frequency bands centered about discrete frequencies called **resonators**.

Below 300 MHz the resonators usually consists of **lumped inductances** and **capacitances**. But the lumped circuit implies that all of its elements are small compared to the wavelength. With the advent of integrated-circuit technology it has become possible to construct elements that behave like lumped inductances and capacitances at microwave frequencies, but these elements are so small that power handling is a problem. Further, dielectric, ohmic and radiative losses can become unacceptable.

LUMPED RESONANT CIRCUITS

a) Series Resonant Circuit



Input Impedance:

$$Z_{in} = R + j\omega L + \frac{1}{j\omega C}$$

$$Z_{in} = R + j\left(\omega L - \frac{1}{\omega C}\right)$$

The complex power delivered (from the source) to the above circuit is:

$$P_{in} = \frac{1}{2}VI^* = \frac{1}{2}Z_{in}II^* = \frac{1}{2}V\frac{V^*}{Z_{in}}$$

$$P_{in} = \frac{1}{2}|I|^2 Z_{in} = \frac{1}{2}|I|^2\left[R + j\left(\omega L - \frac{1}{\omega C}\right)\right]$$

The time-average power dissipated in R is:

$$P_{\text{diss}} = \frac{1}{2}|I|^2 R$$

Complex Power:

$$\mathrm{P_{in}} = \mathrm{P_{loss}} + \mathrm{j}2\omega\left(\mathrm{W_m} - \mathrm{W}_e\right)$$

$$\mathrm{P_{in}} = \frac{1}{2}|I|^2 \mathrm{Z_{in}}$$

Where, average magnetic energy stored in the inductor:

$$\mathrm{W_m} = \frac{1}{4}|\mathrm{I}|^2 L$$

Average electric energy stored in the capacitor:

$$\mathrm{W}_e = \frac{1}{4}|\mathrm{V}_c|^2 C = \frac{1}{4}|\mathrm{I}|^2\frac{1}{\omega^2 C}$$

$$\mathrm{Z}_{in} = \frac{\mathrm{P_{diss}} + \mathrm{j}2\omega\left(\mathrm{W_m} - \mathrm{W}_e\right)}{\frac{1}{2}|I|^2}$$

Resonance occurs when:

$W_e = W_m$

Then,

$$Z_{in} = \frac{P_{diss}}{\frac{1}{2}|I|^2} = R \rightarrow \text{Purely Real}$$

$W_e = W_m$

Implies that,

$$\frac{1}{4}|I|^2 \frac{1}{\omega_o{}^2 C} = \frac{1}{4}|I|^2 L$$

(At resonance $\omega = \omega_o$)

$$\omega_o = \frac{1}{\sqrt{LC}} \rightarrow \text{Resonant Frequency}$$

Quality Factor Q

Q is a measure of the loss of a resonant circuit—lower loss implies a higher Q. Resonator losses may be due to conductor loss, dielectric loss, or radiation loss, and are represented by the resistance, R, of the equivalent circuit. External connected networks may introduce additional loss.

$$Q = \omega_o \frac{\text{time} - \text{average energy stored in the system}}{\text{energy loss per second in the system}}$$

$$Q = \omega_o \frac{W_e + W_m}{P_{diss}}$$

Since, $W_e = W_m$

At resonance,

$$Q = \omega_o \frac{2W_m}{P_{loss}} = \frac{2\frac{1}{4}|I|^2 L}{\frac{1}{2}|I|^2 R} = \omega_o \frac{L}{R} = \frac{1}{\sqrt{LC}} \frac{L}{R} = \frac{1}{\sqrt{LC}} \frac{LC}{RC}$$

$$Q=\frac{\omega_o L}{R}=\frac{1}{\omega_o RC}$$

Q increases as R decreases. (Increases when losses degrease).

Behavior of Z_{in} Near the Resonant Frequency

Let,

$\omega=\omega_o+\Delta\omega$ where $\Delta\omega$ is small

$$Z_{in}=R+j\left(\omega L-\frac{1}{\omega C}\right)=R+j\omega L\left(1-\frac{1}{\omega^2 LC}\right)$$

$$Z_{in}=R+j\omega L\left(1-\frac{\omega_o^2}{\omega^2}\right)=R+j\omega L\left(\frac{\omega^2-\omega_o^2}{\omega^2}\right)$$

$$\omega^2-\omega_o^2=(\omega-\omega_o)(\omega+\omega_o)\simeq\Delta\omega(\omega+\omega-\Delta\omega)\simeq 2\omega\Delta\omega$$

With

$$\omega_o=\omega-\Delta\omega$$

And

$$Z_{in}\simeq R+j\omega L\left(\frac{2\omega\Delta\omega}{\omega^2}\right)=R+j2L\Delta\omega$$

$$Z_{in}\simeq R+j2L\Delta\omega$$

$$Z_{in}\simeq R+j2\frac{QR\Delta\omega}{\omega_o}$$

This form will be useful for identifying equivalent circuits with distributed element resonators.

Bandwidth, Z_{in} versus angular frequency

$$Z_{in} = R + j\left(\omega L - \frac{1}{\omega C}\right)$$

$$\omega_o = \frac{1}{\sqrt{LC}}, \qquad Q = \frac{\omega_o L}{R} = \frac{1}{\omega_o RC}$$

$$Z_{in} = R\left[1 + j\left(\frac{\omega L}{R} - \frac{1}{\omega RC}\right)\right]$$

$$Z_{in} = R\left[1 + j\left(\frac{\omega Q}{\omega_o} - \frac{\omega_o Q}{\omega}\right)\right] = R\left[1 + jQ\left(\frac{\omega}{\omega_o} - \frac{\omega_o}{\omega}\right)\right]$$

$$Z_{in} = R\left[1 + jQ\left(\frac{\omega}{\omega_o} - \frac{\omega_o}{\omega}\right)\right]$$

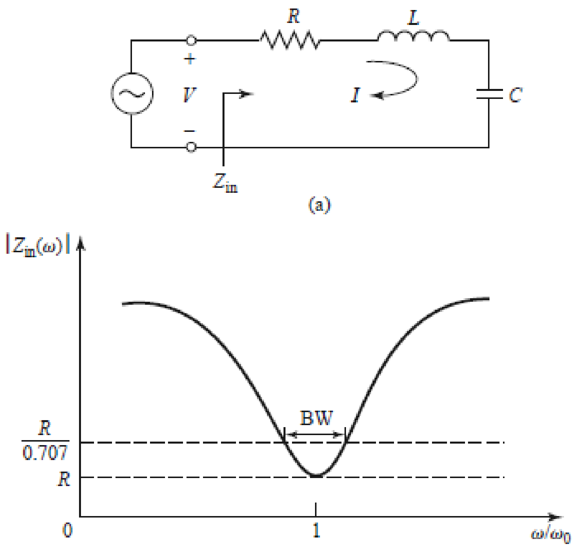
$$\left|Z_{in}\right| = R\left[\left[1 + Q^2\left(\frac{\omega}{\omega_o} - \frac{\omega_o}{\omega}\right)^2\right]\right]^{1/2}$$

Note that, when

$$\omega_o = \omega, \quad \left|Z_{in}\right| = R$$

becomes minimum.

So,



Half-Power Fractional Bandwidth

$$|Z_{in}| = R\left[\left[1+Q^2\left(\frac{\omega}{\omega_o}-\frac{\omega_o}{\omega}\right)^2\right]\right]^{1/2}$$

We see that when:

$$Q\left(\frac{\omega}{\omega_o}-\frac{\omega_o}{\omega}\right) = \pm 1$$

$$|Z_{in}| = R\sqrt{2}$$

Since,

$$P_{in} = \frac{1}{2} Z_{in}\left|\frac{V}{Z_{in}}\right|^2 \rightarrow \text{Complex Power delivered to the circuit by the source}$$

$$\left|\frac{V}{Z_{in}}\right|^2 \rightarrow \text{Real}$$

The time-average (real) power delivered to the circuit is:

$$P_{av} = \mathrm{Re}(P_{in}) = \frac{1}{2} R \left| \frac{V}{Z_{in}} \right|^2$$

Or,

$$P_{av} = \frac{1}{2} R \frac{|V|^2}{|Z_{in}|^2} = \frac{1}{2} R \frac{|V|^2}{2R^2} = \frac{1}{2} \frac{|V|^2}{2R}$$

Real Power delivered to the circuit at resonance:

$$\frac{|V|^2}{2R}$$

So, the average (real) power delivered to the circuit is one-half of that delivered at resonance.

Now consider:

$$Q\left(\frac{\omega}{\omega_o} - \frac{\omega_o}{\omega} \right) = \pm 1$$

The width of the curve is usually small, so we can write:

$$\omega = \omega_o + \Delta\omega$$

$$\frac{\omega_o + \Delta\omega}{\omega_o} - \frac{\omega_o}{\omega_o + \Delta\omega} = \pm \frac{1}{Q}$$

Or,

$$2 \frac{\Delta\omega}{\omega_o} = \pm \frac{1}{Q}$$

$$\Delta\omega = \pm \frac{\omega_o}{2Q}$$

So that the width of the curve at the 3dB is or (half power dB =-10log0.5)

$$\frac{\omega_o}{Q}$$

$$\text{Bandwidth=BW=}\frac{\omega_o}{Q}$$

$$\text{Fractional Bandwidth=}\frac{BW}{\omega_o}=\frac{1}{Q}$$

We can now express the quality factor Q as:

$$Q=\frac{\omega_o}{BW}=\frac{\text{Resonant frequency}}{\text{Bandwidth}}$$

Phase of Z_{in}

$$Z_{in}=R\left[1+jQ\left(\frac{\omega}{\omega_o}-\frac{\omega_o}{\omega}\right)\right]$$

When, $\omega=\omega_o$, Z_{in}=Real and phase(Z_{in})=0

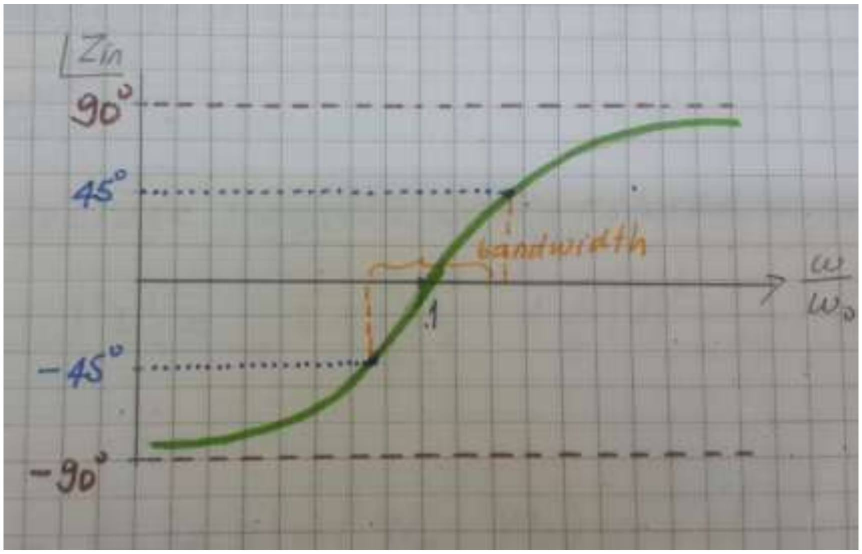
$$\angle Z_{in}=\tan^{-1}\left[Q\left(\frac{\omega}{\omega_o}-\frac{\omega_o}{\omega}\right)\right]$$

When $Q\left(\frac{\omega}{\omega_o}-\frac{\omega_o}{\omega}\right)=1, \quad \angle Z_{in}=45^o$

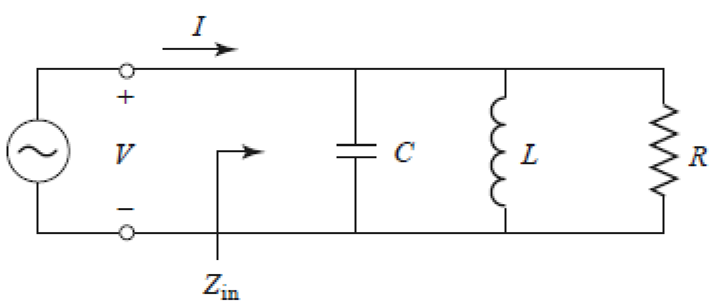
When $Q\left(\frac{\omega}{\omega_o}-\frac{\omega_o}{\omega}\right)=-1, \quad \angle Z_{in}=-45^o$

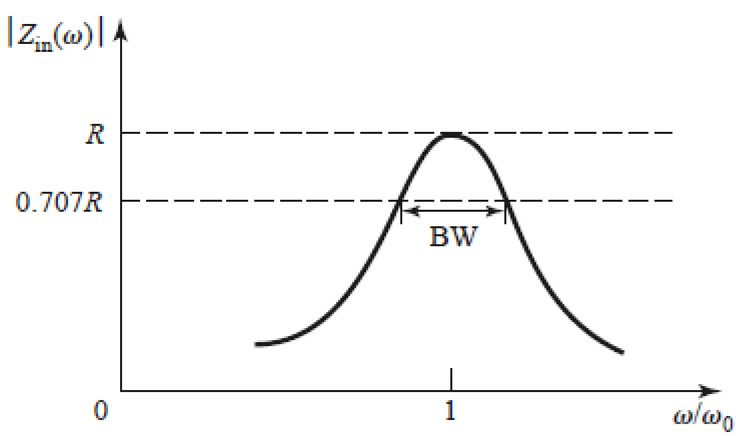
As $\frac{\omega}{\omega_o}\to\infty, \ \angle Z_{in}=\tan^{-1}(\infty)=\frac{\pi}{2}=90^o$

As $\frac{\omega}{\omega_o}\to 0, \ \angle Z_{in}=\tan^{-1}(-\infty)=-\frac{\pi}{2}=-90^o$



b) Parallel Resonant Circuit





$$Z_{in} = \frac{1}{\frac{1}{R} + \frac{1}{j\omega L} + j\omega C}$$

The complex power delivered to the resonator:

$$P_{in} = \frac{1}{2} VI^* = \frac{1}{2} Z_{in} II^* = \frac{1}{2} VV^* \frac{1}{Z_{in}{}^*}$$

$$P_{in} = \frac{1}{2} |V|^2 \left(\frac{1}{R} + \frac{j}{\omega L} - j\omega L \right)$$

The power dissipated in R is:

$$P_{diss} = \frac{1}{2} \frac{|V|^2}{R}$$

The time-average electric energy stored in C is:

$$W_e = \frac{1}{4} |V|^2 C$$

The time-average electric energy stored in L is:

$$W_m = \frac{1}{4} |I_L|^2 L = \frac{1}{4} L \frac{|V|^2}{\omega^2 L^2} = \frac{1}{4} \frac{|V|^2}{\omega^2 L}$$

Then,

$$P_{in} = P_{diss} + 2j\omega(W_e - W_m)$$

$$Z_{in} = \frac{2P_{in}}{|I|^2} = \frac{P_{diss} + 2j\omega(W_m - W_e)}{\frac{|I|^2}{2}}$$

Resonance occurs when:

$W_m = W_e$ and $Z_{in} = R$ in this case.

The resonant frequency ω_o is:

$$\omega_o = \frac{1}{\sqrt{LC}}$$

$$Q = \omega_o \frac{W_m + W_e}{P_{diss}} = \omega_o \frac{2W_m}{P_{diss}} = 2\omega_o \frac{\frac{1}{4} |V|^2 \frac{1}{\omega_o{}^2 L}}{\frac{1}{2} \frac{|V|^2}{R}}$$

Q= $\frac{R}{\omega_o L} = \omega_o RC$ Q increases as R increases.

Z_{in} near resonant frequency:

$$Z_{\text{in}} \simeq \frac{R}{1 + 2j\Delta\omega RC}$$

$$Z_{\text{in}} \simeq \frac{\text{R}}{1+2\text{jQ}\Delta\omega/\omega_o}$$

$$\text{BW}=\frac{1}{\text{Q}}$$

TRANSMISSION LINE RESONATORS

Distributed elements like transmission lines are more commonly used at microwave frequencies. In this section, transmission line sections with various lengths and terminations (open or short circuited) will be used to form resonators.

For lossy transmission lines:

$$Z_{\text{in}} = Z_o \frac{Z_{\text{L}} + Z_o \tanh \gamma \ell}{Z_o + Z_L \tanh \gamma \ell}$$

With $\gamma = \alpha + j\beta$

Short circuited $\lambda / 2$ Line:

$Z_L = 0$

We shell obtain Z_{in} at $\omega = \omega_0 + \Delta\omega$ and compare with series resonant RLC circuit.

$$Z_{\text{in}} = Z_o \tanh \gamma \ell$$

$\gamma = \alpha + j\beta$,

Using the identity for the hyperbolic tangent:

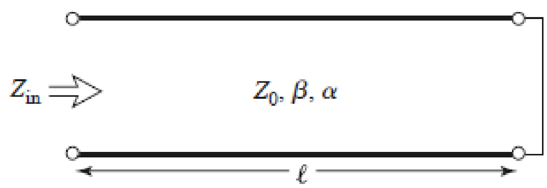
$$\tanh(\alpha + j\beta)\ell = \frac{\tanh\alpha\ell + j\tan\beta\ell}{1 + j\tan\beta\ell\tanh\alpha\ell}$$

$$Z_{\text{in}} = Z_o \frac{\tanh\alpha\ell + j\tan\beta\ell}{1 + j\tan\beta\ell\tanh\alpha\ell}$$

Observe that, if $\alpha = 0$ (lossless):

$$Z_{\text{in}} = jZ_o \tan\beta\ell$$

In practice α is very small, $\tanh\alpha\ell \approx \alpha\ell$ ($\alpha << 1$)



For a certain frequency ($\omega = \omega_o$) $\ell = \dfrac{\lambda}{2}$.

Let $\omega = \omega_0 + \Delta\omega$, assuming $\Delta\omega$ is small ($\dfrac{\Delta\omega}{\omega} << 1$). For a TEM line:

$$\beta\ell = \frac{\omega\ell}{\text{v}_\text{p}} = \frac{\omega_o\ell}{\text{v}_\text{p}} + \frac{\Delta\omega\ell}{\text{v}_\text{p}}$$

$v_p = f_o\lambda_o$

$$\frac{\ell}{v_p} = \frac{\lambda_o/2}{f_o\lambda_o} = \frac{1}{2f_o} = \frac{1}{2\frac{\omega_o}{2\pi}} = \frac{\pi}{\omega_o}$$

$$\beta\ell = \pi + \pi\frac{\Delta\omega}{\omega_o}$$

$$\tan\beta\ell = \tan\left(\pi + \pi\frac{\Delta\omega}{\omega_o}\right) = \tan\left(\pi\frac{\Delta\omega}{\omega_o}\right) = \pi\frac{\Delta\omega}{\omega_o}$$

$$Z_{\text{in}} = Z_o \frac{\alpha\ell + j\left(\pi \dfrac{\Delta\omega}{\omega_o}\right)}{1 + j\alpha\ell\left(\pi \dfrac{\Delta\omega}{\omega_o}\right)}$$

$$Z_{in} \simeq Z_o\left(\alpha\ell + j\frac{\Delta\omega\pi}{\omega_o}\right) \qquad \text{since} \qquad \Delta\omega\alpha\ell / \omega_o << 1$$

For a series resonant circuit we obtained before:

$$Z_{\text{in}} \simeq R + j2L\Delta\omega$$

The equivalent circuit has equivalent R and L:

$$R = Z_o\alpha\ell \qquad Z_o\frac{\Delta\omega\pi}{\omega_o} = 2L\Delta\omega \qquad L = \frac{Z_o\pi}{2\omega_o}$$

As $\omega_o L = \dfrac{1}{\omega_o C}$ $\quad C = \dfrac{2}{\omega_o Z_o \pi}$

At resonance $\Delta\omega = 0$ $\quad Z_{\text{in}} \simeq R$, $Z_{in} \simeq Z_o(\alpha\ell)$

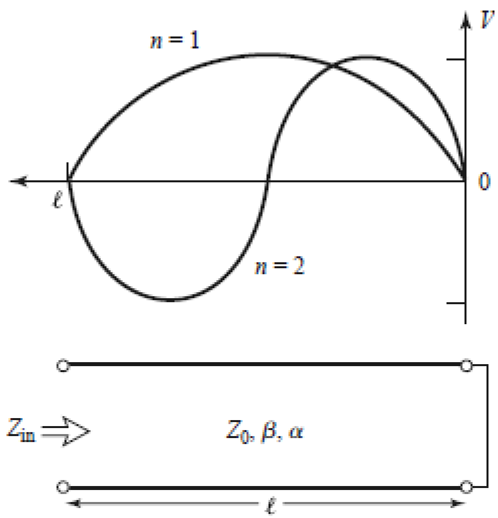
Q for this resonator:

$$\text{Q}=\frac{\omega_o L}{R} = \omega_o\frac{Z_o\pi}{2\omega_o}.\ \frac{\pi}{Z_o\alpha\ell} = \frac{\pi}{2\alpha}\frac{2}{\lambda} = \frac{\beta}{2\alpha}$$

$$\text{Q} = \frac{\beta}{2\alpha}$$

$\beta\ell = \dfrac{2\pi}{\lambda}\dfrac{\lambda}{2} = \pi$ at the first resonance.

Resonance also occurs when $\ell = \lambda, \dfrac{3\lambda}{2}, \ldots., \dfrac{n\lambda}{2}$



A short-circuited length of lossy transmission line, and the voltage distributions for $n = 1$ ($\ell = \lambda/2$) and $n = 2$ ($\ell = \lambda$) resonators.

Short Circuited $\frac{\lambda}{4}$ Line

$$Z_{\text{in}} = Z_o \tanh(\alpha + j\beta)$$

$$Z_{\text{in}} = Z_o \frac{\tanh \alpha\ell + j \tan \beta\ell}{1 + j \tan \beta\ell \tanh \alpha\ell}$$

Multiply the numerator and the denominator by $-\text{j}\cot\beta\ell$

$$Z_{\text{in}} = Z_o \frac{-\text{jcot}\,\beta\ell \tanh \alpha\ell + (-\text{jcot}\,\beta\ell)\, j \tan \beta\ell}{-\text{jcot}\,\beta\ell + (-\text{jcot}\,\beta\ell)\, j \tan \beta\ell \tanh \alpha\ell}$$

Since $\cot \beta\ell = \frac{1}{\tan \beta\ell}$,

$$Z_{\text{in}} = Z_o \frac{1 - j(\cot \beta\ell)\tanh \alpha\ell}{-\text{jcot}\,\beta\ell + \tanh \alpha\ell}$$

At $\omega = \omega_o$, $\ell = \frac{\lambda}{4}$ and let $\omega = \omega_o + \Delta\omega$

[$$\beta\ell = \frac{\omega\ell}{\nu_p} = \frac{\omega_o\ell}{\nu_p} + \frac{\Delta\omega\ell}{\nu_p}$$

$\nu_p = f_o\lambda_o$

$$\frac{\ell}{\nu_p} = \frac{\lambda_o/4}{f_o\lambda_o} = \frac{1}{4f_o} = \frac{1}{4\frac{\omega_o}{2\pi}} = \frac{\pi}{2\omega_o}$$

$$\beta\ell = \frac{\pi}{2} + \pi\frac{\Delta\omega}{2\omega_o}$$

]

$$\beta\ell = \frac{\omega_o\ell}{\nu_p} + \frac{\pi}{2}\frac{\Delta\omega}{\omega_o}$$

$$\cot\beta\ell = \cot\left(\frac{\pi}{2} + \pi\frac{\Delta\omega}{2\omega_o}\right) = -\tan\left(\frac{\pi}{2}\frac{\Delta\omega}{\omega_o}\right) \approx -\pi\frac{\Delta\omega}{2\omega_o}$$

$$\tanh\alpha\ell \approx \alpha\ell \quad as \quad \alpha\ell << 1$$

$$Z_{\text{in}} = Z_o\frac{1 - j(\cot\beta\ell)\tanh\alpha\ell}{-\text{j}\cot\beta\ell + \tanh\alpha\ell}$$

$$Z_{\text{in}} = Z_o\frac{1 + j\pi\frac{\Delta\omega}{2\omega_o}\alpha\ell}{\alpha\ell + j\pi\frac{\Delta\omega}{2\omega_o}}$$

$$\frac{\Delta\omega}{2\omega_o} \approx 0$$

$$Z_{\text{in}} = Z_o\frac{1}{\alpha\ell + j\pi\frac{\Delta\omega}{2\omega_o}}$$

This is similar to parallel resonant circuit.

$$Z_{\text{in}} \approx \frac{R}{1 + j2\Delta\omega RC} = \frac{1}{\frac{1}{R} + j2\Delta\omega C}$$

So equivalent R and C are:

$$R = \frac{Z_o}{\alpha\ell} \quad and \quad C = \frac{\pi}{4\omega_o Z_o}$$

Q=$\omega_o RC$

Q=$\omega_o \dfrac{Z_o}{\alpha\ell}\dfrac{\pi}{4\omega_o Z_o}$

Q=$\dfrac{\beta}{2\alpha}$

Since, $\ell = \dfrac{\pi}{2\beta}$

Open Circuited $\lambda / 2$ Line

$$Z_{\text{in}} = Z_o \frac{Z_L + Z_o \tanh\gamma\ell}{Z_o + Z_L \tanh\gamma\ell}$$

$$Z_{\text{in}} = Z_o \frac{Z_L + Z_o \tanh(\alpha + j\beta)\ell}{Z_o + Z_L \tanh(\alpha + j\beta)\ell}$$

$$\text{As} \quad Z_L \to \infty \quad Z_{in} = Z_o \frac{\frac{Z_L}{Z_L} + \frac{Z_o}{Z_L}\tanh(\alpha + j\beta)\ell}{\frac{Z_o}{Z_L} + \frac{Z_L}{Z_L}\tanh(\alpha + j\beta)\ell}$$

$$Z_{in} = \frac{Z_o}{\tanh(\alpha + j\beta)\ell}$$

$$Z_{\text{in}} = Z_o \frac{1 + j\tan\beta\ell\tanh\alpha\ell}{\tanh\alpha\ell + j\tan\beta\ell}$$

$$\ell = \frac{\lambda}{2} \quad \text{at} \quad \omega = \omega_o \quad \text{and} \quad \omega = \omega_o + \Delta\omega$$

$$\beta\ell = \pi + \frac{\pi\Delta\omega}{\omega_o} \quad \text{(shown before)}$$

$$\tan\beta\ell \approx \frac{\Delta\omega\pi}{\omega_o}, \quad \tanh\alpha\ell \approx \alpha\ell$$

$$Z_{in} \approx \frac{Z_o}{\alpha\ell + j\left(\frac{\Delta\omega\pi}{\omega_o}\right)}$$

Compare with the parallel resonant circuit,

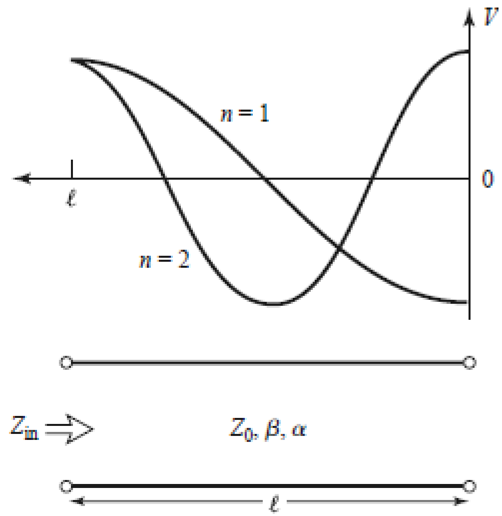
$$Z_{\text{in}} \approx \frac{1}{\frac{1}{R} + j2\Delta\omega C}$$

$$R = \frac{Z_o}{\alpha\ell}, \; C = \frac{\pi}{2\omega_o Z_o}, \; L = \frac{1}{\omega_o^2 C}, \; \omega_o L = \frac{1}{\omega_o C}$$

$$\text{Q} = \omega_o RC = \omega_o \frac{Z_o}{\alpha\ell} \frac{\pi}{2\omega_o Z_o} = \frac{\pi}{2\alpha\ell}$$

$$\beta\ell = \frac{2\pi}{\lambda}\ell = \frac{2\pi}{\lambda}\frac{\lambda}{2} = \pi$$

$$\text{Q} = \frac{\beta}{2\alpha}$$



An open-circuited length of lossy transmission line, and the voltage distributions for $n = 1$ $(\ell = \lambda/2)$ and $n = 2$ $(\ell = \lambda)$ resonators.