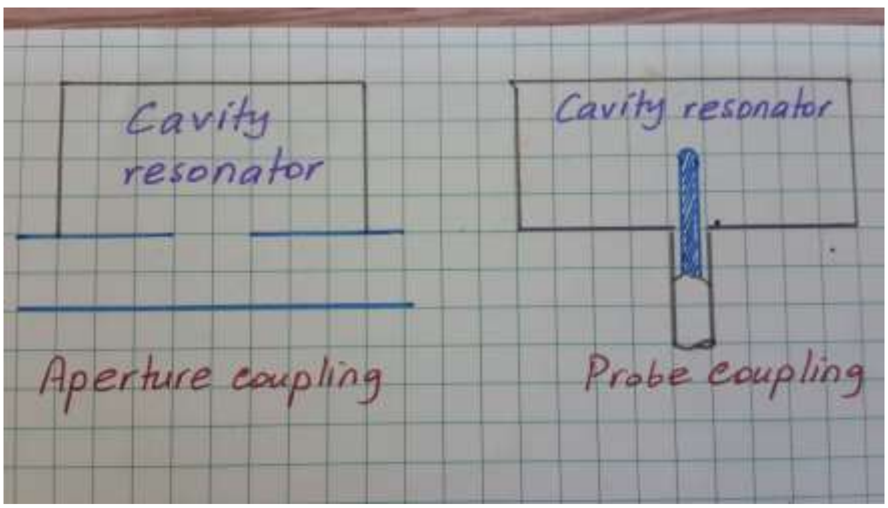
CAVITY RESONATORS

With the advent of the planar guiding structures (stripline, microstrip line, slot-line) and the development of integrated circuitry, the TL resonators were forced to be used in the higher microwave spectrum.

But, TL resonators (coaxial cable, microstrip etc.) have relatively low Q at 3GHz and above. So, cavity resonators are more commonly used at these frequencies. Cavity resonators are air or dielectric filled regions enclosed by metallic walls.

Most common cavity resonator is a rectangular waveguide short circuited at both ends.

They are excited by small probes or apertures.



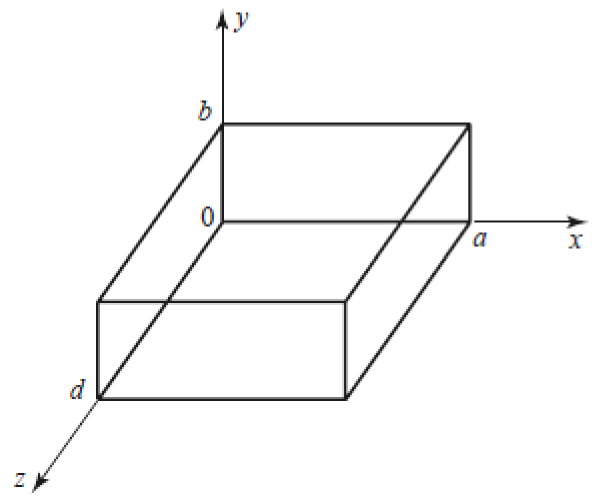
If one of these means is used to excite the cavity at a resonant frequency, the internal fields can build up to a large value.

At resonance the average electric and magnetic energies will be equal, with the total stored energy passing back and forth cyclically between the electric and magnetic fields.

A small amount of this energy is lost per cycle to the finite conductivity of the walls and to dielectric heating (if a dielectric is present).

RECTANGULAR CAVITY RESONATOR

This can be considered as a rectangular waveguide of dimensions $a \times b$ and of length d. Conducting walls are placed at $z=0$ and $z=d$.



Assume that the waveguide can support the TE_{10} mode. The electric field intensity of such a mode is given by:

$$E_y = A\sin\left(\frac{\pi x}{a}\right)e^{-j\beta z}$$

This is a wave traveling in the +z direction, but there must be a wave traveling in the –z direction as well due to the short circuit at z=d with magnitude B for the wave traveling in the –z direction. So:

$$E_y = A\sin\left(\frac{\pi x}{a}\right)e^{-j\beta z} + B\sin\left(\frac{\pi x}{a}\right)e^{j\beta z}$$

We now apply the boundary condition $E_{\text{tangential}} = 0$ at conductor boundaries. (On the other boundaries already satisfied.)

So,

E_y=0 at z=0 and z=d.

At z=0:

$$E_y = \sin\left(\frac{\pi x}{a}\right)(A+B) = 0$$

$$A = -B$$

$$E_y = A\sin\left(\frac{\pi x}{a}\right)\left(e^{-j\beta z} - e^{j\beta z}\right)$$

$$E_y = -j2A\sin\left(\frac{\pi x}{a}\right)\sin\beta z$$

At E_y=0 z=d:

So,

$$-j2A\sin\left(\frac{\pi x}{a}\right)\sin\beta d = 0$$

$$\sin\beta d = 0, \quad \beta d = \ell\pi, \quad \text{where} \quad \ell=1,2,3,4,...$$

$$\beta = \frac{\ell\pi}{d} \quad d = \ell\frac{\lambda}{2} \quad \ell = 1,2,3,...$$

which implies that the cavity must be an integer multiple of a half-guide wavelength long at the resonant frequency.

So,

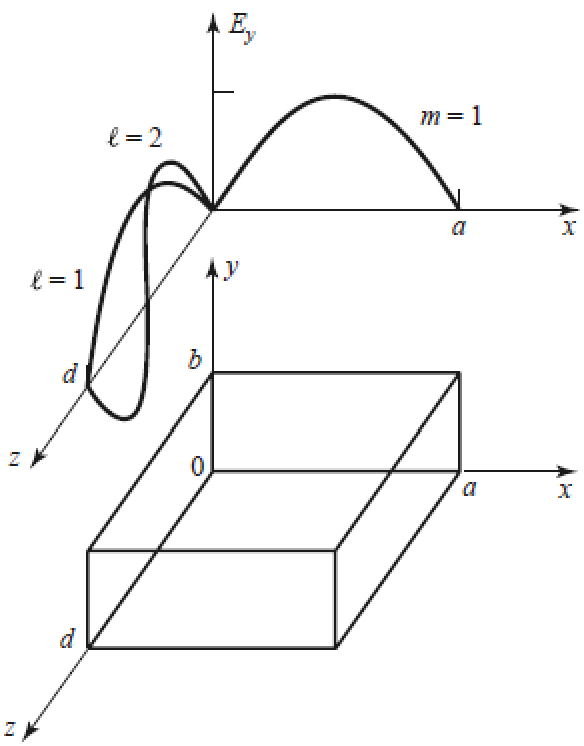
$$E_y = E_{y_o}\sin\left(\frac{\pi x}{a}\right)\sin\left(\frac{\pi z}{d}\right)$$

Take E_{yo}=-j2A

As we have $\ell = 1,2,3,4,......$ the cavity modes are designated as (a third script is added):

$TE_{101}, TE_{102}, TE_{103}, \ldots, TE_{10\ell}$

ℓ determines the number of $\frac{\lambda_g}{2}$ between z=0 and z=d.



A rectangular cavity resonator, and the electric field variations for the TE_{101} and TE_{102} resonant modes.

Obviously TE_{101} mode has the lowest resonant frequency.

The remaining components can be obtained by using the Maxwell's Equations.

$$\nabla X\bar{E} = -j\omega\mu\bar{H}$$

We have only E_y component.

$$-\frac{\partial E_y}{\partial z}\hat{a}_x + \frac{\partial E_y}{\partial x}\hat{a}_z = -j\omega\mu\bar{H}$$

$$H_x = -j\frac{\pi E_{y_o}}{d\omega\mu}\sin\left(\frac{\pi x}{a}\right)\cos\left(\frac{\pi z}{d}\right)$$

$$H_x = j\frac{\pi E_{y_o}}{a\omega\mu}\cos\left(\frac{\pi x}{a}\right)\sin\left(\frac{\pi z}{d}\right)$$

Resonant frequency of $TE_{10\ell}$ Modes

$$\beta = \frac{2\pi}{\lambda_g} \quad \text{but for resonant frequencies } \beta = \frac{\ell\pi}{d} \quad \ell = 1, 2, 3, \ldots$$

Remember $\beta = k\left[1 - \left(\frac{f_c}{f_r}\right)^2\right]^{1/2}$

$f = f_r$ at resonance

$$\frac{2\pi}{\lambda_r}\left[1 - \left(\frac{f_c}{f_r}\right)^2\right]^{1/2} = \ell\frac{\pi}{d}$$

$$\frac{2f_r}{c}\left[1 - \left(\frac{f_c}{f_r}\right)^2\right]^{1/2} = \frac{\ell}{d}$$

$$\left(f_r^2 - f_c^2\right) = \left(\frac{c\ell}{2d}\right)^2$$

$$f_r = \frac{c}{2}\left[\left(\frac{1}{a}\right)^2 + \left(\frac{\ell}{d}\right)^2\right]^{1/2} \rightarrow \text{Resonant Frequencies for } TE_{10\ell} \text{ modes}$$ (air filled)

Other Rectangular Cavity Modes

A waveguide can support other TE and TM modes. When the waveguide is terminated by short circuits at two ends (z=0, and z=d), other TE and TM modes also exist.

TE Modes

By applying boundary conditions at z=0, and z=d walls it can be shown that:

Electric fields:

$$E_x = \left(\frac{n\pi}{k_o b}\right) E_o \cos\left(\frac{m\pi}{a}x\right) \sin\left(\frac{n\pi}{b}y\right) \sin\left(\frac{\ell\pi}{d}z\right)$$

$$E_y = -\left(\frac{m\pi}{k_o a}\right) E_o \sin\left(\frac{m\pi}{a}x\right) \cos\left(\frac{n\pi}{b}y\right) \sin\left(\frac{\ell\pi}{d}z\right)$$

$$E_z = 0$$

Magnetic Fields:

$$H_x = \frac{j\left(\frac{m\pi}{a}\right)\left(\frac{\ell\pi}{d}\right)}{\eta k_o^2} E_o \sin\left(\frac{m\pi}{a}x\right) \cos\left(\frac{n\pi}{b}y\right) \cos\left(\frac{\ell\pi}{d}z\right)$$

$$H_y = \frac{j\left(\frac{n\pi}{b}\right)\left(\frac{\ell\pi}{d}\right)}{\eta k_o^2} E_o \cos\left(\frac{m\pi}{a}x\right) \sin\left(\frac{n\pi}{b}y\right) \cos\left(\frac{\ell\pi}{d}z\right)$$

$$H_z = \frac{j\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2}{\eta k_o^2} E_o \cos\left(\frac{m\pi}{a}x\right) \cos\left(\frac{n\pi}{b}y\right) \sin\left(\frac{\ell\pi}{d}z\right)$$

TM Modes

It can also be shown that:

Magnetic Fields

$$H_x = \frac{n\pi}{k_o b} H_o \sin\left(\frac{m\pi}{a}x\right) \cos\left(\frac{n\pi}{b}y\right) \cos\left(\frac{\ell\pi}{d}z\right)$$

$$H_y = -\frac{m\pi}{k_o a} H_o \cos\left(\frac{m\pi}{a}x\right) \sin\left(\frac{n\pi}{b}y\right) \cos\left(\frac{\ell\pi}{d}z\right)$$

$$H_z = 0$$

Electric Fields

$$E_x = \frac{j\eta\left(\frac{m\pi}{a}\right)\left(\frac{\ell\pi}{d}\right)}{k_o^2} H_o \cos\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right)\sin\left(\frac{\ell\pi}{d}z\right)$$

$$E_y = \frac{j\eta\left(\frac{n\pi}{b}\right)\left(\frac{\ell\pi}{d}\right)}{k_o^2} H_o \sin\left(\frac{m\pi}{a}x\right)\cos\left(\frac{n\pi}{b}y\right)\sin\left(\frac{\ell\pi}{d}z\right)$$

$$E_z = \frac{j\eta\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2}{k_o^2} H_o \sin\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right)\cos\left(\frac{\ell\pi}{d}z\right)$$

Where $\beta = \frac{\ell\pi}{d}$ for both modes.

$TM_{mn\ell}$ modes have the same resonant frequencies as the $TE_{mn\ell}$ modes.

Trivial modes:

The resonator field expressions given above for TE and TM modes are zero for certain combinations of $m, n \text{ and } \ell$ meaning that these modes can never be excited. For the rectangular cavity resonator the trivial modes are:

$TE_{00\ell}, TE_{mn0}, TM_{m0\ell}, TM_{0n\ell}$

The nontrivial mode having the lowest resonant frequency is TE_{101} mode.

Resonant Frequency:

For a general waveguide mode:

$$\beta^2 = \frac{\omega^2}{c^2} - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2$$

$$\beta = \frac{2\pi}{\lambda_g}, \quad \lambda_g = \frac{\lambda}{\sqrt{1 - \left(\frac{f_c}{f}\right)^2}}, \quad f_c = \frac{c}{2}\left(\sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2}\right)$$

For the resonant cavity:

$$\beta = \frac{\ell\pi}{d}$$

$$\frac{\omega^2}{c^2} = \left(\frac{\ell\pi}{d}\right)^2 + \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2$$

As $\omega = 2\pi f$ at resonant frequency $\mathrm{f} \rightarrow \mathrm{f}_r$

$$f_c = \frac{c}{2}\left(\sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2 + \left(\frac{\ell}{d}\right)^2}\right)$$

When $d > a$ the lowest resonant frequency is TE_{101} mode.