Quality Factor Q for the TE_{101} Mode

Quality factor:

$$Q=\omega_o \frac{\text{time}-\text{average energy stored in the system}}{\text{energy loss per second in the system}}$$

$$Q=\frac{\omega_o W_T}{P_\ell}$$

Calculate the total time-averaged stored energy:

The field expression for the TE_{101}

$$E_x = 0$$
$$E_z = 0$$
$$E_y = -\frac{\pi}{k_o a} E_o \sin\left(\frac{\pi x}{a}\right)\sin\left(\frac{\pi z}{d}\right)$$

$$H_x = \frac{j\left(\frac{\pi}{a}\right)\left(\frac{\pi}{d}\right)}{\eta k^2{}_o} E_o \sin\left(\frac{\pi x}{a}\right)\cos\left(\frac{\pi z}{d}\right)$$

$$H_y = 0$$

$$H_z = \frac{j\left(\frac{\pi}{a}\right)^2}{\eta k^2{}_o} E_o \cos\left(\frac{\pi x}{a}\right)\sin\left(\frac{\pi z}{d}\right)$$

$$W_e = \frac{\varepsilon}{4}\int_0^a\int_0^b\int_0^d E_y E_y^* dxdydz = \frac{\varepsilon}{4}\left(\frac{\pi}{k_o a}\right)^2 |E_0|^2 \frac{a}{2} b \frac{d}{2}$$

$$W_e = \frac{\varepsilon \pi^2 |E_0|^2}{16 k_0^2 a} bd$$

$$W_m = \frac{\mu}{4}\int_0^a\int_0^b\int_0^d \left(H_x H_x^* + H_z H_z^*\right) dxdydz$$

$$H_x H_x^* = \frac{\left(\frac{\pi}{a}\right)^2 \left(\frac{\pi}{d}\right)^2 |E_0|^2}{\eta^2 k_0^4} \sin^2\left(\frac{\pi x}{a}\right) \cos^2\left(\frac{\pi z}{d}\right)$$

$$H_z H_z^* = \frac{\left(\frac{\pi}{a}\right)^4 |E_0|^2}{\eta^2 k_0^4} \cos^2\left(\frac{\pi x}{a}\right) \sin^2\left(\frac{\pi z}{d}\right)$$

$$W_m = \frac{\mu}{4} \frac{\left(\frac{\pi}{a}\right)^2 \left(\frac{\pi}{d}\right)^2 |E_0|^2}{\eta^2 k_0^4} \frac{a}{2} b \frac{d}{2} + \frac{\mu}{4} \frac{\left(\frac{\pi}{a}\right)^4 |E_0|^2}{\eta^2 k_0^4} \frac{a}{2} b \frac{d}{2}$$

$$W_m = \frac{\mu}{4} \frac{abd}{4} \frac{|E_0|^2}{\eta^2 k_0^4} \left[\left(\frac{\pi}{a}\right)^2 \left(\frac{\pi}{d}\right)^2 + \left(\frac{\pi}{a}\right)^4 \right]$$

$$W_m = \frac{\mu abd}{16} \frac{|E_0|^2}{\eta^2 k_0^4} \left(\frac{\pi}{a}\right)^2 \left[\left(\frac{\pi}{a}\right)^2 + \left(\frac{\pi}{d}\right)^2 \right]$$

$$W_m = \frac{\varepsilon \pi^2 |E_0|^2 bd}{16 k_0^4 a} \left[\left(\frac{\pi}{a}\right)^2 + \left(\frac{\pi}{d}\right)^2 \right]$$

But,

$$k_o^2 = \left(\frac{\pi}{a}\right)^2 + \left(\frac{\pi}{d}\right)^2$$

For mode TE_{101} mode. Then:

$$W_m = \frac{\varepsilon \pi^2 |E_0|^2 bd}{16 k_0^2 a} = W_e$$

So, the total time-averaged stored energy is:

$$W_T = 2W_e = \frac{\varepsilon \pi^2 |E_0|^2 bd}{8 k_0^2 a}$$

The power loss in the walls is:

$$P_\ell = \frac{R_s}{2} \int_{walls} \bar{J}_s . \bar{J}_s^* ds = \frac{R_s}{2} \int_{walls} \left| \bar{H}_{\tan} \right|^2 ds$$

On the wall z=0

$$H_x = \frac{j\left(\frac{\pi}{a}\right)\left(\frac{\pi}{d}\right)}{\eta k^2_o} E_o \sin\left(\frac{\pi x}{a}\right)\cos\left(\frac{\pi z}{d}\right), \qquad H_y = 0, \qquad H_z = 0$$

So,

$$P_{\ell_1} = \frac{R_s}{2} \int_0^a \int_0^b \left| H_x \right|^2 dxdy = \frac{R_s}{2} \frac{\left(\frac{\pi}{a}\right)^2 \left(\frac{\pi}{d}\right)^2 \left| E_0 \right|^2}{\eta^2 k_0^4} \frac{ab}{2}$$

On the wall z=d

$$H_x = \frac{j\left(\frac{\pi}{a}\right)\left(\frac{\pi}{d}\right)}{\eta k^2_o} E_o \sin\left(\frac{\pi x}{a}\right)(-1), \qquad H_y = 0, \qquad H_z = 0$$

$$P_{\ell_2} = \frac{R_s}{2} \int_0^a \int_0^b \left| H_x (z = d) \right|^2 dxdy = P_{\ell_1}$$

On the wall y=0:

$$P\ell_3 = \frac{R_s}{2} \int_0^a \int_0^b \left(\left| H_x \right|^2 + \left| H_z \right|^2 \right) dxdz$$

$$P_{\ell_3} = \frac{R_s}{2} \left[\frac{\left(\frac{\pi}{a}\right)^2 \left(\frac{\pi}{d}\right)^2 \left| E_0 \right|^2}{\eta^2 k_0^4} \frac{a}{2} \frac{d}{2} + \frac{\left(\frac{\pi}{a}\right)^4}{\eta^2 k_0^4} \left| E_0 \right|^2 \frac{a}{2} \frac{d}{2} \right]$$

On the wall y=b:

$$P_{\ell_4} = P_{\ell_3}$$

On the wall x=0

$$H_x = 0,\ \ H_y = 0,\ \ H_z = \frac{j\left(\frac{\pi}{a}\right)^2}{\eta k_0^2} E_0 \sin\left(\frac{\pi z}{d}\right)$$

$$P_{\ell 5} = \frac{R_s}{2}\frac{\left(\frac{\pi}{a}\right)^4}{\eta^2 k_0^4}\left|E_0\right|^2 \int_0^b\int_0^d \sin^2\left(\frac{\pi z}{d}\right) dydz = \frac{R_s}{2}\frac{\left(\frac{\pi}{a}\right)^a}{\eta^2 k_0^4}\left|E_0\right|^2 b\frac{d}{2}$$

On the wall x=a:

$$H_x = 0,\ \ H_y = 0,\ \ H_z = \frac{j\left(\frac{\pi}{a}\right)^2}{\eta k_0^2} E_0 (-1)\sin\left(\frac{\pi z}{d}\right)$$

$$P_{\ell_6} = P_{\ell_5}$$

So,

$$P_\ell = R_s \frac{\left(\frac{\pi}{a}\right)^2\left(\frac{\pi}{d}\right)^2\left|E_0\right|^2}{\eta^2 k_0^4}\frac{ab}{2} + R_s \frac{\left(\frac{\pi}{a}\right)^2\left(\frac{\pi}{d}\right)^2\left|E_0\right|^2}{\eta^2 k_0^4}\frac{ad}{4}\left[\left(\frac{\pi}{a}\right)^2 + \left(\frac{\pi}{d}\right)^2\right] + R_s \frac{\left(\frac{\pi}{a}\right)^4\left|E_0\right|^2}{\eta^2 k_0^4}\frac{bd}{2}$$

$$P_\ell = R_s\left(\frac{\pi}{a}\right)^2\frac{\left|E_0\right|^2}{\eta^2 k_0^4}\left(\left(\frac{\pi}{d}\right)^2\frac{ab}{2} + \frac{ab}{4}k_0^2 + \left(\frac{\pi}{a}\right)^2\frac{bd}{2}\right)$$

$$P_\ell = R_s\left(\frac{\pi}{a}\right)^2\frac{\left|E_0\right|^2}{\eta^2 k_0^4}\left(\left(\frac{\pi}{d}\right)^2\frac{ab}{2} + \left(\frac{\pi}{a}\right)^2\frac{bd}{2} + \left(\frac{\pi}{a}\right)^2\frac{ad}{4} + \left(\frac{\pi}{d}\right)^2\frac{ad}{4}\right)$$

$$P_\ell = R_s\left(\frac{\pi}{a}\right)^2\frac{\left|E_0\right|^2}{\eta^2 k_0^4}\left(\frac{ab}{2d^2} + \frac{bd}{2a^2} + \frac{d}{4a} + \frac{a}{4d}\right)$$

The quality factor:

$$Q=\frac{\omega_o W_T}{P_\ell}$$

$$Q=\frac{(k_o ad)^3 b\eta}{2\pi^2 R_s\left(2a^3b+2d^3b+a^3d+d^3a\right)}$$

This Q is Q due to the finite conductivity of the walls, when the cavity is air filled or filled with a lossless dielectric.

It can be shown that Q due to lossy dielectrics given by loss tangent $\tan\delta$ (when the walls are lossless) is:

$$Q_d=\frac{1}{\tan\delta}$$

Remember that $\tan\delta$ is the loss tangent, $\tan\delta=\frac{\sigma}{\omega\varepsilon}$.

When we have both:

$$\frac{1}{Q}=\frac{1}{Q_c}+\frac{1}{Q_d}$$

CYLINDRICAL CAVITY

The cylindrical cavity is a circular WG of length **d** and radius **a** with a short circuiting plates at each end.

This type of cavity is commonly used in practice in wave meters to measure the frequency, because of its high Q and wide frequency range of operation it provides. A high Q is necessary in a frequency meter in order to obtain a high degree of resolution or accuracy in the measurement of unknown frequency. When the cavity is tuned to frequency of the unknown source, it absorbs a maximum of power from the input line. A crystal detector coupled to the input line can be used to indicate this dip in power at resonance.

Field Expressions for the $TE_{nm\ell}$ Modes

$$E_\rho = j\omega_{nm\ell}\mu \frac{H_o}{k_c^2}\frac{n}{\rho} J_n\left(\frac{p^1_{nm}}{a}\rho\right)\sin(n\phi)\sin\left(\frac{\ell\pi}{d}z\right)$$

$$E_\phi = j\omega_{nm\ell}\mu \frac{H_o}{k_c} J_n'\left(\frac{p^1_{nm}}{a}\rho\right)\cos(n\phi)\sin\left(\frac{\ell\pi}{d}z\right)$$

$$E_z = 0$$

$$H_\rho = \frac{\beta}{k_c} H_o J_n\left(\frac{p^1_{nm}}{a}\rho\right)\cos(n\phi)\cos\left(\frac{\ell\pi}{d}z\right)$$

$$H_\phi = -\frac{\beta}{k_c^2}\frac{n}{\rho} H_o J_n\left(\frac{p'_{nm}}{a}\rho\right)\sin(n\phi)\cos\left(\frac{\ell\pi}{d}z\right)$$

$$H_z = H_o J_n\left(\frac{p^1_{nm}}{a}\rho\right)\cos(n\phi)\sin\left(\frac{\ell\pi}{d}z\right)$$

Since ℓ cannot take the zero value and it must start from one and since the lowest possible p'_{nm} is 1.841 which corresponds to n=1, m=1, the lowest possible TE mode is TE_{111} .

We had $k_c^2 = k^2 - \beta^2$ $\quad k_c = \frac{p'_{nm}}{a}$ $\quad \beta = \frac{\ell\pi}{d}$

The resonance frequencies are:

$$f_{nm\ell} = \frac{1}{2\pi\sqrt{\varepsilon\mu}}\sqrt{\left(\frac{p'_{nm}}{a}\right)^2 + \left(\frac{\ell\pi}{d}\right)^2}$$

$$\lambda_{nm\ell} = \frac{2\pi}{\sqrt{\left(\frac{p'_{nm}}{a}\right)^2 + \left(\frac{\ell\pi}{d}\right)^2}}$$

TE_{111} has the lowest resonant frequency:

$$f_{nm\ell} = \frac{1}{2\pi\sqrt{\varepsilon\mu}}\sqrt{\left(\frac{1.841}{a}\right)^2 + \left(\frac{\pi}{d}\right)^2}$$

$$\lambda_{nm\ell} = \frac{2\pi}{\sqrt{\left(\frac{1.841}{a}\right)^2 + \left(\frac{\pi}{d}\right)^2}}$$

Field Expressions for the $TM_{nm\ell}$ Modes

$$E_\rho = -\frac{\frac{\ell\pi}{d}}{\frac{p_{nm}}{a}} E_o J_n^{'}\left(\frac{p_{nm}}{a}\rho\right)\cos(n\phi)\sin\left(\frac{\ell\pi}{d}z\right)$$

$$E_\phi = -\frac{\frac{\ell\pi}{d}}{\frac{p_{nm}}{a}}\frac{n}{\rho} E_o J_n\left(\frac{p_{nm}}{a}\rho\right)\sin(n\phi)\sin\left(\frac{\ell\pi}{d}z\right)$$

$$E_\rho = E_o J_n\left(\frac{p_{nm}}{a}\rho\right)\cos(n\phi)\cos\left(\frac{\ell\pi}{d}z\right)$$

$$H_\rho = -j\omega_{nm\ell}\varepsilon\frac{E_o}{\frac{p_{nm}}{a}} J_n\left(\frac{p_{nm}}{a}\rho\right)\sin(n\phi)\cos\left(\frac{\ell\pi}{d}z\right)$$

$$H_\phi = -j\omega_{nm\ell}\varepsilon\frac{E_o}{\frac{p_{nm}}{a}} J_n^{'}\left(\frac{p_{nm}}{a}\rho\right)\cos(n\phi)\cos\left(\frac{\ell\pi}{d}z\right)$$

$$H_z = 0$$

p_{nm} is the m^th^ root of the Bessel Function, J_n.

p'_{nm} is the m^th^ root of the derivative of the Bessel Function, J_n.

n is the order of the Bessel Function or the derivative of the Bessel Function,

$\ell = 0,1,2,....$ $\ell \neq 0$ for the TE modes.

$$k^2 = \left(\frac{p_{nm}}{a}\right)^2 + \left(\frac{\ell\pi}{d}\right)^2$$

So, the resonant frequencies:

$$f_{nm\ell} = \frac{1}{2\pi\sqrt{\varepsilon\mu}}\sqrt{\left(\frac{p_{nm}}{a}\right)^2 + \left(\frac{\ell\pi}{d}\right)^2}$$

The lowest TM resonant frequency is:

$$f_{011} = \frac{1}{2\pi\sqrt{\varepsilon\mu}}\sqrt{\left(\frac{2.405}{a}\right)^2 + \left(\frac{\pi}{d}\right)^2}$$