MICROWAVE FILTERS

Resonators are important because of their ability to select and reject frequencies. This property finds particular application in the fabrication of filters.

A filter is a two-port network used to control the frequency response at a certain point in an RF or microwave system by providing transmission at frequencies within the pass band of the filter and attenuation in the stop band of the filter.
The filters are used in the following fields:

1) Communications
2) Radars
3) Laboratory measurement equipment
4) Harmonic reflection

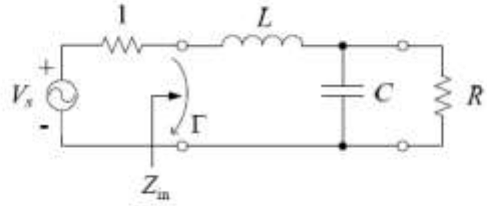
Idealized transfer functions for filters:

1) Low pass
2) High Pass
3) Band Pass
4) Band stop

At low frequencies we use L and C for building filters (lumped elements).

In microwave frequencies we use low frequency techniques (L and C) and then convert to distributed elements (transmission lines appropriate for narrow bands).

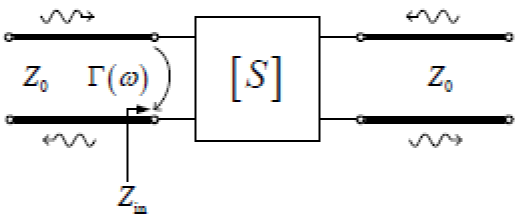
An example of a low pass filter:



Two methods are commonly used for the filter design.

1) The Image Parameter Method
2) The Insertion Loss Method

We shall see the insertion loss method.



It compares the incident power on the input terminals to the power delivered to the load.

$$\text{Power Loss Ratio}=\mathrm{P_{LR}}=\frac{\text{Power incident at the Input}}{\text{Power delivered to load}}=\frac{P_{in}}{P_{load}}$$

This ratio ideally should be unity at frequencies in a pass-band and infinity at frequencies in a stop-band.

A wave of amplitude V_o^+ is incident on the filter. The reflected wave amplitude is represented by V_o^-, returning back to the generator.

Since the Reflection Coefficient is:

$$\Gamma_{in}=\frac{V_o^-}{V_o^+}$$

The incident power is:

$$P_{in}=\frac{1}{2}\left(\frac{V_o^+V_o^{+*}}{Z_o}\right)$$

The delivered power is:

$$P_{del}=\left(1-\left|\Gamma_{in}\right|^2\right)P_{in}$$

$$P_{del}=P_{in}-P_{ref}=\frac{1}{2Z_o}\left(V_o^+V_o^{+*}-V_o^-V_o^{-*}\right)$$

Power Loss Ratio:

$$P_{LR}=\frac{P_{in}}{P_{del}}=\frac{\frac{1}{2}\frac{V_o^+V_o^{+*}}{Z_o}}{\frac{1}{2Z_o}\left(V_o^+V_o^{+*}-V_o^-V_o^{-*}\right)}=\frac{1}{1-\left(\frac{V_o^-V_o^{-*}}{V_o^+V_o^{+*}}\right)}$$

$$P_{LR}=\frac{1}{1-\left|\Gamma_{in}\right|^2}$$

If $P_{LR}(\omega_1)$ is close to unity, we can say that the angular frequency ω_1 is “passed” by the filter. Alternatively, if $P_{LR}(\omega_2)$ is large, it can argued that ω_2 is stopped by the filter.

The Insertion Loss:

The insertion loss measured in decibels is:

$$L=10\log\left(P_{LR}\right)\ dB$$

Specifications of the Power Loss Ratio

The insertion loss method of the filter design begins by:

- Specifying the power loss ratio P_{LR} (or the magnitude of the reflection coefficient $\left|\Gamma_{in}\right|$) as a function of ω.
- Then a network that will give the desired power loss ratio is synthesized.

It must be kept in mind, however, that a completely arbitrary $\Gamma(\omega)$ as a function of ω cannot be chosen since it may not correspond to a physical network.

Restrictions on $\Gamma(\omega)$

The restrictions to be imposed on Γ are known as the conditions for realizability which are the following:

1) For a passive network the reflected power cannot exceed the incident power.

$$\Gamma(\omega) \leq 1$$

2) $\left|\Gamma(\omega)\right|^2$ must be an even function of ω.

$$\left|\Gamma(\omega)\right|^2 = \rho^2(\omega) = \Gamma(\omega)\Gamma^*(\omega) = \Gamma(\omega)\Gamma(-\omega)$$

$$\rho^2(\omega) = \rho^2(-\omega) \quad \rightarrow \rho^2(\omega) \text{ is an even function of } \omega.$$

If the normalized input impedance of the filter network is

$$Z_{in}(\omega) = \frac{Z_{in}}{Z_o} = \bar{R}(\omega) + j\bar{X}(\omega)$$

We have:

$$\Gamma(\omega) = \frac{\bar{z}_{in} - 1}{\bar{z}_{in} + 1} = \frac{\bar{R}(\omega) - 1 + j\,\bar{\mathrm{x}}(\omega)}{\bar{R}(\omega) + 1 + j\,\bar{\mathrm{x}}(\omega)}$$

It is known that $\bar{R}$ is an even function of ω and $\bar{\mathrm{x}}$ is an odd function of ω.
Hence

$$\Gamma(-\omega) = \frac{\bar{R}(\omega) - 1 - j\,\bar{\mathrm{x}}(\omega)}{\bar{R}(\omega) + 1 - j\,\bar{\mathrm{x}}(\omega)} = \Gamma^*(\omega)$$

3) Any low frequency impedance function (*impedance of a network made up of resistors, capacitors and inductors) can be expressed as the ratio of two polynomials in* ω. Consequently, Γ can also be expressed as the ratio of two polynomials. It follows that $\rho^2(\omega)$ can then be expressed in the form:

$$\rho^2(\omega) = \frac{M(\omega^2)}{M(\omega^2) + N(\omega^2)}$$

Where M and N are non-negative real polynomials in ω^2.

$$P_{LR} = 1 + \frac{M\left(\omega^2\right)}{N\left(\omega^2\right)} = 1 + \frac{\left[\bar{R}(\omega) - 1\right]^2 + \left[\bar{X}(\omega)\right]^2}{4\bar{R}(\omega)}$$

Power Loss Ratios

Within the restrictions we can choose any P_{LR} function. There are two important pass-band functions used for microwave filters.

1) Maximally flat (Butturworth or binomial)
2) Chebyshev (Equal Ripple)

Maximally flat (Butturworth) Filter Characteristics

We choose :

$$N\left(\omega^2\right) = 1$$

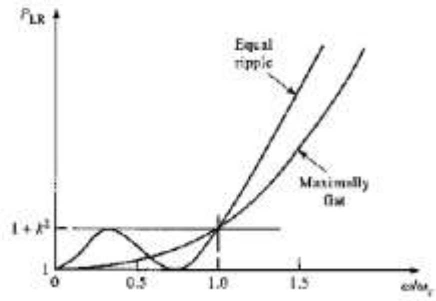
$$M\left(\omega^2\right) = k^2 \left(\frac{\omega}{\omega_c}\right)^{2N}$$

$$P_{LR} = 1 + k^2 \left(\frac{\omega}{\omega_c}\right)^{2N}$$

where N=Number of filter sections (order of the filter)

Passband is from $\omega = 0$ to ω_c (cutoff).

- This power loss ratio acquires its name maximally flat from the fact that P_{LR} has (2N-1) zero derivatives (slope zero, flat) at $\omega = 0$.
- The *maximum value of P_{LR} in the passband is* $k^2 + 1$, for this reason k^2 *is called the passband tolerance.*
- For $\omega > \omega_c$, P_{LR} increases indefinitely at a rate dependent on the exponent 2N, in turn is related to the number of filter sections employed.



If we denote the value of the insertion loss at $\omega_c = \omega$ as L_a then:

$$L_a = 10\log\left(1+k^2\right)$$

Maximum value is $1+k^2$ in passband, k^2 is called the passband tolerance. The band edge (ω_c) is usually such that:

$L_a = 3dB$ or $L_a = 10\log 2$, $k^2 = 1, k = 1$.

Chebyshev Filter Characteristics

(Equal Ripple)

The power loss ratio is:

$$P_{LR} = 1 + k^2 T_N^2\left(\frac{\omega}{\omega_c}\right)$$

$T_N\left(\frac{\omega}{\omega_c}\right)$ is the Chebyshev polynomial of degree N.

$$T_N\left(\frac{\omega}{\omega_c}\right) = \cos\left(N\cos^{-1}\frac{\omega}{\omega_c}\right)$$

$T_N\left(\frac{\omega}{\omega_c}\right)$ oscillates between +1 and -1 for $\left|\frac{\omega}{\omega_c}\right| \le 1$ and increases monotonically for $\left|\frac{\omega}{\omega_c}\right| > 1$.

N	Chebyshev Polynomial
0	1
1	x
2	$2x^2 - 1$
3	$4x^3 - 3x$
4	$8x^4 - 8x^2 + 1$
5	$16x^5 - 20x^3 + 5x$
6	……

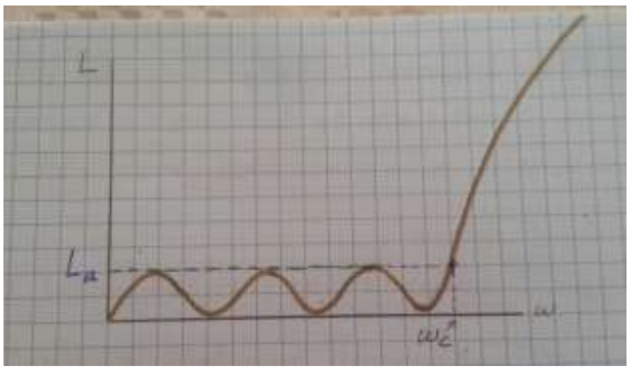
Can also be expressed as:

$$T_N\left(\frac{\omega}{\omega_c}\right)=\cos\left(\mathrm{N}\cos^{-1}\left(\frac{\omega}{\omega_c}\right)\right) \qquad -1\le\frac{\omega}{\omega_c}\le 1$$

$$T_N\left(\frac{\omega}{\omega_c}\right)=\cosh\left(\mathrm{N}\cos^{-1}\left(\frac{\omega}{\omega_c}\right)\right) \qquad \frac{\omega}{\omega_c}\ge 1$$

[Recurrence Relation for $T_n(x)$:

$T_{n+1}(x)=2xT_n(x)-T_{n-1}(x) \qquad -\infty<x<\infty$]



L_a = maximum ripple dB attenuation in the passband, ω_c is the equal-ripple band edge.

The insertion loss may be specified mathematically as:

$$L(\omega)=10\log\left\{1+\varepsilon\cos^2\left[N\cos^{-1}\left(\frac{\omega}{\omega_c}\right)\right]\right\} \qquad \omega\le\omega_c$$

$$L(\omega)=10\log\left\{1+\varepsilon\cosh^2\left[N\cosh^{-1}\left(\frac{\omega}{\omega_c}\right)\right]\right\} \qquad \omega\ge\omega_c$$

Where, $\varepsilon=\left[\text{antilog}\left(\frac{L_a}{10}\right)\right]-1$

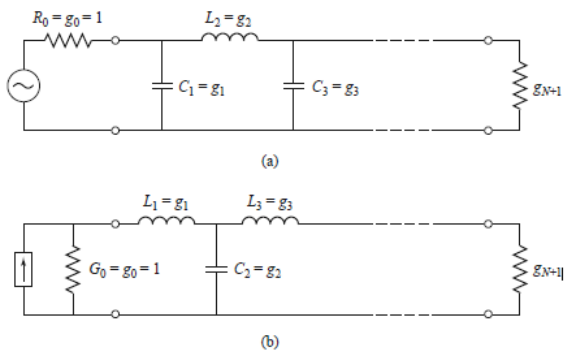
N=Number of reactive elements in the filter.

If N is even there will be $\frac{N}{2}$ frequencies where L=0.

If N is odd there will be $\frac{N+1}{2}$ frequencies where L=0.

Some Low Pass Filter Designs

The *maximally flat* and *Chebyshev* low-pass filter *power loss ratios* discussed up to now can be realized by means of a ladder network of capacitors and inductors in the forms shown below.



Ladder circuits for low-pass filter prototypes and their element definitions. (a) Prototype beginning with a shunt element. (b) Prototype beginning with a series element.

- Circuit (b) is the dual of circuit (a). Both can be designed to give the same power loss ratio.
- The load impedance is chosen to 1Ω and the generator impedance as R (can be taken as 1Ω).
- The element values are denoted by g_k and are the same in both circuits.

We shall use (a).

Power Loss Ratio: $P_{LR} = \dfrac{1}{1-|\Gamma|^2}$

$$\Gamma = \frac{Z_{in}-1}{Z_{in}+1}$$

Where Z_{in} is the input impedance at the plane. (a function of ω)

$$P_{LR} = \frac{|Z_{in}+1|}{2\left(Z_{in}+Z_{in}^*\right)} \qquad \text{(a function of } \omega \text{)}$$

(At $\omega = 0$, all capacitors appear as open circuits and all inductors appear as short circuits, hence $Z_{in} = 1$ if $R_L = 1$.)

Maximally Flat (Butterworth) Filter

$$P_{LR} = 1 + k^2\left(\frac{\omega}{\omega_c}\right)^{2N}$$

$P_{LR} = 1$ at $\omega = 0$ $|Z_{in} - 1| = 0$)

If we have maximally flat (Butterworth) filter this P_{LR} is now equated to P_{LR} required. We equate the two P_{LR} and obtain g_k values (N equations and N g_k) (equate the coefficients of ω^2, ω^4 *etc.*)

Here,

g_o : the generator resistance.

$g_k,\ k = 1:N$ Inductance L for series inductance, capacitance C for shunt capacitors.

g_{N+1} : Load resistance if the last one is capacitor.

: Load conductance if the last one is inductor.

The results are tabulated and g values are obtained from these tables.

TABLE 8.3 Element Values for Maximally Flat Low-Pass Filter Prototypes ($g_0 = 1$, $\omega_c = 1$, $N = 1$ to 10)

N	g_1	g_2	g_3	g_4	g_5	g_6	g_7	g_8	g_9	g_{10}	g_{11}
1	2.0000	1.0000									
2	1.4142	1.4142	1.0000								
3	1.0000	2.0000	1.0000	1.0000							
4	0.7654	1.8478	1.8478	0.7654	1.0000						
5	0.6180	1.6180	2.0000	1.6180	0.6180	1.0000					
6	0.5176	1.4142	1.9318	1.9318	1.4142	0.5176	1.0000				
7	0.4450	1.2470	1.8019	2.0000	1.8019	1.2470	0.4450	1.0000			
8	0.3902	1.1111	1.6629	1.9615	1.9615	1.6629	1.1111	0.3902	1.0000		
9	0.3473	1.0000	1.5321	1.8794	2.0000	1.8794	1.5321	1.0000	0.3473	1.0000	
10	0.3129	0.9080	1.4142	1.7820	1.9754	1.9754	1.7820	1.4142	0.9080	0.3129	1.0000

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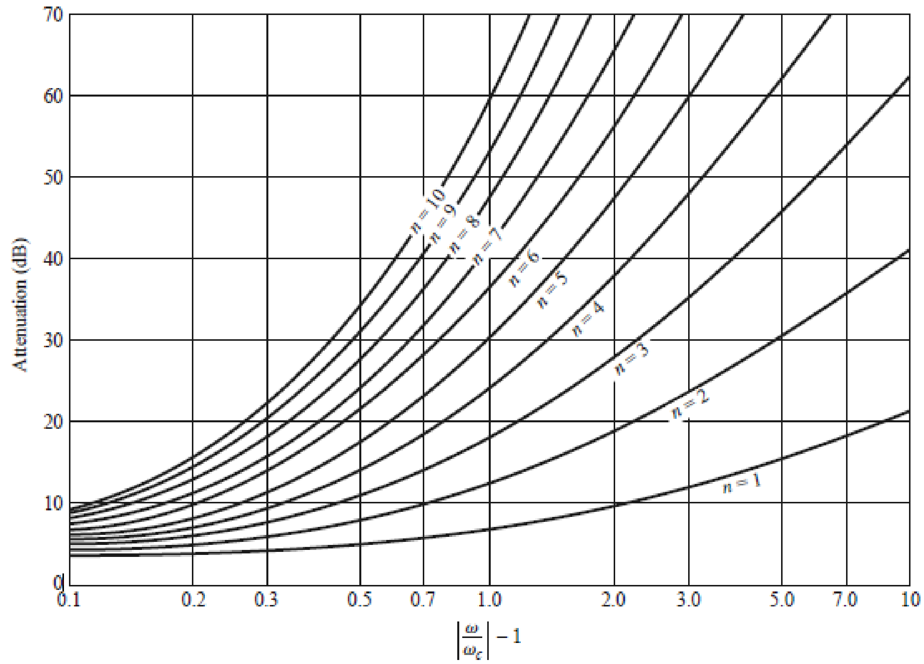
N is usually determined by choosing a performance (…dB obtained at ….Hz) at the stopband.

The table below gives the element values for maximally flat low-pass filter prototypes for $N = 1$ to 10.



Attenuation versus normalized frequency for maximally flat filter prototypes.

Adapted from G. L. Matthaei, L. Young, and E. M. T. Jones, *Microwave Filters, Impedance-Matching Networks, and Coupling Structures*, Artech House, Dedham, Mass., 1980, with permission.

Chebyshev Filter:

Similar solutions and tabulated results are also obtained for Chebyshev filters.

$$\mathrm{P_{LR}} = 1 + \mathrm{k}^2 T_N^2\left(\frac{\omega}{\omega_c}\right)$$

$\omega = 0$, $\mathrm{P_{LR}} = 1 + \mathrm{k}^2 T_N^2(0)$

$$T_N(0) = \cos\left(N\cos^{-1}0\right)$$

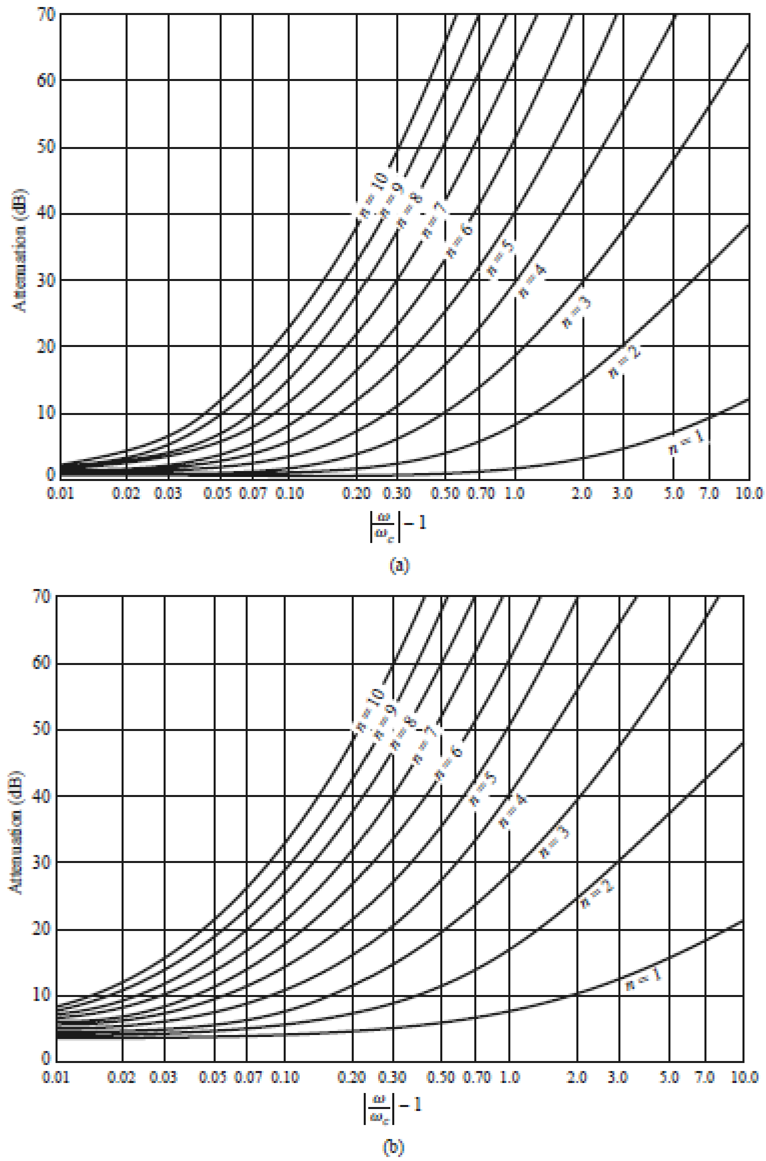
$$T_N(0) = \cos\left(N\frac{\pi}{2}\right)$$

If N is odd, then $T_N(0) = 0$, R=1

If N is even then $T_N{}^2(0) = 1$, $\mathrm{P_{LR}} = 1 + \mathrm{k}^2$

For a Maximally Flat (Butterworth) Filter choose initially $\omega = 1$ and k=1 (half power cutoff).

$$P_{LR} = 1 + k^2\left(\frac{\omega}{\omega_c}\right)^{2N} \qquad P_{LR} = 1 + \omega^{2N}$$



Attenuation versus normalized frequency for equal-ripple filter prototypes. (a) 0.5 dB ripple level. (b) 3.0 dB ripple level.

Element Values for Equal-Ripple Low-Pass Filter Prototypes ($g_0 = 1, \omega_c = 1$, $N = 1$ to 10, 0.5 dB and 3.0 dB ripple)

					0.5 dB Ripple						
N	g_1	g_2	g_3	g_4	g_5	g_6	g_7	g_8	g_9	g_{10}	g_{11}
1	0.6986	1.0000									
2	1.4029	0.7071	1.9841								
3	1.5963	1.0967	1.5963	1.0000							
4	1.6703	1.1926	2.3661	0.8419	1.9841						
5	1.7058	1.2296	2.5408	1.2296	1.7058	1.0000					
6	1.7254	1.2479	2.6064	1.3137	2.4758	0.8696	1.9841				
7	1.7372	1.2583	2.6381	1.3444	2.6381	1.2583	1.7372	1.0000			
8	1.7451	1.2647	2.6564	1.3590	2.6964	1.3389	2.5093	0.8796	1.9841		
9	1.7504	1.2690	2.6678	1.3673	2.7239	1.3673	2.6678	1.2690	1.7504	1.0000	
10	1.7543	1.2721	2.6754	1.3725	2.7392	1.3806	2.7231	1.3485	2.5239	0.8842	1.9841
					3.0 dB Ripple						
N	g_1	g_2	g_3	g_4	g_5	g_6	g_7	g_8	g_9	g_{10}	g_{11}
1	1.9953	1.0000									
2	3.1013	0.5339	5.8095								
3	3.3487	0.7117	3.3487	1.0000							
4	3.4389	0.7483	4.3471	0.5920	5.8095						
5	3.4817	0.7618	4.5381	0.7618	3.4817	1.0000					
6	3.5045	0.7685	4.6061	0.7929	4.4641	0.6033	5.8095				
7	3.5182	0.7723	4.6386	0.8039	4.6386	0.7723	3.5182	1.0000			
8	3.5277	0.7745	4.6575	0.8089	4.6990	0.8018	4.4990	0.6073	5.8095		
9	3.5340	0.7760	4.6692	0.8118	4.7272	0.8118	4.6692	0.7760	3.5340	1.0000	
10	3.5384	0.7771	4.6768	0.8136	4.7425	0.8164	4.7260	0.8051	4.5142	0.6091	5.8095

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Filter Transformations

Impedance Scaling:

In the prototype design the load impedance (R_L) and source impedance (R_s) are chosen as 1Ω (except for the even order Chebyshev). If we now denormalise by changing the R_s to R_o, then the impedance scaled values become:

$$R_s^{'} = R_o$$

$$R_L^{'} = R_L R_o = R_o$$ (For Chebyshev R_L may not be 1.)

$$C^{'} = \frac{C}{R_o}$$

$$L^{'} = LR_o$$

where L, C, and R_L are the component values for the original prototype.

Frequency Scaling:

The cutoff frequency of the low pass filter was chosen as 1. If we now choose the cut off frequency as ω_c, in all expressions we replace ω with $\frac{\omega}{\omega_c}$.

$$\omega \to \frac{\omega}{\omega_c}$$

$$P'_{LR}(\omega) = P'_{LR}\left(\frac{\omega}{\omega_c}\right)$$

Cutoff now occurs at $\omega = \omega_c$ not at $\omega = 1$.

The reactance's are:

$$jX_k = j\frac{\omega}{\omega_c}L' = \mathrm{j}\omega\left(L'_k\right)$$

$$jB_k = j\frac{\omega}{\omega_c}\mathrm{L'} = \mathrm{j}\omega\left(C'_k\right)$$

So, when we scale for both frequency and impedance:

$$L'_k = \frac{R_o L_k}{\omega_c} \qquad C'_k = \frac{C_k}{R_o \omega_c}$$

L_k and C_k here are the values g_k obtained from tables.

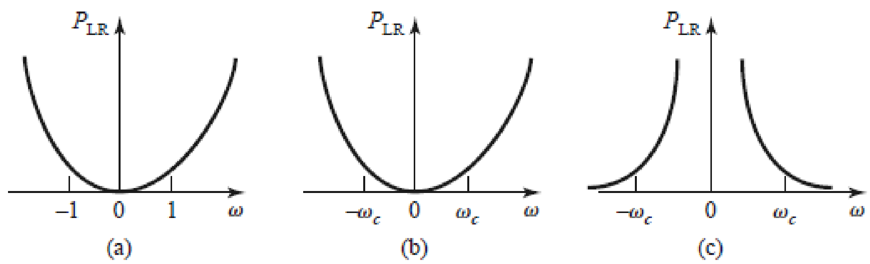
Low Pass Filter Design Procedure:

1) Determine N from requirements,
2) Determine the type of filter to be used (Butterworth, Chebyshev),
3) Determine g_k for $R_s = 1\Omega$ and $\omega_c = 1$,
4) Scale for frequency and impedance,
5) Convert C and L to transmission lines for microwaves.

Low Pass to High Pass Transformation

$$f(\omega) = -\frac{\omega_c}{\omega} \qquad P_{LR} = 1 + P\left(\frac{\omega_c}{\omega}\right)^2 \qquad \omega = -\frac{\omega_c}{\omega}$$

1) The point $\omega' = 0$ is mapped into the points $\omega = \pm\infty$, and vice versa; cutoff occurs when $\omega = \pm\omega_c$
2) The points $\omega = \pm 1$ are mapped into $\omega = \mp\omega_c$
3) The points $\omega = \pm\infty$ are mapped into $\omega = 0$
4) The effect is to interchange the passband and stopband regions.



Frequency scaling for low-pass filters and transformation to a high-pass response. (a) Low-pass filter prototype response for $\omega_c = 1$ rad/sec. (b) Frequency scaling for low-pass response. (c) Transformation to high-pass response.

The frequency substitution:

$$\omega \leftarrow -\frac{\omega_c}{\omega}$$

can be used to convert a low-pass response to a high-pass response, as shown in the figure above.

1) This substitution maps $\omega = 0$ to $\omega = \pm\infty$, and vice versa; cutoff occurs when $\omega = \pm\omega_c$
2) The negative sign is needed to convert inductors (and capacitors) to realizable capacitors (and inductors).

Serious Reactance:

Low Pass | High Pass

$$jX_k = j\omega L_k \qquad -j\frac{\omega_c}{\omega}L_k = \frac{1}{j\omega C_k'}$$

Shunt Susceptance

$$jB_k = j\omega C_k \qquad -j\frac{\omega_c}{\omega}C_k = \frac{1}{j\omega L_k'}$$

So L_k is replaced by $C'_k = \dfrac{1}{\omega_c L_k}$ and C_k is replaced by $L'_k = \dfrac{1}{\omega_c C_k}$.

Capacitances become inductance and inductances become capacitance.

If we include the impedance scaling:

$$C_k^{'} = \frac{1}{R_o \omega_c L_c} \qquad L_k^{'} = \frac{R_o}{\omega_c L_c}$$

(L_k and C_k are g values).

So:

1) We obtain low pass prototype.
2) By converting $C_k \rightarrow L_k$, $L_k \rightarrow C_k$ and using the formulae above we obtain the high pass filter.

Low Pass to Band Pass Transformation

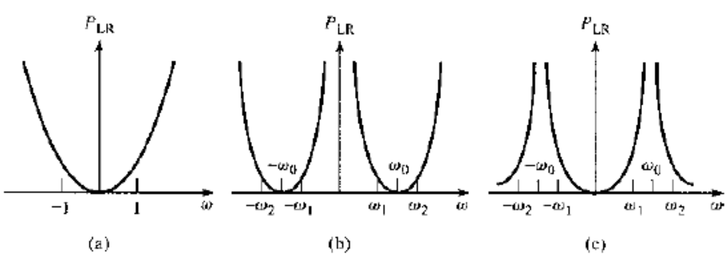
Low-pass prototype filter designs can also be transformed to have the bandpass responses.

$$\omega \leftarrow \frac{1}{\Delta}\left(\frac{\omega}{\omega_c} - \frac{\omega_c}{\omega}\right) ,$$

$\Delta = \dfrac{\omega_2 - \omega_1}{\omega_o}$ is the fractional bandwidth of the bassband.

$\omega 1$ and $\omega 2$ denote the edges of the passband.

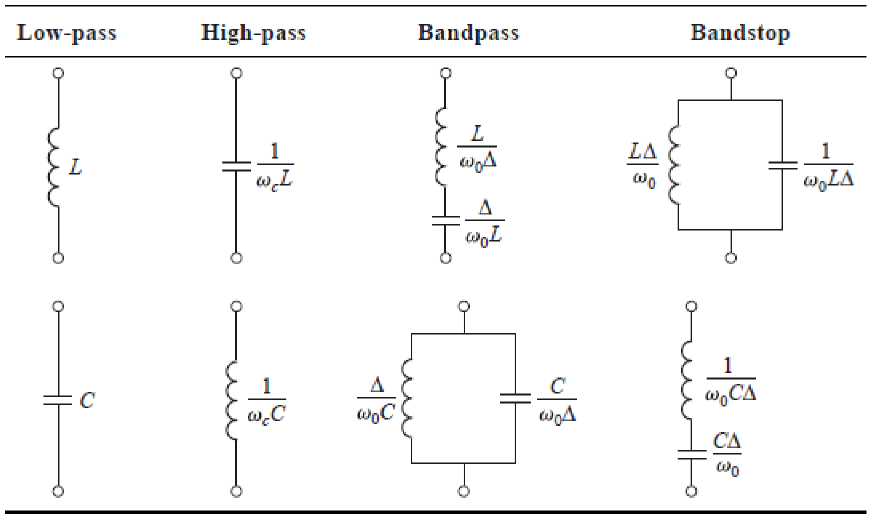
The center frequency: $\omega_o = \sqrt{\omega_1 \omega_2}$



Bandpass and bandstop frequency transformations. (a) Low-pass filter prototype response for $\omega_c = 1$. (b) Transformation to bandpass response. (c) Transformation to bandstop response.

The shunt capacitances are converted to a shunt LC and series L converted to a series LC.

Summary of Prototype Filter Transformations $\left(\Delta = \frac{\omega_2 - \omega_1}{\omega_0}\right)$



Then carry out impedance scaling. Similar equations are obtained for the bandstop filters.

Filter Design for Microwave Frequencies

At microwave frequencies, it is not very practical to manufacture lumped element capacitors and inductors. So transmission lines are used instead.

Richard's Transformation:

Using this transformation we can use the designs with C and L to obtain filters with s/c and o/s transmission line.

We define a new variable Ω.

$$\omega \leftarrow \Omega \qquad \Omega = \tan \beta \ell$$

This transformation maps the ω plane to the Ω plane, which repeats with a period of $\omega / v_p = 2\pi$. It is used to synthesize an *LC* network using open- and short-circuited transmission line stubs.
So the reactance of an inductor:

$$jX_c = j\Omega L = jL \tan \beta \ell$$

But we know that for a short circuited line of length ℓ,

$$Z_{in} = jZ_o \tan \beta \ell$$

So the inductor will be replaced by a short circuited line of characteristic impedance of L.

The susceptance of a capacitor as:

$$jB_C = j\Omega C = jC\tan\beta\ell$$

For an open circuited line $Y_{in} = jY_o \tan\beta\ell$

We can replace a C with an open circuited line of $Y_o\left(\frac{1}{Z_o}\right) = C$

These results indicate that an **inductor** can be replaced with a **short-circuited stub** of length $\beta\ell$ and **characteristic impedance** *L*, while a **capacitor** can be replaced with an **open-circuited stub** of length $\beta\ell$ and **characteristic impedance** 1/*C*.

The cutoff occurs at $\omega = 1$.

$$\tan\beta\ell = 1$$

$$\frac{2\pi}{\lambda}\ell = \frac{\pi}{4}$$

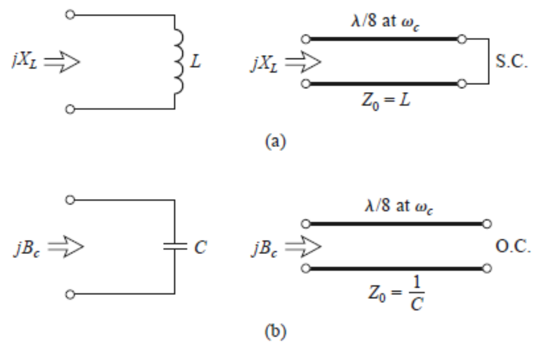
$$\ell = \frac{\lambda}{8}$$

So:

$\ell = \frac{\lambda}{8}$ where λ is the wavelength of the line at the cutoff frequency ω_c .

So we must use o/c or s/c lines of length at the cutoff frequency.

At frequencies away from ω_c , the impedances of the stubs will no longer match the original lumped-element impedances, and the filter response will differ from the desired prototype response. In addition, the response will be periodic in frequency, repeating every $4\omega_c$.



Richards' transformation. (a) For an inductor to a short-circuited stub. (b) For a capacitor to an open-circuited stub.