FINITE DIFFERENCE APPROXIMATIONS OF PARTIAL DIFFERENTIAL EQUATIONS

Introduction

In general real life EM problems cannot be solved by using the analytical methods, because:

1) The PDE is not linear,
2) The solution region is complex,
3) The boundary conditions are of mixed types,
4) The boundary conditions are time dependent,
5) The medium is inhomogeneous or anisotropic.

For such complicated problems numerical methods must be employed. The basic approach for solving PDE numerically is to transform the continuous equations into discrete equations, which can be solved using a computational algorithm to obtain an approximate solution of the PDE.

Finite Difference Method (FDM) is one of the available numerical methods which can easily be applied to solve PDE's with such complexity.

FD method is based upon the discretization of differential equations by finite difference equations.

Finite difference approximations *have algebraic forms* and relate the value of the dependent variable at a point in the solution region, to the values at some neighboring points. It is easy to understand and apply.

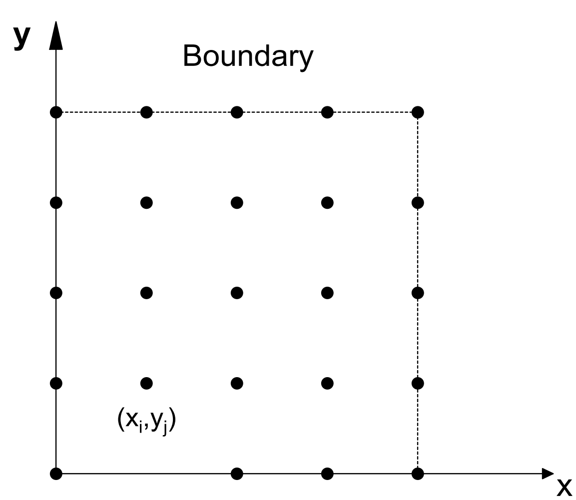
Steps of finite difference solution:

- Divide the solution region into a grid of nodes or list of points spanning the computational domain,
- Approximate the given differential equation by finite difference equivalent,
- Apply a source or excitation,
- Solve the differential equations subject to the boundary conditions.

When the PDE includes time as an independent variable, the FD approach is referred as the finite difference time-domain (FDTD) algorithm.

Rectangular Grid

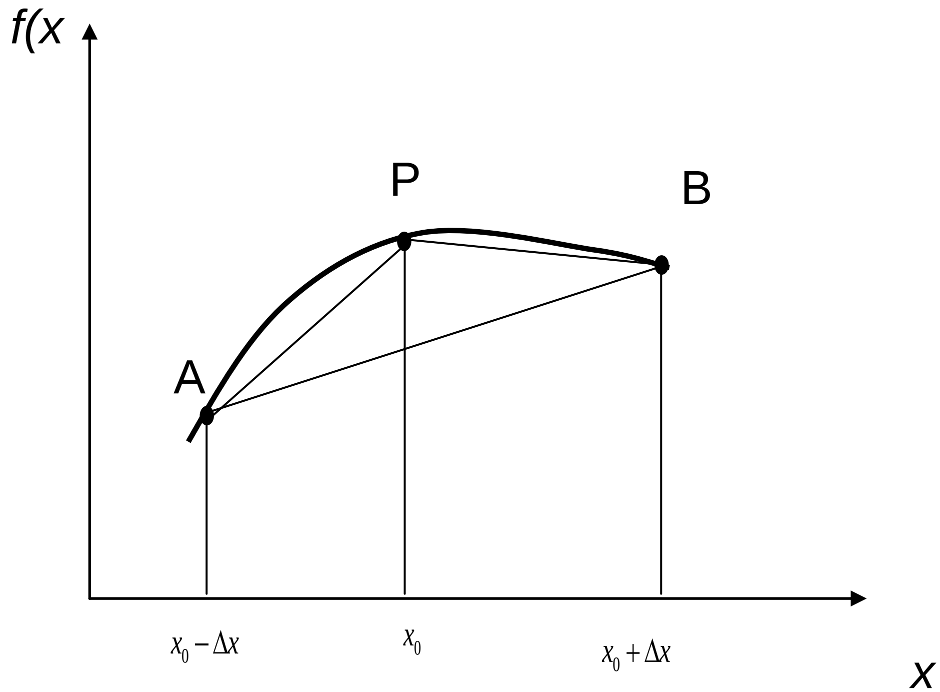
A grid is list of points at which the unknown function in the PDE is sampled.



Two dimensional rectangular grids.

Finite Difference Schemes

Look at the construction of the finite difference approximations from the given differential equation.



The derivative of a given function f(x) can be approximated in different ways. Higher order approximations can be used to obtain more accurate results by using many sample values at neighboring points. But higher order approximations increase the computational cost.

Three of the approximations are:

$$\frac{df(x_0)}{dx} \simeq \frac{f(x_0+\Delta x) - f(x_0)}{\Delta x} \quad (1) \qquad \textit{Forward Difference Formula}$$

$$\frac{df(x_0)}{dx} \simeq \frac{f(x_0) - f(x_0-\Delta x)}{\Delta x} \quad (2) \qquad \textit{Backword Difference Formula}$$

$$\frac{df(x_0)}{dx} \simeq \frac{f(x_0+\Delta x) - f(x_0-\Delta x)}{2\Delta x} (3) \qquad \textit{Central Difference Formula}$$

These approximations are derived from the Taylor series expansions:

$$f(x_0+\Delta x) = f(x_0) + \Delta x \frac{\partial f(x_0)}{\partial x} + \frac{1}{2!}(\Delta x)^2 \frac{\partial^2 f(x_0)}{\partial x^2} + \frac{(\Delta x)^3}{3!}\frac{\partial^3 f(x_0)}{\partial x^3} + \ldots (4)$$

$$f(x_0-\Delta x) = f(x_0) - \Delta x \frac{\partial f(x_0)}{\partial x} + \frac{1}{2!}(\Delta x)^2 \frac{\partial^2 f(x_0)}{\partial x^2} - \frac{(\Delta x)^3}{3!}\frac{\partial^3 f(x_0)}{\partial x^3} + \ldots (5)$$

Subtracting these equations and neglecting the higher order terms we obtain the central difference approximation:

$$\frac{\partial f(x_0)}{\partial x} \simeq \frac{f(x_0+\Delta x) - f(x_0-\Delta x)}{2\Delta x}$$

The error is second order.

The error in the forward and backward approximation formulas is first order.

As long as the derivatives of *f* are well behaved and the step size is not too large, the central difference formula is more accurate compared to the other two. i.e. backward and forward difference approximations.

The error in central difference decreases quadratically as the step size decreases, whereas the decrease is only linear for the other two formulas.
In general, central difference formula is to be preferred. Situations where the data is not available on both sides of the point where the numerical derivative is to be calculated are exceptions.

Similarly the approximation of the second derivative $\frac{\partial^2 f}{\partial x^2}$ can be obtained by adding equation 4 to 5 and neglecting higher order terms:

Adding 4 and 5 :

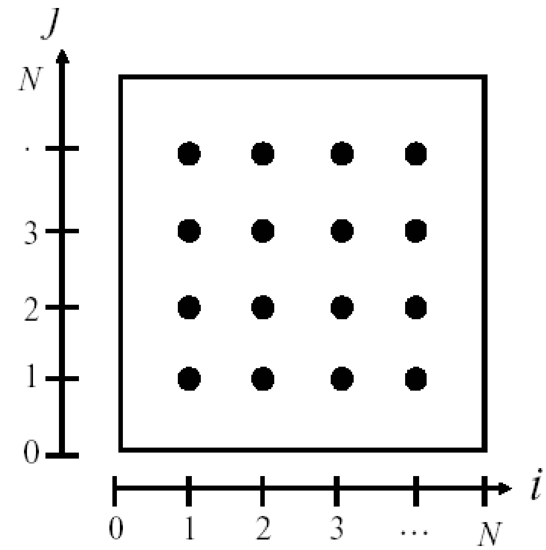
$$f(x_0+\Delta x)+f(x_0-\Delta x)=2f(x_0)+(\Delta x)^2\frac{\partial^2 f(x_0)}{\partial x^2}+O(\Delta x)^4$$

Neglecting higher order terms:

$$\frac{\partial^2 f}{\partial x^2}=\frac{f(x_0+\Delta x)-2f(x_0)+f(x_0-\Delta x)}{(\Delta x)^2}$$

This approximation has second order error.

To apply the difference method to find the solution of a function *f*, the solution region is divided into rectangles:



Let the coordinates (x, t) of a typical grid point or a node be:

$$x = i\Delta x,\ \ i = 0,1,2,\ldots$$
$$t = j\Delta t,\ \ \ j = 0,1,2,\ldots$$

The value of Φ at a point is:

$$\Phi(i, j) = \Phi(i\Delta x, j\Delta t)$$

Using this notation, the central difference approximations of Φ at (i, j) are:

First derivative:

$$\left.\frac{\partial \Phi}{\partial x}\right|_{(i,j)} \simeq \frac{\Phi(i+1,j)-\Phi(i-1,j)}{2\Delta x}$$

$$\left.\frac{\partial \Phi}{\partial t}\right|_{(i,j)} \simeq \frac{\Phi(i,j+1)-\Phi(i,j-1)}{2\Delta t}$$

Second derivative:

$$\left.\frac{\partial^2 \Phi}{\partial x^2}\right|_{(i,j)} \simeq \frac{\Phi(i+1,j)-2\Phi(i,j)+\Phi(i-1,j)}{(\Delta x)^2}$$

$$\left.\frac{\partial^2 \Phi}{\partial t^2}\right|_{(i,j)} \simeq \frac{\Phi(i,j+1)-2\Phi(i,j)+\Phi(i,j-1)}{(\Delta t)^2}$$