METHOD OF MOMENT

(continuation)

Consider again the finite straight wire at a constant potential. From statics we know that a linear electric charge distribution $\rho_l(\bar{r}')$ will create an electric potential, V(r):

$$V(\bar{r}) = \frac{1}{4\pi\varepsilon_0}\int_l \frac{\rho_\ell(r')}{\left|\bar{r}-\bar{r}'\right|}d\ell'$$

This equation may be used to find potentials due to the known line charge densities. However, for most of the practical cases the charge distributions are unknown even when the potential on the source is given.

As defined, the wire has a length of ℓ along the y direction. Its radius is a, and it is connected to a battery of 1 Volts.

Remember that, choosing the observation point along the wire axis gives:

$$1 = \frac{1}{4\pi\varepsilon_0}\int_0^\ell \frac{\rho_\ell(y')}{R(y,y')}dy'$$

Where,

$$\begin{aligned} R(y,y') &= \left[x'^2 + z'^2 + (y-y')^2\right]^{1/2} \\ &= \left[a^2 + (y-y')^2\right]^{1/2} \end{aligned}$$

Approximate the unknown charge distribution $\rho(y')$ by an expansion of N known terms with constant, but unknown coefficients, that is:

$$\rho(y') = \sum_{n=1}^{N} a_n g_n(y')$$

Thus the integral equation can be written as:

$$4\pi\varepsilon_0 = \int_0^{\ell} \frac{1}{R(y,y')} \sum_{n=1}^{N} [a_n g_n(y')] dy'$$

The above equation is a nonsingular integral; its integration can be changed to summation:

$$4\pi\varepsilon_0 = \sum_{n=1}^{N} a_n \int_0^{\ell} \frac{g_n(y')}{R(y-y')} dy'$$

The $g_n(y')$ functions in the expansion are chosen to model accurately the unknown quantity and minimize the computation. These functions are called **basis (or expansion)** functions.

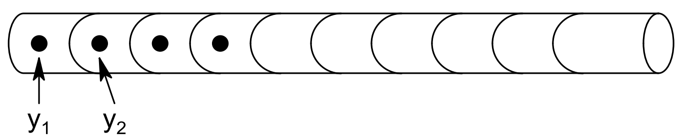
The choice of the **basis** function is one of the most critical parts of the MoM. A large variety of basis functions exists.

Popular choices include functions with the following spatial variations:

1) Constant (subdomain piecewise constant) (also known as a pulse or stair-step),
2) Linear,
3) Polynomial,
4) Piecewise Sinusoidal, etc.

In this solution pulse basis functions will be used since it is an introductory example.

Divide the wire into N segments having lengths of $\Delta = \ell / N$.



These functions are defined to be of constant value over one segment and zero elsewhere. Or:

$$g_n(y') = \begin{Bmatrix} 0 & y' < (n-1)\Delta \\ 1 & (n-1)\Delta \le y' \le n\Delta \\ 0 & y' > n\Delta \end{Bmatrix}$$

In other words, n^{th} function is unity in one segment (segment *n*) and zero elsewhere.

Using these pulse basis functions and interchanging the order of integral and summation, we obtain:

$$4\pi\varepsilon_0 = a_1\int_0^{\Delta}\frac{g_1(y')dy'}{R(y_1-y')} + a_2\int_{\Delta}^{2\Delta}\frac{g_2(y')dy'}{R(y_2-y')} + \ldots + a_N\int_{(N-1)\Delta}^{\ell}\frac{g_N(y')dy'}{R(y_N-y')}$$

The matrix form:

$$4\pi\varepsilon_0 = a_1\int_0^{\Delta}\frac{g_1(y')dy'}{R(y_1-y')} + a_2\int_{\Delta}^{2\Delta}\frac{g_2(y')dy'}{R(y_1-y')} + \ldots + a_N\int_{(N-1)\Delta}^{\ell}\frac{g_N(y')dy'}{R(y_1-y')}$$

$$4\pi\varepsilon_0 = a_1\int_0^{\Delta}\frac{g_1(y')dy'}{R(y_2-y')} + a_2\int_{\Delta}^{2\Delta}\frac{g_2(y')dy'}{R(y_2-y')} + \ldots + a_N\int_{(N-1)\Delta}^{\ell}\frac{g_N(y')dy'}{R(y_2-y')}$$

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$$4\pi\varepsilon_0 = a_1\int_0^{\Delta}\frac{g_1(y')dy'}{R(y_N-y')} + a_2\int_{\Delta}^{2\Delta}\frac{g_2(y')dy'}{R(y_N-y')} + \ldots + a_N\int_{(N-1)\Delta}^{\ell}\frac{g_N(y')dy'}{R(y_N-y')}$$

The above set of equations is a system of linear equations. The original integral equation inversion problem has been reduced to a matrix equation inversion problem which can be written as:

$$[V_m] = [A_{mn}][a_n]$$

$$A_{mn} = \int \frac{g_n(y')}{\sqrt{(y_m - y')^2 + a^2}} dy' = \int_{(n-1)\Delta}^{n\Delta} \frac{1}{\left[a^2 + (y_m - y')^2\right]^{1/2}} dy'$$

$$\left[a_n\right] = \left[a_1 \;\; a_2 \;\; ...a_N\right]^T$$

$$\left[V_n\right] = \left[4\pi\varepsilon_o \;\; 4\pi\varepsilon_o \;\; ...4\pi\varepsilon_o\right]^T$$

Where, *n* subscribe refers to the source point and *m* refers to the testing (sampling) points.

$[A_{mn}]$: Matrix to be generated (NXN)

$[V_m]$: Excitation column vector (known) (NX1).

$[a_n]$: Unknown response column vector to be found (NX1).

The solution is:

$$\left[a_n\right] = \left[A_{mn}\right]^{-1} \left[V_m\right]$$

The integrals involved may be solved by using appropriate approximations. But this may not be possible for complicated problems.

Efficient numerical integration computer subroutines are available.

Summary:

The solution of the integral equation for the charge distribution on a wire has been discussed.

The unknown charge was approximated with some basis functions, dividing the wire into segments and then sequentially enforcing at the center of each segment to form a set of linear equations.

MoM Basics

MoM can be used in the solution of linear operational equations of the form:

$$Lf = g$$

MoM application begins by approximating the unknown function by a linear combination of unknown functions in the form:

$$f = \sum_{i=1}^{N} \alpha_i f_i$$

The functions α_i are called basis or expansion functions.

They are selected with the appropriate values of the α_i parameters that the right side of this equation is a reasonably accurate approximation to the left side. We have:

$$\sum_{i=1}^{N} \alpha_i L(f_i) = g$$

We define a set of linearly independent "testing functions" or "weighting functions" $\{w_1, w_2, ..., w_N\}$ in the range of L. Taking the inner product (usually an integral) of the above equation and using the linearity of the inner product:

$$\sum_{i=1}^{N} \alpha_i \langle Lf_i, w_j \rangle = \langle g, w_j \rangle, \quad j = 1, 2, ..., M$$

It is common practice to select *M=N*, but this is not necessary.

Any function can be used as a testing function. But we should be aware that if the function is complicated the evaluation of the elements will be difficult. Dirac-Delta function $\delta(x)$ is one of the most commonly used testing function because the result of integration of the inner product of $\delta(x)$ with any function over any region, is conditionally equal to the magnitude of the function itself at the place of δ.

For *M=N:*

$$[A]\underline{\alpha} = \underline{g}$$

where,

$$[A] = [\langle w_j, Lf_i \rangle]$$

$$\underline{\alpha} = [\alpha_i]$$

$$\underline{g} = [\langle w_j, g \rangle]$$

Or

$$[A] = \begin{bmatrix} \langle w_1, Lf_1 \rangle & \langle w_1, Lf_2 \rangle & \ldots & \langle w_1, Lf_N \rangle \\ \langle w_2, Lf_1 \rangle & \langle w_2, Lf_2 \rangle & \ldots & \langle w_2, Lf_N \rangle \\ . & . & \ldots & . \\ . & . & \ldots & . \\ \langle w_N, Lf_1 \rangle & \langle w_N, Lf_2 \rangle & \ldots & \langle w_N, Lf_N \rangle \end{bmatrix}$$

$$\underline{\alpha} = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ . \\ . \\ \alpha_N \end{bmatrix} \leftarrow (NX1)$$

$$\underline{g} = \begin{bmatrix} \langle w_1, g \rangle \\ \langle w_2, g \rangle \\ \\ \langle w_N, g \rangle \end{bmatrix} \leftarrow (NX1)$$

Or

$$\underline{\alpha} = [A]^{-1} \underline{g}$$

If A is non-singular.

Inner Product

Inner product can be defined as:

$$\langle u,v \rangle = \int_{\Omega} uv d\Omega$$

<u>Example:</u>
Find the inner product of u(x)=1-x and v(x)=2x in the interval (0,1).
<u>Solution:</u>

$$\langle u,v \rangle = \int_0^1 (1-x) 2x dx$$

$$\langle u,v \rangle = 2\left(\frac{x^2}{2} - \frac{x^3}{3} \right) \Big|_0^1 = 0.333$$