PLANE WAVES IN A CONDUCTING MEDIUM

II. GOOD CONDUCTING CASE

If \( \varepsilon'' >> \varepsilon' \) or \( \left( \frac{\sigma}{\varepsilon \mu} \right) >> 1 \), conduction current dominates considerably over the displacement current. Then,

\[
\alpha = \sqrt{\frac{\varepsilon \mu}{2}} \left[ \sqrt{1 + \left( \frac{\sigma}{\varepsilon \mu} \right)^2} - 1 \right]^{\frac{1}{2}} \approx \varepsilon \mu \left( \frac{\sigma}{\varepsilon \mu} \right) - 1^{\frac{1}{2}}
\]

\[
\alpha \approx \varepsilon \mu \sqrt{\frac{\varepsilon \mu}{2}} = \sqrt{\frac{\varepsilon \mu \sigma}{2}} = \sqrt{\frac{2\pi f \mu \sigma}{2}} = \sqrt{\pi f \mu \sigma}
\]

Then, \( \alpha = \sqrt{\pi f \mu \sigma} \), \( \alpha \) is a function of frequency.

\[
\beta = \sqrt{\frac{\varepsilon \mu}{2}} \left[ \sqrt{1 + \left( \frac{\sigma}{\varepsilon \mu} \right)^2} + 1 \right]^{\frac{1}{2}} \approx \varepsilon \mu \left( \frac{\sigma}{\varepsilon \mu} + 1 \right)^{\frac{1}{2}}
\]

\[
\beta = \varepsilon \mu \sqrt{\frac{\varepsilon \mu}{2}} = \sqrt{\pi f \mu \sigma}
\]

\[
\alpha = \beta = \sqrt{\pi f \mu \sigma} \). Then,
\[
\gamma = \alpha + j\beta = \sqrt{\pi f \mu \sigma} (1 + j)
\]

Intrinsic Impedance:

\[
\eta_e = \sqrt{\frac{\mu}{\varepsilon_e}} = \sqrt{\frac{\mu}{\varepsilon - j \frac{\sigma}{\varepsilon}}} = \sqrt{\frac{\mu}{\varepsilon}} \left( 1 - j \frac{\sigma}{\varepsilon} \right)^{\frac{1}{2}}
\]

\[
\eta_e \approx \sqrt{\frac{\mu}{\varepsilon}} \left( -j \frac{\sigma}{\varepsilon} \right)^{\frac{1}{2}} = \sqrt{\frac{\mu}{\varepsilon}} \frac{1}{\sqrt{-j \frac{\sigma}{\varepsilon}}} = \sqrt{\frac{\mu \varepsilon}{\sigma}} = \sqrt{\frac{\mu \varepsilon}{\sigma}} \sqrt{j} = \sqrt{\frac{\mu \varepsilon (1 + j)}{\sigma}}
\]

\[
\eta_e \approx (1 + j) \sqrt{\frac{\mu \varepsilon}{2\sigma}}
\]
\[ \sqrt{\frac{w \mu}{2\sigma}} = \sqrt{\frac{w\mu_0}{2\sigma_0^2}} = \sqrt{\frac{w\mu_0}{2}} \cdot \frac{1}{\sigma} = \frac{\alpha}{\sigma} \]

So, \( \eta_0 = (1 + j) \frac{\alpha}{\sigma} = \frac{\alpha \sqrt{2}}{\sigma} \angle 45^\circ \)

The results show that \( \vec{H} \) lags \( \vec{E} \) by 45° in time.

**Phase Velocity**

\[ u_p = \frac{w}{\beta} = \frac{w}{\sqrt{\frac{w\mu\sigma}{2}}} = \sqrt{\frac{2w}{\mu\sigma}} \]

The wavelength,

\[ \lambda = \frac{2\pi}{\beta} = \frac{2\pi}{\sqrt{\frac{\pi f \mu \sigma}{2}}} = 2\sqrt{\frac{\pi f \mu \sigma}{2}} \]

\[ \lambda = \frac{2\pi}{\beta} = \frac{2\pi}{\alpha} = 2\pi \left( \frac{1}{\alpha} \right) = 2\pi \delta \]

**Example:**

Consider copper for which \( \sigma = 5.80 \times 10^7 (S/m) \) \( \epsilon = \epsilon_0 \), \( \mu = \mu_0 \). Take \( f = 3 \text{MHz} \).

**Solution:**

Let us first find the wavelength of a wave in free space at this frequency:

\[ \lambda_0 = \frac{c}{f} = \frac{3 \times 10^8}{3 \times 10^6} = 100 \text{meters} \]

Calculate the loss tangent:

\[ \frac{\sigma}{\omega \epsilon c} = \frac{5.8 \times 10^7}{2\pi \times 3 \times 10^6 \times \frac{1}{36\pi} \times 10^{-9}} = 3.48 \times 10^1 \gg 1 \]

Copper behaves like a good conductor at \( f = 3 \text{MHz} \).

The wavelength in copper will then be:
\[ \lambda = 2 \left( \frac{\pi}{f \mu \sigma} \right)^{1/2} = 2 \left( \frac{\pi}{3 \times 10^8 \times 4 \pi \times 10^{-3} \times 5.8 \times 10^7} \right)^{1/2} = 0.239 \times 10^{-3} \text{m} . \]

or
\[ \lambda = 0.239 \text{mm} \] (compare with 100 meters in air!)

\[ \alpha = \sqrt{\pi f \mu \sigma} = \left( \pi \times 3 \times 10^6 \times 4 \pi \times 10^{-3} \times 5.8 \times 10^7 \right)^{1/2} = 2.62 \times 10^4 \text{(Np/m)} \]

The depth of penetration:

\[ \delta = \frac{1}{\alpha} = 0.38 \times 10^{-4} \text{m} = 0.038 \text{mm} . \]

\[ \delta \text{ at } f = 10 \text{GHz} \text{ is } 0.66 \mu \text{m} . \]

The frequency at which \( \sigma = \omega \varepsilon \) is \( f = \frac{5.8 \times 10^7}{2 \pi \varepsilon_0} = 1.04 \times 10^{16} \text{Hz} \)

Thus at frequencies of even several gigahertz, copper behaves like an excellent conductor. The material is actually a good conductor well into the ultraviolet region. The skin-depth at optical frequencies (\( w = 10^7 \)) is roughly \( 10^{-8} \text{m} \), which explains why metals are opaque.

Example:

Consider sea-water, for which \( \sigma = 4 \text{mhos/m} \), \( \varepsilon = 80 \varepsilon_0 \), \( \mu = \mu_0 \). Take \( f = 25 \text{kHz} \).

Solution:

\[ \frac{\sigma}{\omega \varepsilon} = \frac{4}{2 \pi \times 25 \times 10^3 \times 80 \times \frac{1}{36 \pi} \times 10^{-9}} = 36000 >= 1 \]

Sea-water is good conductor at \( f = 25 \text{kHz} \).

\[ \lambda_{\text{air}} = \frac{3 \times 10^8}{25 \times 10^3} = 12 \text{km} \]

\[ \lambda_{\text{sea-water}} = 2 \left( \frac{\pi}{f \mu \sigma} \right)^{1/2} = 2 \left( \frac{\pi}{25 \times 10^3 \times 4 \pi \times 10^{-3} \times 4} \right)^{1/2} = \frac{1}{4 \pi \times 10^{-2}} \left( \frac{1}{f_{\text{air}}} \right)^{1/2} \text{ or} \]
\[ \delta = \frac{8}{\sqrt{f_{\text{kHz}}}} \text{ (meter)} \]

<table>
<thead>
<tr>
<th>( f (\text{kHz}) )</th>
<th>( \delta (\text{m}) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>2.531</td>
</tr>
<tr>
<td>15</td>
<td>2.067</td>
</tr>
<tr>
<td>20</td>
<td>1.789</td>
</tr>
<tr>
<td>25</td>
<td>1.6</td>
</tr>
<tr>
<td>30</td>
<td>1.462</td>
</tr>
<tr>
<td>35</td>
<td>1.353</td>
</tr>
<tr>
<td>100</td>
<td>0.8</td>
</tr>
</tbody>
</table>