# EENG 428 Laboratory --- Lab Session 1

## Description:

In this session, an oriented review on linear algebra and matrix analysis is present to serve the aim of this course.

## Prerequisites:

Attending students are expected to know the following matrix operations:

- Matrix-Matrix addition, multiplication.
- Determinant of a matrix.
- Transpose of a matrix.
- Matrix inversion.
- Basic Matlab commands.

## Contents:

1- A review on vector-vector operations

2- A review on point and vector representations in 3D space.

3- Vector representation with additional scalar.

4- Inversion of deficient matrices (Moore-Penrose pseudo-inverse).

5- Orthogonal matrices, and the General Transformation Matrices.

6- Inversion of block-matrices.

## 1- Vector-Vector product operations:

### 1.1- Inner product:

The inner product between the vectors $\boldsymbol{a} = a_x\hat{\imath} + a_y\hat{\jmath} + a_z\hat{k}$ and $\boldsymbol{b} = b_x\hat{\imath} + b_y\hat{\jmath} + b_z\hat{k}$ is defined as:

$\boldsymbol{a} \,.\, \boldsymbol{b} = a_x * b_x + a_y * b_y + a_z * b_z$ and it results in a scalar.

Important Properties:

- The inner product is commutative in real vector subspaces.
- The inner product between two non-zero vectors is zero if and only if the vectors are perpendicular (orthogonal)
- The practical way to find the inner product between two vectors is by using the normal matrix multiplication of one vector transpose by the other:

$$\boldsymbol{a} \,.\, \boldsymbol{b} = \boldsymbol{a}^{\boldsymbol{T}} * \boldsymbol{b} = [a_x \quad a_y \quad a_z] * \begin{bmatrix} b_x \\ b_y \\ b_z \end{bmatrix}$$

### 1.2- Cross Product:

The cross product between the vectors $\boldsymbol{a} = a_x\hat{\imath} + a_y\hat{\jmath} + a_z\hat{k}$ and $\boldsymbol{b} = b_x\hat{\imath} + b_y\hat{\jmath} + b_z\hat{k}$ can be found by:

$\boldsymbol{a} \times \boldsymbol{b} = \det\left(\begin{bmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{bmatrix}\right)$ and it results in a vector.

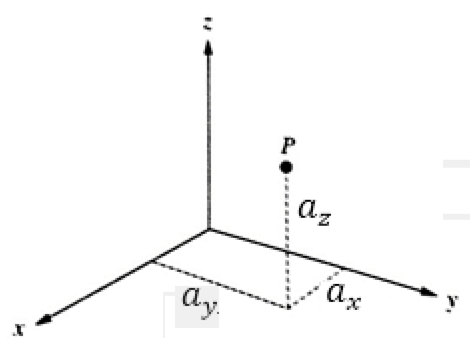
Important properties:

- The cross product is not a commutative operation, the order of the operation effects the direction of the resultant vector ($a \times b = -b \times a$)
- The inner product between two non-zero vectors is zero if and only if the vectors are parallel (or anti-parallel).
- The practical way of finding the cross product between two vectors is by using the normal matrix multiplication of one vector represented in a skew-symmetric matrix by the other:

$$\boldsymbol{a} \times \boldsymbol{b} = \begin{bmatrix} 0 & -a_z & a_y \\ a_z & 0 & -a_x \\ -a_y & a_x & 0 \end{bmatrix} * \begin{bmatrix} b_x \\ b_y \\ b_z \end{bmatrix}$$

The last vector-vector product operation known as the outer-product is less used in the context of this course.

2- **Point and Vector representation in 3D space:**



A point can be represented in space as a vector starting from the origin and ending at the point location using Cartesian coordinates as:

$$P = a_x\hat{\imath} + a_y\hat{\jmath} + a_z\hat{k}$$

<u>Where:</u>

- $\hat{\imath}$ ,$\hat{\jmath}$ and $\hat{k}$ are the unit vectors in x, y and z axes respectively.
- $a_x$, $a_y$ and $a_z$ are scalars represent the locations (projections) of the point (of the vector) on x, y and z respectively.

An alternative and a more preferable representation for points and vectors using matrices is given by:

$$P = \begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix}$$

The length of a vector is defined as: (the Euclidian norm of a vector)

$$L_p = \sqrt{a_x{}^2 + a_y{}^2 + a_z{}^2}$$

Matrix representations of vectors are more flexible in vector-vector and vector-matrix operations as we will discuss later on.

### 3- Representing vectors and unit-vectors using an additional scalar:

The representation of vectors can be sligtly modified to include a scale factor *w* such that if x, y and z are divided by *w* we can reobtain the original vector. This representation will allow us to generalize a transformation matrix as we will discuss later.

Given $P = \begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix}$, we can represent this vector in another form as:

$$\overline{P} = \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} \qquad \text{Where} \qquad a_x = \frac{x}{w}\,,\ a_y = \frac{y}{w} \text{ and } a_z = \frac{z}{w}$$

Adding the scalar $w$ is very useful in representing vectors with very small dimensions (for both humans and computers), in addition, it will be very helpful in representing a set of coordinates w.r.t another set, using what is known as “Transformation Matrices”.

The scalar $w$ can take any real number, where a special case when $w$ is zero has a significant usage in representing unit vectors; unit vectors are vectors with unit length, and represent directions starting from the origin to infinity.

As an example, consider the vector $\overline{P} = \hat{\imath} + \hat{\jmath} + \hat{k}$.

If we choose $w$ to be unity, we can represent the vector as $\overline{P} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$

Where if we choose $w$ to be zero, we obtain a unit vector in the direction of $\overline{P}$ as:

$$\hat{P} = \begin{bmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ 0 \end{bmatrix}$$

Exercise (1) (Homework):

Consider the vectors a,b and c given below.

1- Give a scaled representation for each of these vectors using the scalars $w = 1, 3\ and\ 9$.
2- Find a unit vector to represent the direction of a, b and c (unit vector for each).
3- Find the unit vector in the direction of (a-b) x c (comment on the answer)
4- Find the result of (axb).(a-b) (comment on the answer)
5- Verify your answer using matlab { use the functions *length()*, *cross()* }

Hint: Refer to Example 2.1 in the book.

$\overline{a} = \hat{\imath} + 2\hat{\jmath} + \hat{k}$ | $\overline{b} = 2\hat{\imath} + \hat{\jmath} - \hat{k}$ | $\overline{c} = 2\hat{\imath} - 2\hat{\jmath} - 4\hat{k}$

### 4- Moore-Penrose Pseudo Inverse:

In real life, a deficient of a redundant system of linear equations is frequently noticed. Such type of systems might possess no unique solution, or they might not possess a solution at all. The solution of linear systems is usually expressed in terms of the four fundamental subspaces of the Matrix representing the linear system.

Many robotic applications (analysis or real-time control), involve the inversion of what is known as the "Jacobian Matrix", which in turn can be deficient (or non-square as well). For this purpose, introducing the pseudo-inverse helps in generalizing solutions as you will see later on in this course.

Given a matrix $A$, the pseudo-inverse of $A$ is defined as:

$$A^{+} = \lim_{\delta \to 0}(A^{T}A + \delta^{2}I)^{-1}A^{T} = \lim_{\delta \to 0} A^{T}(AA^{T} + \delta^{2}I)^{-1}$$

Using one of the formulas, we can obtain a unique inverse for any given matrix.

<u>Useful properties:</u>

- If the matrix is already invertible, its pseudo-inverse is identical to its inverse.
- If a matrix $A$ is not-invertible, but $A^{T}A$ is invertible, the pseudo-inverse will be easy to find (respectively), and is called as the left inverse.
- If a matrix $A$ is not-invertible, but $AA^{T}$ is invertible, the pseudo-inverse will be easy to find (respectively), and is called as the right inverse.

Exercise (2) (Homework):

1- Find by hand the pseudo-inverse of the Matrices *A* ,*B* and *C* givin below (use the right inverse and the left inverse when appropriate).

$$A = \begin{bmatrix} 1 & 1 \\ 3 & 3 \end{bmatrix} \quad | \quad B = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 3 & 3 \end{bmatrix} \quad | \quad C = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

2- Check your answer using Matlab. (hint: use the function *pinv()*)

Show all your steps clearly, and give comments when necessary.

## 5- Orthogonal matrices, and the General Transformation Matrices:

### 5.1- Orthogonal matrices:

A matrix is said to be orthogonal if and only if:

1- All its columns (rows) are unit vectors.

2- Each of its columns (rows) is perpendicular to the rest of the columns (rows).

These two conditions can be expressed in terms of a simple matrix equation:

$$R * R^T = R^T * R = I$$

Where: $I$ is the identity matrix.

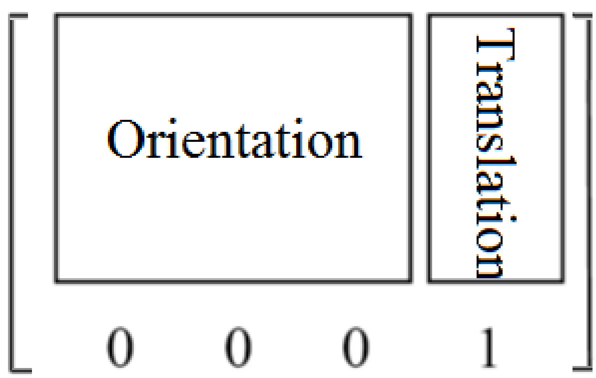
Important properties:

- If $R$ is an orthogonal matrix, $R^T$ is its inverse.
- If $R$ is an orthogonal matrix, $\det(R) = 1$

### 5.3- General Transformation Matrices:

A general transformation matrix is usually used to represent one set of Cartesian coordinates (coordinate frame) w.r.t another set.

A general transformation matrix is a Block matrix that combines two sets of information, the orientation of one coordinate frame (the non-reference frame) w.r.t another (the reference frame), and the position of the non-reference frame w.r.t the reference frame as follows:




Simply, the Orientation part is represented by a special orthogonal matrix, and the translation part is represented by a vector.

<u>Important Properties:</u>

- The inverse a Transformation matrix, is another transformation matrix, and it interchange the role of the reference frame and the non-reference frame, i.e. if ${}^{A}T_{B}$ represents the frame *B* w.r.t the frame *A*, $({}^{A}T_{B})^{-1} = {}^{B}T_{A}$

- The cross product between each two columns of the orientation part (the rotational matrix) should follow the common right-hand-rule for x, y and z Cartesian coordinates as follows:

If: $R = [\boldsymbol{n} \quad \boldsymbol{o} \quad \boldsymbol{a}] = \begin{bmatrix} n_x & o_x & a_x \\ n_y & o_y & a_y \\ n_z & o_z & a_z \end{bmatrix}$

Then: $\times\, o = a$ , $o \times a = n$ , and $a \times n = o$

Exercise (3) (Homework):

Find all the possible sets of the unkowns, that makes the matrix $F$ a transformation matrix.

$$F = \begin{bmatrix} ? & 0 & ? & 5 \\ 1/\sqrt{2} & ? & ? & 3 \\ ? & ? & 0 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Hints:

1- Use the following approuch to solve the problem:

- Formulate a system of 6 equations, with 6 unkowns using the equations:

$$\boldsymbol{n^T * n = o^T * o = a^T * a = 1}$$

$$\boldsymbol{n^T * o = o^T * a = a^T * n = 0}$$

- Solve the system (you should obtain 16 sets of solutions).

- Accept the solutions that obay the followin three conditions:

$$\boldsymbol{n \times o = a}\,, \ \boldsymbol{o \times a = n}\,, \text{ and } \boldsymbol{a \times n = o}$$

2- There is a part of the solution in the course book (pages 36-37).

3- You may use the Matlab command *solve()*. (search for it)

4- The problem has 8 acceptable sets, find them all.

Show all your work clearly.

**6- Inversion of Block matrices:**

In general, matrix inversion is computationally complex, we always tend to use certain properties to simplify the process of matrix inversion.

In the context of this course, the inversion of 4x4 general transformation matrix appears frequently.

In previous, the general transformation matrix was introduced as a block-type matrix from the form:

$T = \begin{bmatrix} R & P \\ 0 & 1 \end{bmatrix}$; where $R$ is an orthogonal matrix, and P is a vector.

This matrix is similar to the block matrix $\begin{bmatrix} A & B \\ 0 & D \end{bmatrix}$ which is known to have the property:

$$\begin{bmatrix} A & B \\ 0 & D \end{bmatrix}^{-1} = \begin{bmatrix} A^{-1} & -A^{-1}BD^{-1} \\ 0 & D^{-1} \end{bmatrix}$$

Using the fact $R^{-1} = R^T$, the inverse of the general transformation matrix simplifies to:

$$T^{-1} = \begin{bmatrix} R & P \\ 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} R^T & -R^T P \\ 0 & 1 \end{bmatrix}$$

Which is very easy to compute in comparison with the general method of inverting a 4x4 matrix.

Exercise (4) (Homework):

Given the following transformation matrices, find the inverse of each by hand, then verify your answer using Matlab. (use the command *inv()* )

$$T_1 = \begin{bmatrix} 0.5403 & -0.8415 & 0 & 5 \\ 0.8415 & 0.5403 & 0 & 5 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad T_2 = \begin{bmatrix} 0.5403 & 0.8330 & 0.1187 & 3 \\ 0.8415 & -0.5349 & -0.0762 & 5 \\ 0 & 0.1411 & -0.9900 & 9 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

**Only Email submissions are accepted (you have exactly 6 days to submit)**

**Paper solutions are not accepted under any circumstances**

- Take clear photos of your paper solutions and combine them all in a single pdf file.
- Send your Matlab codes in a separate file (m-file or word document).

  Submit to the Email: lab.eeng428@gmail.com
- In case of emergency, contact me on the Email: (Don't visit me in the office before writing an Email explaining your problem)

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