EENG 428 Introduction to Robotics Laboratory

EXPERIMENT 4

The Use of Laplace Transformation to Solve Differential Equations

Objectives:

This experiment introduces using MATLAB for defining and solving differential equation using the Laplace transformation approach, as well as plotting the obtained solutions.

1. Introduction

The Laplace transform is one of the most useful tools for solving differential equations with initial conditions. In fact, applying a Laplace transform to a differential equation with initial conditions effectively hides the fact that we're differentiating and turns everything into expressions without any derivatives in sight. We can then manipulate those expressions using quite algebraic operations. Finally, an inverse Laplace transform takes us back to our original differential equation, except that the algebraic manipulations we did cause the solution of our equation to pop out with little extra work on our part. The chart below illustrates this process:

(1) Start with differential equation with initial conditions: $y'' + y = e^t$ $y(0) = 0$ $y'(0) = 0$	→	(2) Take Laplace transform and obtain subsidiary equation: $(s^2 + 1)Y(s) = 1/(s - 1)$
		↓
(4) Use inverse Laplace transform to get back to solution: $y(t) = (1/2)e^t - (1/2)\cos(t) - (1/2)\sin(t)$	←	(3) Solve subsidiary equation for Y: $Y(s) = (1/2)/(s - 1) + (-1/2)s/(s^2 + 1) + (-1/2)/(s^2 + 1)$

2. Forward and inverse Laplace Transform with MATLAB

a. Forward Laplace Transformation

The command ***laplace*** is used.

Syntax

laplace(F)

L = laplace(F) is the Laplace transform of the scalar symbol F with default independent variable ***t***. The default return is a function of ***s***. The Laplace transform is applied to a function of ***t*** and returns a function of ***s***. It should be noted that the Laplace transform is done on symbolic variables, so the *sym* or *syms* command has to be previously used.

Example 1

Find the Laplace transform of:

$$g(s) = \frac{1}{\sqrt{s}}$$

```
syms s

g = 1/sqrt(s)

laplace(g)
```

, returns:

```
pi^(1/2)/t^(1/2)
```

b. inverse Laplace transform:

The command ***ilaplace*** is used.

Syntax

F = ilaplace(L)

F= ilaplace(L) is the inverse Laplace transform of the scalar symbolic object L with default independent variable ***s***. The default return is a function of ***t***. The inverse Laplace transform is applied to a function of ***s*** and returns a function of ***t***. Again, all variables must be symbolic.

Example:

Evaluate the inverse Laplace transform of:

$$f(u) = \frac{1}{u^2 - a^2}$$

The code:

```
syms x u

syms a real

f = 1/(u^2-a^2)

simplify(ilaplace(f,x))
```

, returns:

```
sinh(a*x)/a
```

3. Differntial Equation Solution via Laplace Transform:

As mentioned in the introduction section, solution of a differential equation with Laplace transform involves firstly transforming the whole equation into the Laplace domain. Next, $Y(s)$ is written in terms of $X(s)$. Finally, $Y(s)$ is transformed back to the time domain to end up with $y(t)$ being the solution to the differential equation.

These steps are illustrated in the following example.

Example:

Solve the following IVP(Initial-Value Problem).

$$y'' - 10y' + 9y = 5t, \quad y(0) = -1, \quad y'(0) = 2$$

The first step in using Laplace transforms to solve an IVP is to take the transform of every term in the differential equation.

$$L(y'') - 10L(y') + 9L(y) - L(5t) = 0$$

Herein, MATLAB is used to perform this step , as follows:

```
pretty(laplace( diff( sym('y(t)'),2 )-10*diff( sym('y(t)'),1 )+ 9*sym('y(t)')-5*sym('t') ))
```

which returns the transformation of the whole equation as:

```
                       2                           5
10 y(0) - D(y)(0) - s y(0) + s  transform::laplace(y(t), t, s) - -- -
                                                  2
                                                 s

   10 s transform::laplace(y(t), t, s) + 9 transform::laplace(y(t), t, s),
```

Or, in the standard form:

$$10y(0) - y'(0) - sy(0) + s^2Y(s) - \frac{5}{s^2} - 10sY(s) + 9Y(s) = 0$$

Next, Plug in the initial conditions and collect all the terms that have a *Y(s)* in them.

$$(s^2 - 10s + 9)Y(s) + s - 12 = \frac{5}{s^2}$$

Then, solve for *Y(s)*.

$$Y(s) = \frac{5}{s^2(s-9)(s-1)} + \frac{12-s}{(s-9)(s-1)}$$

At this stage, the next step is to transform back Y(s) to the time domain, as follows:

```
>> syms s

>> ilaplace( 5/( s^2*(s-9)*(s-1) ) +(12-s)/( (s-9)*(s-1) ) )

 ans =

 (5*t)/9 + (31*exp(9*t))/81 - 2*exp(t) + 50/81
```

So, the solution to the equation is:

$$y(t) = \frac{5t}{9} + \frac{31}{81}e^{9t} - 2e^{t} + \frac{50}{81}$$

3. Homework

Considering the following questions, provide your answers in a report including your codes and associated outputs and/or plots by copying them and pasting on the report pages. To be submitted as a hard copy on next lab session. You should work individually and group work will be penalized.

1. Solve the following IVP, by of Laplace transform method via MATLAB.

$$y'' + 4y' = \cos(t-3) + 4t, \quad y(3) = 0 \;\; y'(3) = 7$$

2. Solve the following IVP, by of Laplace transform method via MATLAB.

$$y'' - 6y + 15y = 2\sin(3t)\,, y(0) = -1, \;\; y'(0) = -4$$