PLANE WAVES IN A CONDUCTING MEDIUM

Consider a linear, homogeneous, lossy medium with permittivity ε, permeability μ and conductivity σ. In such a medium we have:

$$\nabla X\overline{E} = -jw\mu\overline{H} \qquad \nabla.\overline{E} = 0$$
$$\nabla X\overline{H} = \sigma\overline{E} + jw\varepsilon\overline{E} \qquad \nabla.\overline{H} = 0$$

Consider

$$\nabla X\overline{H} = \sigma\overline{E} + jw\varepsilon\overline{E} = jw(\varepsilon + \frac{\sigma}{jw})\overline{E} = jw\varepsilon_c\overline{E}$$

where , $\varepsilon_c = \varepsilon - j\frac{\sigma}{w} = \varepsilon' - \varepsilon''$ is defined as the complex permittivity of the conducting

medium with $\varepsilon' = \varepsilon$ and $\varepsilon'' = \frac{\sigma}{w}$.

Define the loss tangent $\tan\delta_c = \frac{\varepsilon''}{\varepsilon'} = \frac{\sigma}{w\varepsilon}$ with loss angle δ_c.

<u>The Wave Equations:</u>

$\nabla X\nabla X\overline{E} = -jw\mu\nabla X\overline{H} = -jw\mu\, jw\varepsilon_c\overline{E}$ or

$\nabla(\nabla.\overline{E}) - \nabla^2\overline{E} = w^2\mu\varepsilon_c\overline{E}$ or,

$\nabla^2\overline{E} + k_c^2\overline{E} = 0$ where $k_c = w\sqrt{\mu\varepsilon_c}$ the complex wave number.

$$k_c = w\sqrt{\mu(\varepsilon - j\frac{\sigma}{w})}$$

Now define, the propagation constant as $\gamma = jk_c = jw\sqrt{\mu\varepsilon_c} = jw\sqrt{\mu\left(\varepsilon - j\frac{\sigma}{w}\right)}$

Since γ is complex, let

$$\gamma = \alpha + j\beta = jk_c$$

then

$$k_c = \beta - j\alpha$$

Where α is the attenuation constant (*Nepers/m*) and β is the phase constant in *rad/m*.

For a lossless medium, $\sigma = 0 \Rightarrow \alpha = 0$ and $\beta = k = w\sqrt{\varepsilon\mu}$. In term of γ the wave equation becomes: $\nabla^2 \bar{E} - \gamma^2 \bar{E} = 0$.

<u>A Simple Solution to the Wave Equation</u>

Let $\bar{E} = \hat{a}_x E_x(z)$, then $\frac{d^2 E_x}{dz^2} + k_c^2 E_x = 0$

The general solution:

$$E_x(z) = E_{x_0}^+ e^{-jk_c z} + E_{x_0}^- e^{jk_c z}$$

Consider only the (+) solution:

$$E_x^+(z) = E_{x_0}^+ e^{-jk_c z}$$

$-jk_c = -j(\beta - j\alpha) = -\alpha - j\beta$, so $E_x^+(z) = E_{x_0}^+ e^{-\alpha z} e^{-j\beta z}$

Assume that $E_{x_0}^+$ is real without any loss of generality, the real physical $\bar{E}$ field will then be

$$E_x^+(z,t) = E_{x_0}^+ e^{-\alpha z} \cos(wt - \beta z)$$

The corresponding $\bar{H}$ -field:

$H_y^+(z) = \frac{E_{x_0}^+}{\eta_c} e^{-\alpha z} e^{-j\beta z}$ where $\eta_c = \left(\frac{\mu}{\varepsilon_c}\right)^{\frac{1}{2}}$ is the complex intrinsic impedance.

Let $\eta_c = |\eta_c| e^{j\theta\eta}$ then

$H_y^+(z) = \frac{E_{x_0}^+}{|\eta_c|} e^{-\alpha z} e^{j(\theta_\eta - \beta z)}$ and

$$H_y^+(z,t) = \frac{E_{x_0}^+}{|\eta_c|} e^{-\alpha z} \cos(wt - \beta z + \theta_\eta)$$

The expressions for α and β are exact. α and β are seen to depend on frequency. The phase velocity:

$$u_p = \frac{w}{\beta} = \sqrt{\frac{2}{\varepsilon\mu}} \left[\sqrt{1 + \left(\frac{\sigma}{w\varepsilon}\right)^2} + 1 \right]^{-1/2}$$

u_p is a function of frequency, so we have dispersion.

Consider now two limiting cases:

I. LOW-LOSS DIELECTRIC CASE

If $\varepsilon'' << \varepsilon'$ or $\left(\frac{\alpha}{w\varepsilon} << 1\right)$(Displacement current dominates considerably on conduction current), then the attenuation constant:

$$\alpha = w\sqrt{\frac{\varepsilon\mu}{2}}\left[\sqrt{1+\left(\frac{\sigma}{w\varepsilon}\right)^2} - 1\right]^{1/2} \approx w\sqrt{\frac{\varepsilon\mu}{2}}\left[1+\frac{1}{2}\left(\frac{\sigma}{w\varepsilon}\right)^2 - 1\right]^{1/2}$$

$$\alpha \approx w\sqrt{\frac{\varepsilon\mu}{2}}\frac{1}{\sqrt{2}}\left(\frac{\sigma}{w\varepsilon}\right) = \frac{\sigma}{2}\sqrt{\frac{\mu}{\varepsilon}}$$

So $\alpha \approx \frac{\sigma}{2}\sqrt{\frac{\mu}{\varepsilon}}$ independent of frequency.

Penetration depth $\delta = \frac{1}{\alpha} = \frac{2}{\sigma}\sqrt{\frac{\varepsilon}{\mu}}$

The phase constant:

$$\beta = w\sqrt{\frac{\varepsilon\mu}{2}}\left[\sqrt{1+\left(\frac{\sigma}{w\varepsilon}\right)^2} + 1\right]^{1/2} \approx w\sqrt{\frac{\varepsilon\mu}{2}}\left[1+\frac{1}{2}\left(\frac{\sigma}{w\varepsilon}\right)^2 + 1\right]^{1/2}$$

$$\beta \approx w\sqrt{\frac{\varepsilon\mu}{2}}\left[2+\frac{1}{2}\left(\frac{\sigma}{w\varepsilon}\right)^2\right]^{1/2} = w\sqrt{\frac{\varepsilon\mu}{2}}\sqrt{2}\left[1+\frac{1}{4}\left(\frac{\sigma}{w\varepsilon}\right)^2\right]^{1/2}$$

$$\beta \approx w\sqrt{\varepsilon\mu}\left[1+\frac{1}{8}\left(\frac{\sigma}{w\varepsilon}\right)^2\right]$$ β is a function of frequency.

Intrinsic Impedance:

$$\eta_c = \sqrt{\frac{\mu}{\varepsilon_c}} = \left[\frac{\mu}{\varepsilon - j\frac{\sigma}{w}}\right]^{1/2} = \sqrt{\frac{\mu}{\varepsilon}}\left(1 - j\frac{\sigma}{w\varepsilon}\right)^{-1/2}$$

$$\eta_c \approx \sqrt{\frac{\mu}{\varepsilon}}\left(1 + j\frac{\sigma}{2w\varepsilon}\right)$$

Phase velocity:

$$u_p = \frac{w}{\beta} = \frac{1}{\sqrt{\varepsilon\mu}}\left[1 - \frac{1}{8}\left(\frac{\sigma}{w\varepsilon}\right)^2\right]$$ u_p is a function of frequency.

Consider again $\bar{E}(z)$ again:

$\bar{E}(z) = E_{x_0}^{+} e^{-jk_c z}\hat{a}_x = E_{x_0}^{+} e^{-\alpha z} e^{-j\beta z}\hat{a}_x$ with

$$\bar{E}(z,t) = \hat{a}_x E_{x_0}^{+} e^{-\alpha z}\cos(wt - \beta z)$$

The corresponding magnetic field:

$$\bar{H}(z) = \frac{1}{\eta_c}\hat{a}_y E_{x_0}^{+} e^{-\alpha z} e^{-j\beta z}$$

$$\frac{1}{\eta_c} \cong \frac{1}{\sqrt{\frac{\mu}{\varepsilon}}\left(1 + j\frac{\sigma}{2w\varepsilon}\right)} \cong \frac{1}{\eta}\left(1 - j\frac{\sigma}{2w\varepsilon}\right) = \frac{1}{\eta}\left[1 + \left(\frac{\sigma}{2w\varepsilon}\right)^2\right]^{1/2} e^{-j\tan^{-1}\left(\frac{\sigma}{2w\varepsilon}\right)}$$

So,

$$\bar{H}(z) = \hat{a}_y \frac{E_{x_0}^{+}}{\eta}\left[1 + \left(\frac{\sigma}{2w\varepsilon}\right)^2\right]^{1/2} e^{-\alpha z} e^{-j\left(\beta z + \tan^{-1}\left(\frac{\sigma}{2w\varepsilon}\right)\right)}$$

$$\bar{H}(z,t) = \hat{a}_y \frac{E_{x_0}^{+}}{\eta}\left[1 + \left(\frac{\sigma}{2w\varepsilon}\right)^2\right]^{1/2} e^{-\alpha z}\cos\left(wt - \beta z - \tan^{-1}\left(\frac{\sigma}{2w\varepsilon}\right)\right)$$

We see that $\bar{E}(z,t)$ and $\bar{H}(z,t)$ are not in phase in time. This phase difference is small since ($\frac{\alpha}{2w\varepsilon} << 1$) and is a function of frequency.