Fundamental Period of Continuous Time Signals

To identify the period T, the frequency $f = \frac{1}{T}$ or the angular frequency $w = 2\pi f = 2\pi/T$ of a given sinusoidal or complex exponential signal, it is always helpful to write it in any of the following forms

$$\sin(wt) = \sin(2\pi ft) = \sin(2\pi t/T)$$

The fundamental frequency of a signal is the Greatest Common Divisor (GCD) of all the frequency components contained in a signal and equivalently, the fundamental period is the Least Common Multiple (LCM) of all individual periods of the components.

Example 1

Find the fundamental frequency of the following continuous signal

$$x(t) = cos\left(\frac{10\pi}{3}t\right) + sin\left(\frac{5\pi}{4}t\right)$$

The frequencies and periods of the two terms are, respectively,

$$w_1 = \frac{10\pi}{3}, f_1 = \frac{5}{3}, T_1 = \frac{3}{5}$$

$$\text{and } w_2 = \frac{5\pi}{4}, f_2 = \frac{5}{8}, T_2 = \frac{8}{5}$$

The fundamental frequency f_0 is the GCD of $f_1 = 5/3$ **and** $f_2 = 5/8$

$$f_0 = GCD\left(\frac{5}{3}, \frac{5}{8}\right) = GCD\left(\frac{40}{24}, \frac{15}{24}\right) = \frac{5}{24}$$

Alternatively, the period of the fundamental T_0 is the LCM of $T_1 = \frac{3}{5}$ **and** $T_1 = \frac{8}{5}$

$$T_0 = LCM\left(\frac{3}{5}, \frac{8}{5}\right) = \frac{24}{5}$$

Now we get $w_0 = 2\pi f_0 = \frac{2\pi}{T_0} = \frac{5\pi}{12}$ and the signal can be written as

$$x(t) = cos\left(8\frac{5\pi}{12}t\right) + sin\left(3\frac{5\pi}{12}t\right) = cos(8w_0t) + sin(3w_0t)$$

i.e., the two terms are the 3rd and 8th harmonic of the fundamental frequency w_0, respectively.

Fundamental Period of Discrete Time Signals

Example 1

Considering a discrete-time signal

$$x[n] = cos\left(\frac{\pi}{2}n\right) cos\left(\frac{\pi}{4}n\right), \quad n \in Z,$$

The time period of the signal $x[n]$ can be found empirically as $\left(\frac{2\pi}{\pi/4}\right) = 8$ since the smaller sub-period is $\pi/4$. However, since the mathematical method will always give us a more precise result, we shall refer to the trigonometric identities.

From the identity

$$\cos(a)\cos(b) = \frac{1}{1}[\cos(a-b) + \cos(a+b)]$$

Hence, the waveform adopts the period of the lowest frequency because

$$\frac{\pi}{2}n - \frac{\pi}{4}n = \frac{\pi}{4}n \Rightarrow \frac{\pi}{4}n = \frac{2\pi}{8}n = \frac{2\pi}{N}n$$

Hence the period is $N = 8$.

Example 2

$$x[n] = sin\left(\frac{5\pi}{6}n\right) + cos\left(\frac{3\pi}{4}n\right)$$

The Least-Common-Multiplier of the denominator is 12. Therefore

$$x[n] = sin\left(\frac{10\pi}{12}n\right) + cos\left(\frac{9\pi}{12}n\right)$$

Hence, the fundamental frequency is $w_0 = \frac{\pi}{12}$, the fundamental period is $T = \frac{2\pi}{w_0} = 24$ and the two terms are the 9th and 10th harmonic of the fundamental frequency w_0.