2.1 INTRODUCTION

Define the physical system and its components;

- ⇨ Formulate the mathematical models and list the necessary assumption;
- ⇨ Write the differential equation describing the model;
- ⇨ Linearized the equation if necessary;
- ⇨ Perform Laplace transform on the linearized equations;
- ⇨ Solve the equations for the desired output variables;
- ⇨ Examine the solutions and the assumptions;
- ⇨ If necessary, reanalyze or redesign the system;

2.2 DIFFRENTIAL EQUATIONS OF PHYSICAL SYSTEMS

The differential equations describing the dynamic performance of physical system are obtained by utilizing the physical laws of the process. This approach applies equally well to mechanical, electrical, fluid, and thermodynamics systems.

Physical Laws commonly used in control:

Newotn's Law: $\sum forces = Ma.$

If we treat the inertial force Ma as a "force", then according to D'Alembert's principle, by bring the Ma to the left-hand side of the equation yield

$$\sum forces = 0.$$

Physical Laws commonly used in control

Newtown's Law:
The sum of forces on a body equal to zero; the sum of torques on a body equals zero.

$$\sum forces = 0.$$

Kirchhoff's voltage law:
The sum of voltages around a closed path equals zero.

$$\sum voltages = 0.$$

Kirchhoff's current law:

The sum of electric currents flowing from a node equals zero. $\sum currents = 0.$

Variables of physical system

Through-variable: a variable is transmitted through a physical element, such as force, torque, current, fluid flow rate, etc..

Across-variable: a variable is measured the difference across the physical element, such as velocity, displacement, voltage, pressure, temperature.

On Building Models

- Control-relevant models are often quite simple compared to the true system and generally combine physical reasoning with experimental data
- Actuators should be included as they often are nonlinear and have their own dynamic behavior

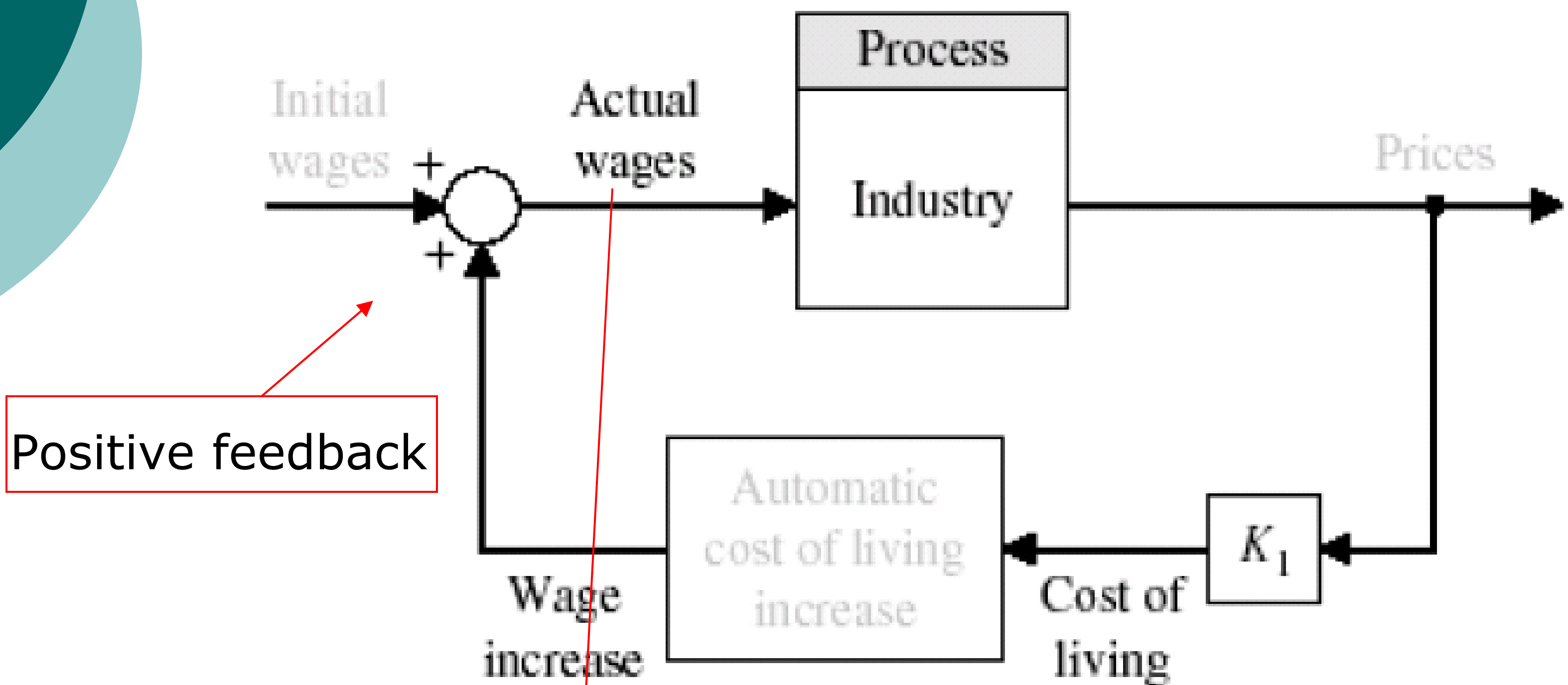
Model Development

- Essential step of the “Design Procedure”
- Three different choices for modeling:
  - Analysis: Mathematical models based on first principles, including differential equations.
  - Grey-Box: Model is developed and then parameters are inferred from experiments.
  - Black-Box: Input/Output data is used to infer a dynamic relationship.
- Nonlinear models can be linearized.
- Linear models are most often used in analysis and design.

Examples of Models

- Nonlinear time varying models
- Linear Time Invariant (LTI) models
- Continuous time /Discrete time models
- Transfer functions/State space models
- Analogies between mechanical, electrical, hydraulic and biological models

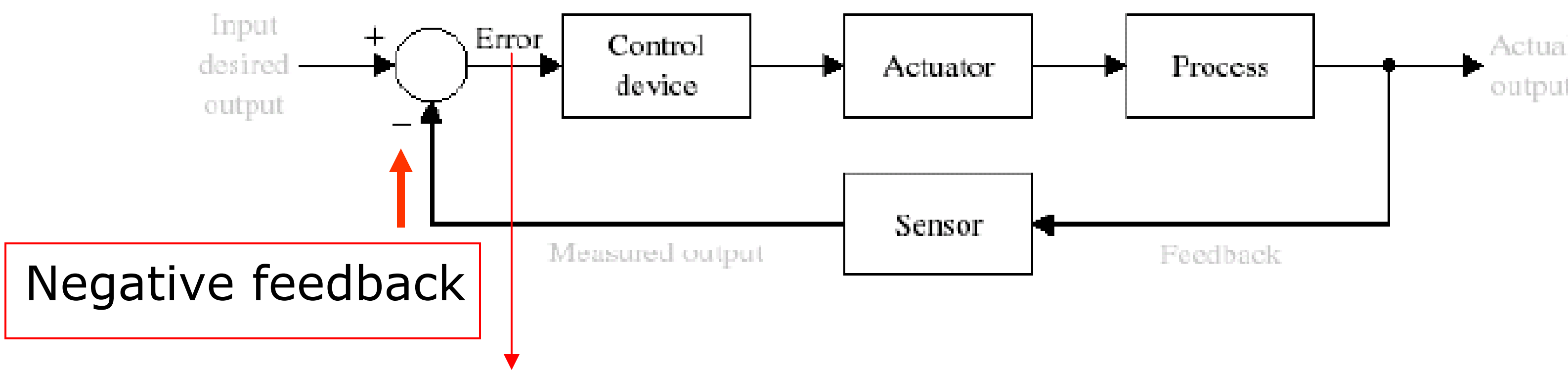
Positive Feedback and Inflation



Actual wages=Initial wages ⊕ Wage increase

Typical Feedback Loop

- The sensor and the actuator are key components of the feedback loop

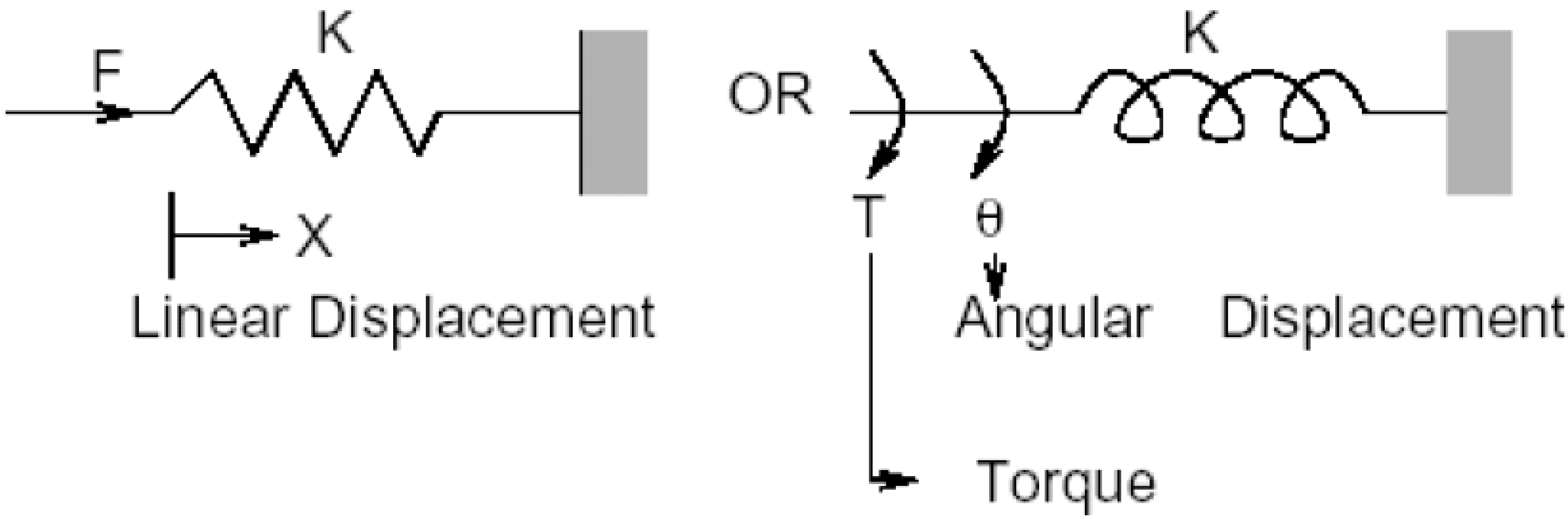


Error=input (desired output) - Measured output

Modelling by Analysis

- Given a physical system and its principles of operation
  - Look up relevant physical laws (conservation principles)
  - Define inputs and outputs
  - Derive balance equations and establish initial conditions
  - Verify existence of solutions
- Works well for some mechanical and electrical situations

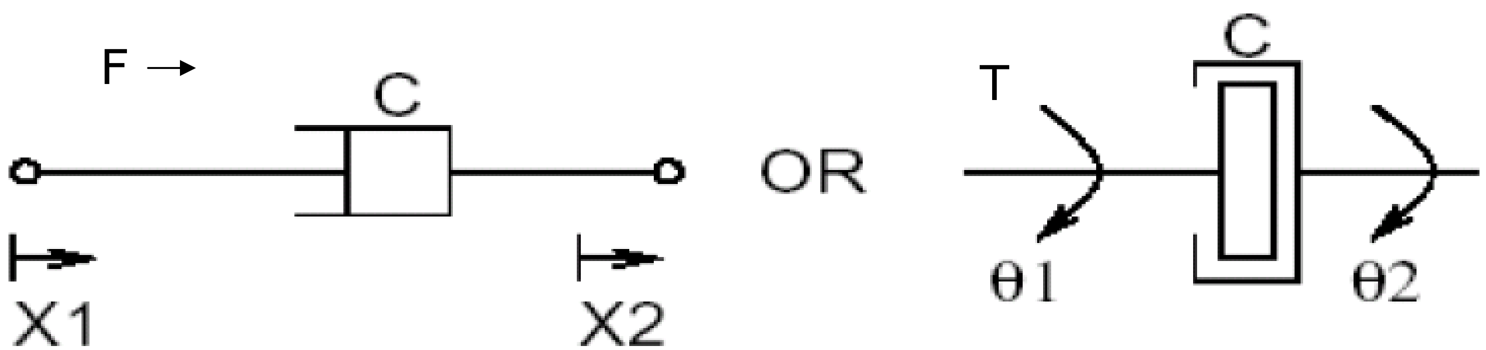
Springs



Physical Laws: F=Kx T=K θ

Spring Constants

Dampers and Dashpots



Linear Displacement

Angular Displacements

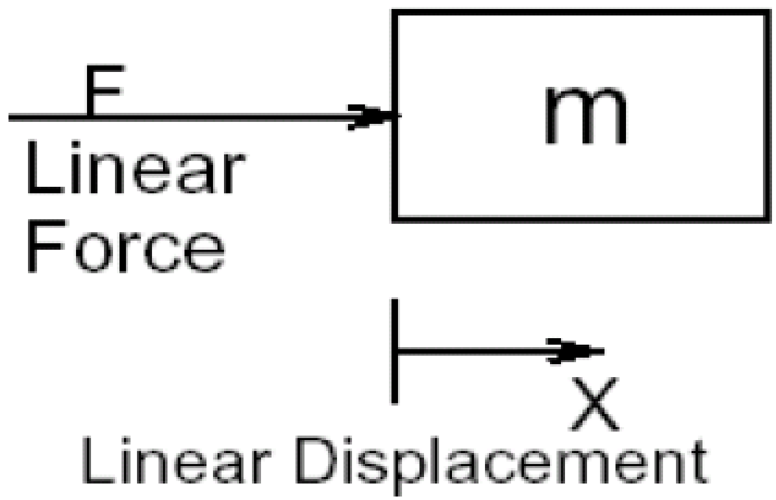
Physical Laws:

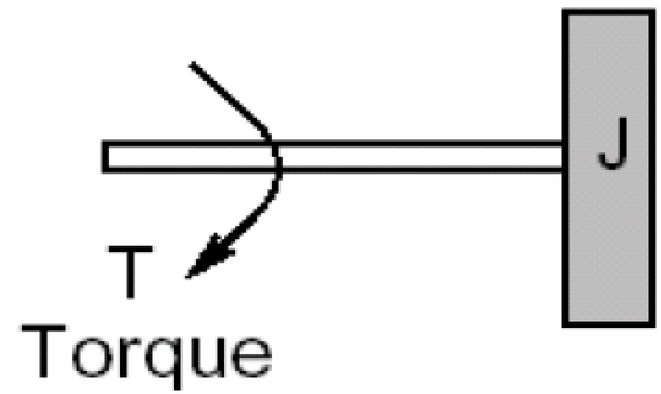
$F=C(\dot{X}1-\dot{X}2)$

$T=C(\dot{\theta}1-\dot{\theta}2)$

Damping Coefficients

Mass and Inertia





Physical Laws:

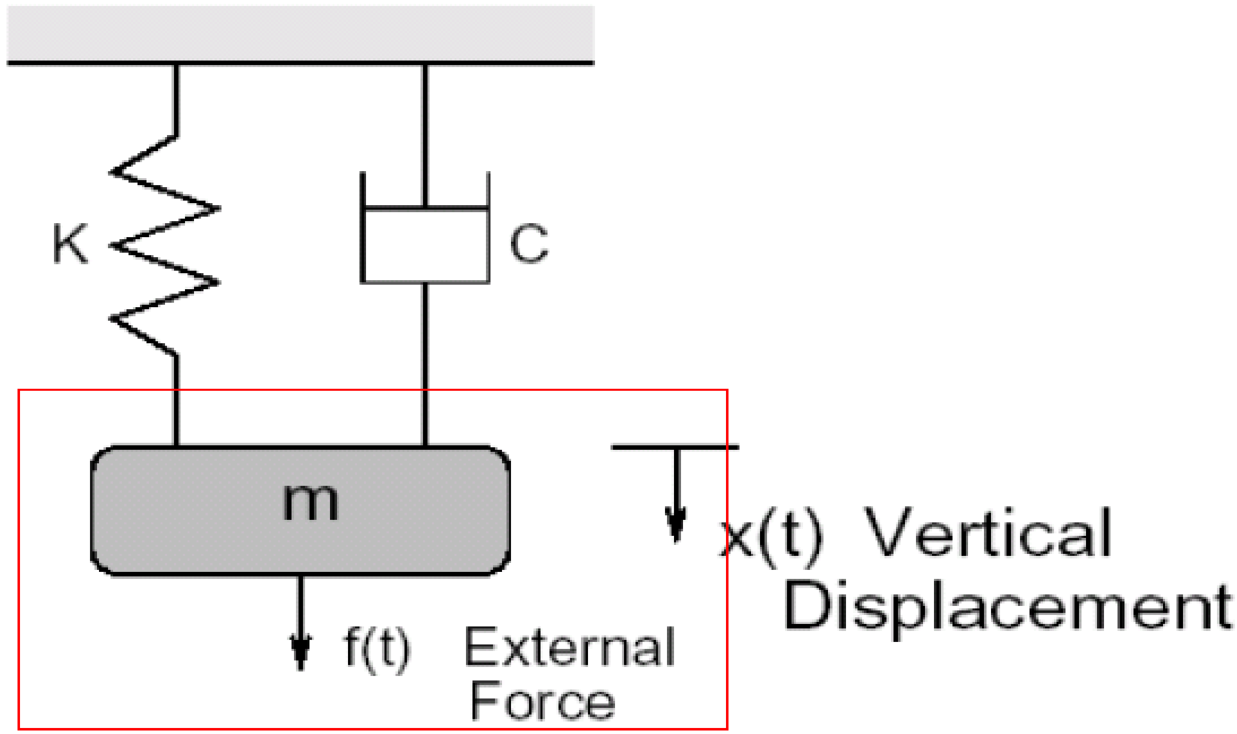
$$F = m\ddot{X}$$

$$T = J\ddot{\theta}$$

m: mass

J: moment of inertia

Spring-Mass-Damper System



$$m\ddot{x} = F(t)$$

Spring-Mass-Damper System

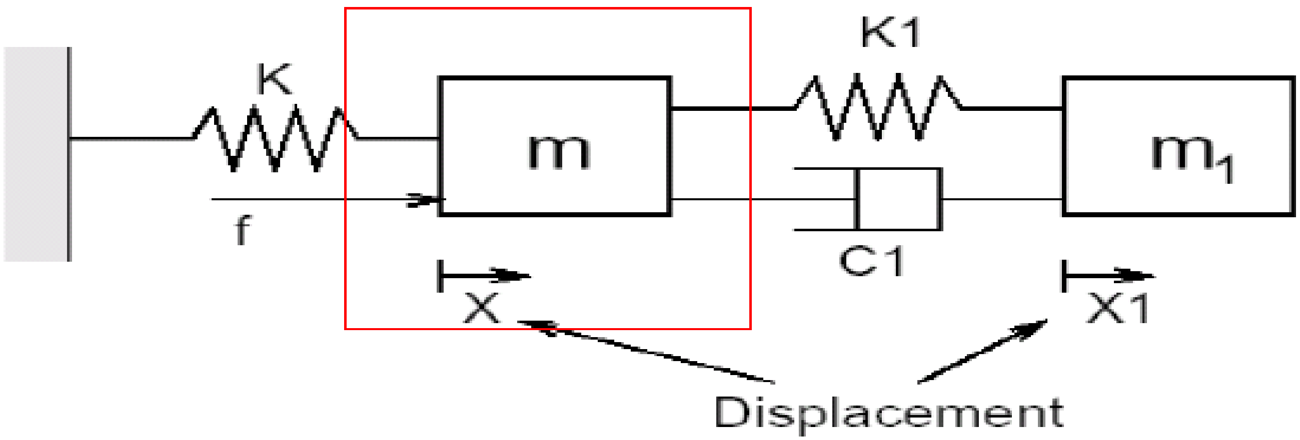
$F(t) \longleftarrow$ equivalent force on mass m.

The signs of $-kx$ and $-c\dot{x}$ are negative because these forces oppose the motion of m.

$$\Longrightarrow \quad m\ddot{x} + c\dot{x} + kx = f(t)$$

Note: (i) Second order system (energy storage), (ii) dc gain

Two-Mass System



$$\begin{cases} m\ddot{x} = \\ m_1\ddot{x}_1 = \end{cases}$$

Note: 4^{th} order system

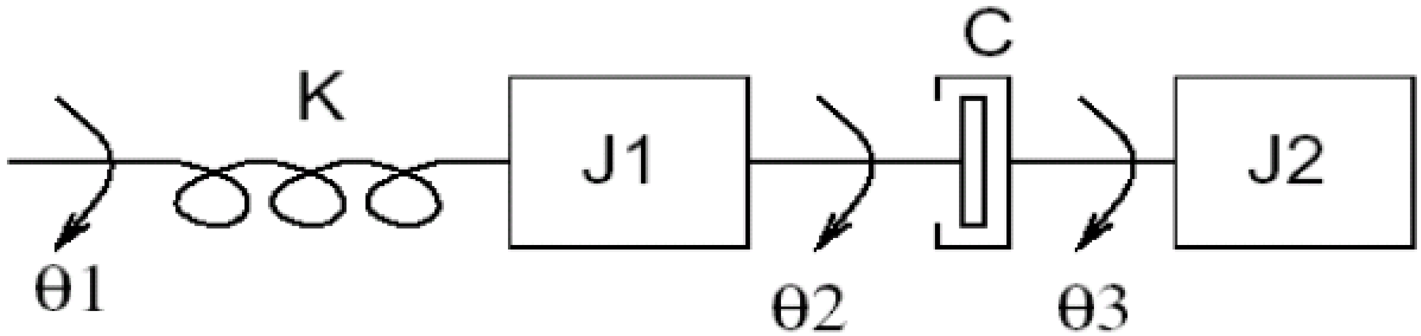
Two-Mass System

<u>Note:</u>

- forces due to the spring k, and damper c_1 oppose the motion of m,
- external force f causes the motion of m.

Initial conditions needed: $x(0), x_1(0), \dot{x}(0), \dot{x}_1(0)$.

Rotating Drive System



θi := Angular Positions

The model of the dynamical system:

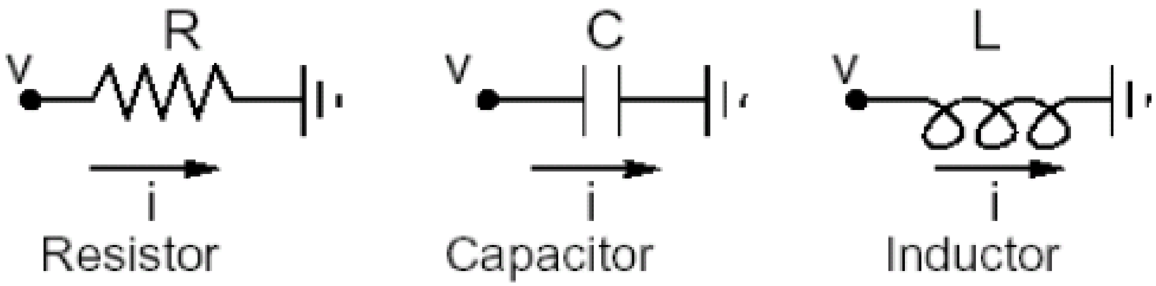
$$\begin{cases} J_1\ddot{\theta}_2 = C(\dot{\theta}_3 - \dot{\theta}_2) - K(\theta_2 - \theta_1) \\ J_2\ddot{\theta}_3 = -C(\dot{\theta}_3 - \dot{\theta}_2) \end{cases}$$

Rotating Drive System

$T(t)$	$\longleftarrow$	equivalent torque acting on J_1
$-c(\dot{\theta}_2 - \dot{\theta}_3)$	$\longleftarrow$	torque opposing motion of J_1
$k(\theta_1 - \theta_2)$	$\longleftarrow$	torque causing motion of J_1

Initial conditions needed: $\theta_2(0), \theta_3(0), \dot{\theta}_2(0), \dot{\theta}_3(0)$.

Basic Electric Components

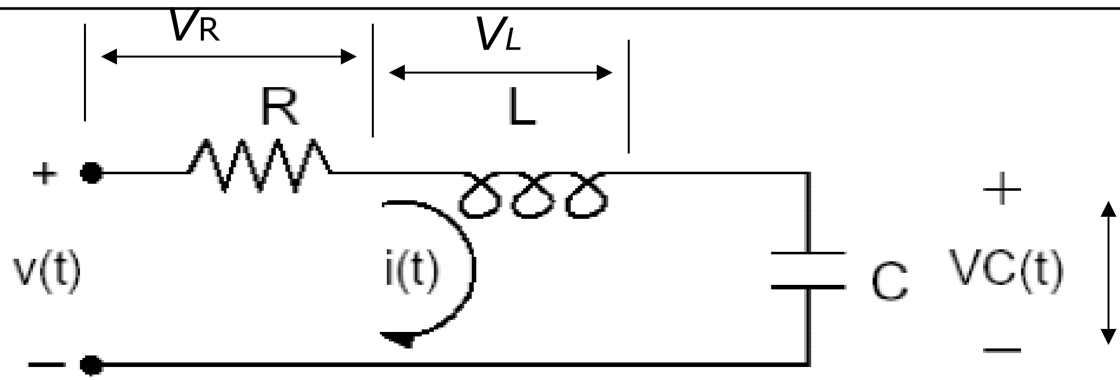


$v \longleftarrow$ voltage

$i \longleftarrow$ current

$\Longrightarrow$ Models : $v = iR$, $i = C\frac{dv}{dt}$, $v = L\frac{di}{dt}$

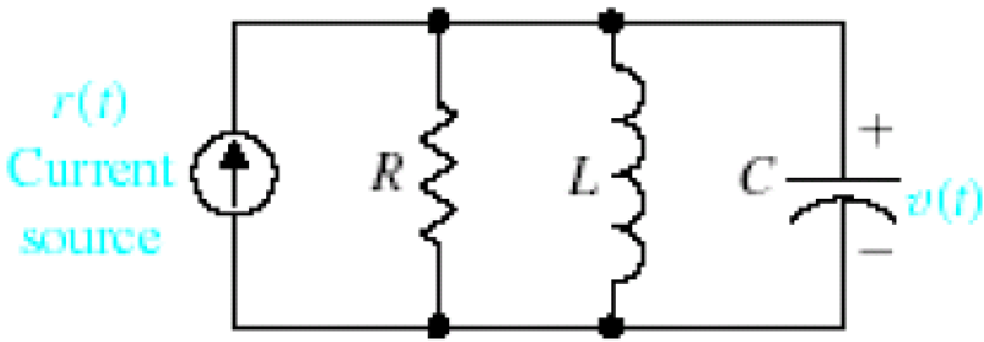
Series RLC Circuit



Applying Kirchhoff's voltage law yields an integro-differential model:

$$\begin{aligned} v(t) &= v_R(t) + v_L(t) + v_C(t) \\ &= Ri(t) + L\frac{di(t)}{dt} + \frac{1}{C}\int_0^t i(\tau)\, d\tau \end{aligned}$$

Parallel RLC Circuit

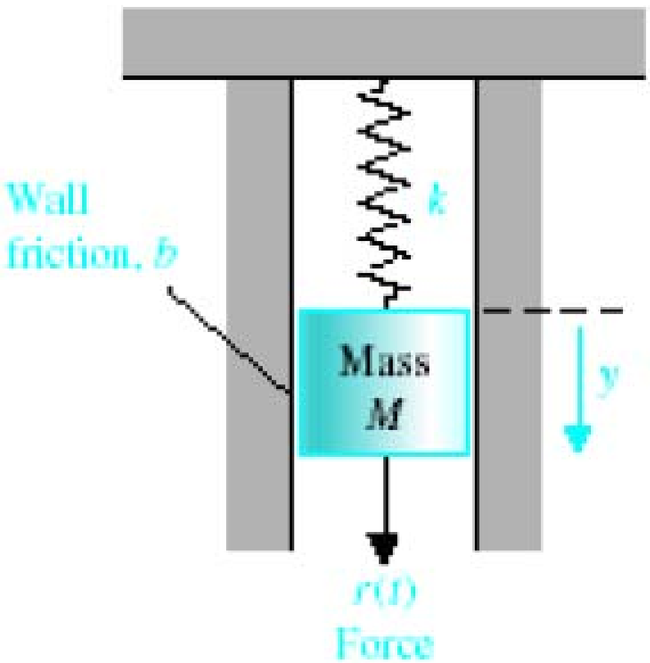


Applying Kirchoff's current law:

$r(t) = r_R + r_L + r_C$, r(t) is current.

$$\frac{v(t)}{R} + C\frac{dv(t)}{dt} + \frac{1}{L}\int_0^t v(t)\,dt = r(t)$$

Analogous Systems



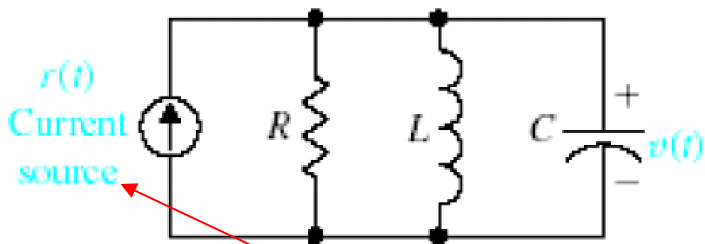
$$M\frac{d^2y(t)}{dt^2}+b\frac{dy(t)}{dt}+ky(t)=r(t)$$

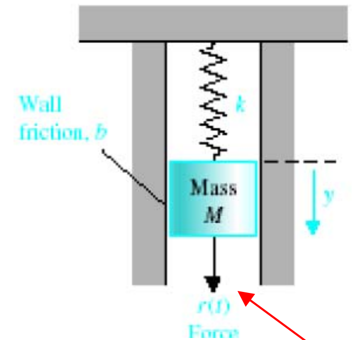
Rewriting in terms of velocity:

$$M\frac{dv(t)}{dt}+bv(t)+k\int_0^t v(t)dt=r(t)$$

We see that it is equivalent to the previous RLC circuit where the velocity and the voltage are analogous variables. Those systems are called analogous systems.

Analogous Systems





$$\frac{v(t)}{R} + C\frac{dv(t)}{dt} + \frac{1}{L}\int_0^t v(t)dt = r(t)$$

$$M\frac{dv(t)}{dt} + bv(t) + k\int_0^t v(t)dt = r(t)$$

volatge

velocity

M=C b=1/R K=1/L

Velocity=Voltage Force =current

Modelling Issues

- To design systematically and verify a controller, one needs a mathematical description of the process to be controlled
- A model is never perfect, hence modeling errors are a fact of life
- Some models can be developed from first principles, others are obtained from observing the process in action.
- Simple mechanical systems and electrical circuits lead to similar differential equations and transfer function relations – they are analogous

Summary

- Design involves simulation and often requires an iterative design process
- To design systematically and verify a controller, one needs a mathematical description of the process to be controlled
- A model is never perfect, hence modeling errors are a fact of life
- Some models can be developed from first principles, others are obtained from observing the process in action.
- Simple mechanical systems and electrical circuits lead to similar differential equations and transfer function relations – they are analogous

Linearization of Nonlinear Models

- Although many systems are nonlinear, they can be approximated by linear systems for small deviations around a chosen operating point
- Note that the derived approximation will be valid only around this operating point
- Control theory for Linear Time-Invariant (LTI) systems is more complete and much simpler than that for nonlinear systems

Linearization of Nonlinear Models

A linear system satisfied the properties of superposition and homogeneity.

Principle of superposition:

$$y = f(x)$$

$$y_1 = f(x_1) \quad and \quad y_2 = f(x_2)$$

$$y_1 + y_2 = f(x_1 + x_2)$$

Homogeneity:

$$y = f(x)$$

$$\beta y = f(\beta x)$$

Linearization of Nonlinear Models

A system represented by

$$y = mx + b$$

is not linear, because it does not satisfy the homogeneity property.

However, this second device may be considered linear about an operating point x_0, y_0 for small changes Δx and Δy.

$$x = x_0 + \Delta x \qquad y = y_0 + \Delta y$$

$$y = mx + b$$

$$y_0 + \Delta y = mx_0 + m\Delta x + b$$

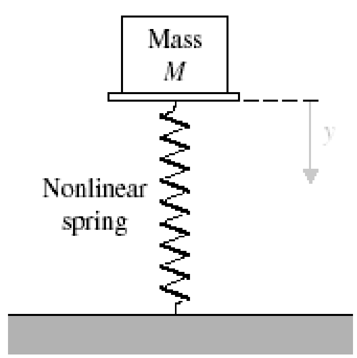
Since $y_0 = mx_0 + b$, we have

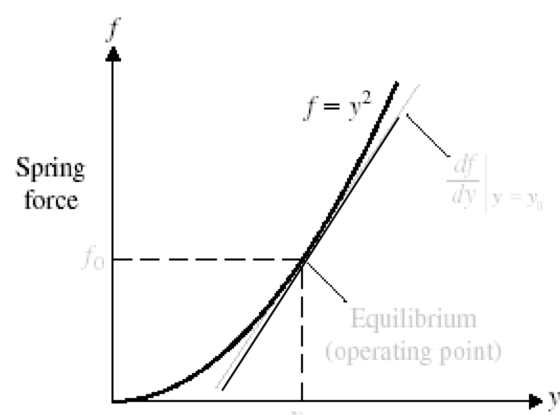
$$\boxed{\Delta y} = m\Delta x + (mx_0 + b - y_0) \boxed{= m\Delta x}$$

How to Linearize

- Identify an operating point
- Perform Taylor series expansion and keep only constant and 1st derivative terms

$$y = \underbrace{f(x_0) + \frac{\partial f(x_0)}{\partial x}\bigg|_{x=x_0}(x-x_0)} + \frac{\partial^2 f(x_0)}{\partial x^2}\bigg|_{x=x_0}\frac{(x-x_0)^2}{2!} + \underset{\text{H.O.T.}}{\ldots}$$





How to Linearize

$$f = y^2$$

How to Linearize

$$f = y^2$$

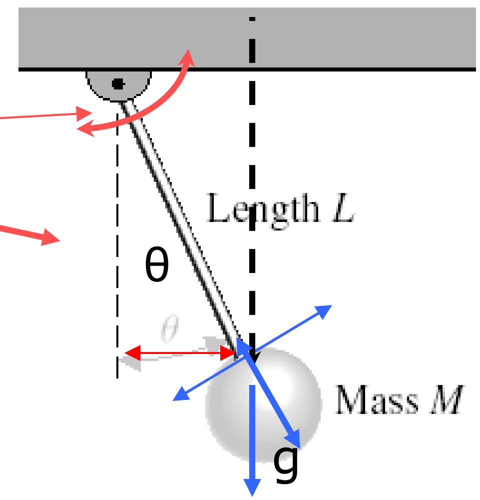
$$f_0 = Mg = y_0^2 \quad \Rightarrow \quad y_0 = (Mg)^{1/2}$$

$$\Delta f = m\Delta y, \quad \Rightarrow \quad m = \left.\frac{df}{dy}\right|_{y_0} = 2y_0$$

$$\Delta f = 2y_0\Delta y$$

Example of Linearization

- Pendulum



$$T = MgL\sin\theta$$

Equilibrium: $\theta_0 = 0 \quad T_0 = 0$

Taylor Series

$$T - T_0 = MgL \left.\frac{\partial \sin\theta}{\partial\theta}\right|_{\theta_0} (\theta - \theta_0)$$

$$T = MgL(\cos\theta_0)(\theta - \theta_0)$$

$$T = MgL\theta$$